Lecture 10

(1) Let $V = (v_1, v_2, \ldots, v_m)$ be a basis for K_m. Then

$$x^{(m)} \in x^{(0)} + K_m \iff x^{(m)} = x^{(0)} + Vy,\ y \in \mathbb{R}^m$$

The residual is

$$r^{(m)} = b - Ax^{(m)} = b - A(x^{(0)} + Vy) = r^{(0)} - AVy$$

Orthogonality $r^{(m)} \perp L_m$ means $0 = W^T r^{(m)}$

where $W = (w_1, \ldots, w_m)$ is a basis for L_m.

We get $0 = W^T r^{(m)} = W^T(r^{(0)} - AVy) \Rightarrow$

$$(W^T A V) y = W^T r^{(0)}$$

and

$$x^{(m)} = x^{(0)} + V(W^T A V)^{-1} W^T r^{(0)}$$

Works if $(W^T A V)$ is non-singular

(2) proof Def A is pos. def if $x^T A x > 0\ \forall x$.

$L_m = K_m \Rightarrow$ Let $V = (v_1, \ldots, v_m)$ be a basis for both. Then

$$x^T (V^T A V) x = (Vx)^T A (Vx) = y^T A y > 0$$

for $y \neq 0$. Also $y \neq 0$ if $x \neq 0$ since V is a basis.

(3) $K_1 = L_1 = \{r^{(k)}\}$ means

$$x^{(k+1)} = x^{(k)} + K_1 = x^{(k)} + \alpha_k r^{(k)}$$

Orthogonality means $r^{(k+1)} \perp L_1 = \{r^{(k)}\}$ so

$$0 = (r^{(k)})^T (b - Ax^{(k+1)}) = (r^{(k)})^T (r^{(k)} - \alpha_k A r^{(k)})$$

Solve for α_k gives

$$x^{(k+1)} = x^{(k)} + \frac{(r^{(k)})^T A r^{(k)}}{\|r^{(k)}\|_2^2} r^{(k)}$$

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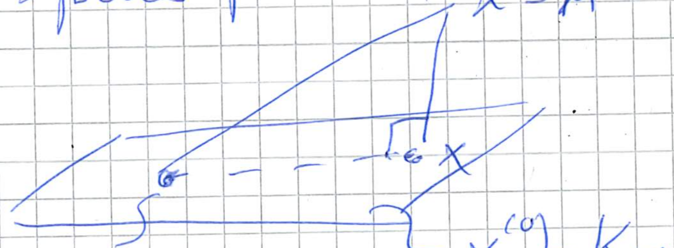
(4) Recall $(x,y)_A = x^T A y$ is a scaler product.

proof Consider the least squares problem $x^* = A^{-1}b$

$$\min_{x \in x^{(0)} + K_m} \|x^* - x\|_A$$



so $x - x^*$ is A-orthogonal to $x^{(0)} + K_m$ or

$$0 = (x - x^{(0)})^T A (x^* - x) = \underbrace{(x - x^{(0)})}_{\in K_m}{}^T \underbrace{(b - Ax)}_{r}$$

so $r \perp K_m$. Same condition that defines the projection method.

(5) proof Let $V = (v_1, .., v_m)$ be a basis for K_m then AV is a basis for $\mathcal{L}_m = AK_m$. We get

$$r^{(m)} = r^{(0)} - AVy \perp \mathcal{L}_m \Rightarrow 0 = (AV)^T (r^{(0)} - AVy)$$

$$\Rightarrow \quad V^T A^T A V y = V^T A^T r^{(0)} \;//$$

Normal equations for

$$\min_{y \in \mathbb{R}^m} \| AVy - r^{(0)} \|_2 = \min_{y \in \mathbb{R}^m} \| A \underbrace{(x^{(0)} + Vy)}_{\in K_m + x^{(0)}} - b \|_2$$

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(6) Want to solve $Ax = b$. Can do matrix-vector multiply and liner combinations

$$\Rightarrow \quad x = x^{(0)} + \underbrace{c_1 r^{(0)} + c_2 A r^{(0)} + \dots + c_m A^m r^{(0)}}_{\text{Krylov subspace.}}$$

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(7) proof If $\dim(K_m) = k < m$ then

$$r^{(0)} = \alpha_1 A r^{(0)} + \dots + \alpha_m A^{m-1} r^{(0)} \qquad (*)$$

The residual is

$$r^{(m)} = b - Ax = b - A\left(x^{(0)} + c_1 r^{(0)} + \dots + c_m A^{m-1} r^{(0)}\right)$$

$$= r^{(0)} - c_1 A r^{(0)} - \dots - c_m A^m r^{(0)}$$

Can pick $c_1, .., c_m$ so $r^{(m)} = 0$ due to $(*)$ and we have the exact solution.

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$x = pcg(A' * A, A' * b)$

means forming $A^T * A$ and $A^T b$. $\Rightarrow$ cgls