Lecture 9

(1) Let f_1, f_2 be two functions. Then

$$K(f_1+f_2) = \int_a^b k(x,x')\left(f_1(x') + f_2(x')\right)dx' =$$

$$\int_a^b k\cdot f_1 + \int_a^b k f_2 = Kf_1 + Kf_2$$

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(2) Let $f = (f_1, f_2, \ldots, f_n)$ then

$$(Kf)(x_j) = \int_a^b k(x_j, x')f(x')dx' \approx$$

$$\approx \sum_{i=1}^{n} \underbrace{\left(\frac{1}{n}\right)}_{\text{stepsize}} \underbrace{k(x_j, x_i) f(x_i)}_{\text{function value}} = g_j$$

So

$$g = Kf, \quad (K)_{ij} = \frac{1}{n} k(x_j, x_i)$$

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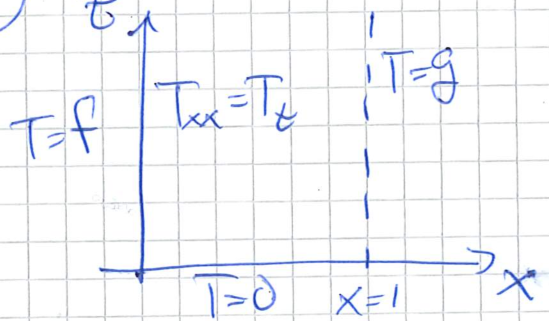
(3) Model



$T(x,t)$ satisfies

$$\begin{cases} T_{xx} = T_t & 0<x<\infty,\ 0<t<\infty \\ T(x,0) = 0 & 0<x<\infty \\ T(0,t) = f(t) & 0<t<\infty \end{cases}$$

$T\big|_{x\to\infty}$ bounded

Given $f(t)$ we solve the problem and compute $g(t) = T(1,t)$

(4) Let $T_1(x,r,t)$ and $T_2(x,r,t)$ be two solutions. Then $T = T_1 + T_2$ satisfies the PDE and also the boundary conditions $\Rightarrow$ the problem is linear

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(5) Let $\{x_i, y_j\}$ $i = 1,..,n$ and $j = 1,..,m$ be a grid. The matrix

$$(f)_{ij} = f(x_i, y_j) \text{ is } n \times m$$

Store as a vector $f \in \mathbb{R}^{m \cdot n \times 1}$ and

$$\underset{mn \times mn}{K} : \underset{m \cdot n \times 1}{f} \Rightarrow \underset{m \cdot n \times 1}{g}$$

K is <u>very large</u>!

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(6) The elements on row 3 of A are

$$\text{Elements}(\underbrace{\text{RowEndIndex}(3-1)+1}_{6+1=7} : \underbrace{\text{RowEndIndex}(3)}_{8})$$

$$= \text{Elements}(7:8) = (2.1 \quad -1.8)$$

How to find elements in a column?

```
c=[];
for i=1:nnz(A)
    if ColumnIndex(i)==3
        c=[c; Elements(i)];
    end
end
```

More difficult!

(7)

```
y= zeros(n,1);
for k=1:n
   for i= RowEndIndex(k-1)+1:RowEndIndex(k)
      y(k)=y(k)+Elements(i)*x(ColumnIndex(i))
   end
end
```

Very simple matrix-vector multiply.

(8) Finite differences

$$u_{xx} = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + O(h^2)$$

(9) proof

$$Ax = b \Rightarrow (M-N)x = b \Rightarrow$$
$$x = M^{-1}Nx + M^{-1}b$$

proof

$$\|x^{(k)} - x\| = \|Gx^{(k-1)} + c - (Gx + c)\|$$
$$= \|G(x^{(k-1)} - x)\| \leq \underbrace{\|G\|}_{\leq \rho(G)} \cdot \|x^{(k-1)} - x\|$$
$$\leq \dots \leq (\rho(G))^k \|x^{(0)} - x\|$$

(10)

Proof

Landweber is $x^{(k+1)} = \underbrace{(I - \omega A^T A)}_{G} x^{(k)} + \underbrace{\omega A^T b}_{c}$

Use $A = U \Sigma V^T \Rightarrow$

$$G = \underbrace{I}_{= VV^T} - \omega A^T A = VV^T - \omega V \Sigma^T \Sigma V^T =$$

$$= V \underbrace{(I - \omega \Sigma^T \Sigma)}_{= \operatorname{diag}(1 - \omega \sigma_i^2)} V^T$$

so

$$\|G\|_2 = \max_{1 \le i \le n} |1 - \omega \sigma_i^2| \text{ and } |1 - \omega \sigma_i^2| < 1$$

$$\text{if } 1 - \omega \sigma_1^2 > -1 \Rightarrow \omega < \frac{2}{\sigma_1^2} \quad //$$

Always possible to get convergence.