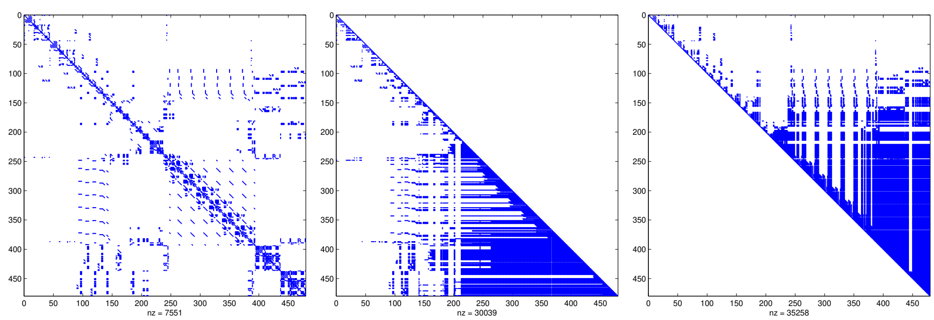
TANA15 Numerical Linear Algebra

Seminar Problems



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1 Basic Matrix Computations

Exercise 1.1 Let A and B be two $n \times n$ matrices. Prove that $(AB)^T = B^T A^T$. Also prove that if both A and B are non-singular then $(AB)^{-1} = B^{-1}A^{-1}$.

Exercise 1.2 Prove that if $A \in \mathbb{R}^{n\times n}$ is non-singular then $(A^{-1})^T = (A^T)^{-1}$. Hence the notation A^{-T} makes sense.

Exercise 1.3 Prove that $\dim(\text{Null}(A)) + \dim(\text{Range}(A)) = n$.

Hint Pick a basis for $\text{Null}(A)$ and complete to a basis for all of $\mathbb{R}^n$.

Exercise 1.4 Suppose $A \in \mathbb{R}^{m\times m}$ and $B \in \mathbb{R}^{m\times n}$, $m > n$. How many operations are required to evaluate the formula $z = (A + I)Bx + y$, where x and y are vectors.

Exercise 1.5 An $n \times n$ matrix is said to be *elementary* if it can be written as $A = I - uv^T$, where u and v are vectors. What condition on u and v ensures that A is non-singular? Also prove that A^{-1} is also elementary (by showing $A^{-1} = I - \alpha uv^T$).

Exercise 1.6 Prove the *Sherman-Morrison* formula

$$(A - uv^T)^{-1} = A^{-1} + A^{-1}u(1 - v^T A^{-1}u)^{-1}v^T A^{-1}.$$

Exercise 1.7 Prove the inequality $\|x\|_\infty \le \|x\|_2 \le \sqrt{n}\|x\|_\infty$.

Exercise 1.8 Let $u \in \mathbb{R}^m$ and $v \in \mathbb{R}^n$. Show that

$$\|uv^T\|_2 = \|u\|_2\|v\|_2.$$

Exercise 1.9 Prove that $\|I\| = 1$ and $\|A\|\|A^{-1}\| \ge 1$ for all matrix norms induced by a vector norm.

Exercise 1.10 Suppose we implement matrix-vector multiplication by a loop:

```
y=zeros(n,1);
for i=1:n
  for j=1:n
    y(i)=y(i)+A(i,j)*x(j);
  end
end
```

on a machine where matrices are stored by column in main memory.

a) Suppose one memory block corresponds exactly to the size of one column `A(:,j)` or the vectors x and y. Further assume that only a couple of memory blocks fit in Cache memory. Clearly explain why the above code is inefficient. Also check the ratio between the number of memory blocks loaded into Cache memory and the number of floating point operations needed.

b) Propose an alternative implementation of matrix-vector multiply and clearly explain why it is better.

2 Linear Systems of Equations

Exercise 2.1 Suppose we have a linear system $Ax = b$ where

$$A = \begin{pmatrix} 2 & 1 & -2 \\ -1 & 0 & 3 \\ 1 & 2 & -1 \end{pmatrix} \text{ and } b = \begin{pmatrix} 6 \\ 1 \\ -3 \end{pmatrix}.$$

During the first step of Gaussian elimination we multiply the system with a matrix L_1 such that the new system $L_1Ax = L_1b$ is

$$\begin{pmatrix} 2 & 1 & -2 \\ 0 & 0.5 & 2 \\ 0 & 1.5 & 0 \end{pmatrix} x = \begin{pmatrix} 6 \\ 4 \\ -6 \end{pmatrix}.$$

Give the Gausstransformation L_1.

Exercise 2.2 Let $r = b - A\widehat{x}$ be the residual for an approximate solution to the linear system $Ax = b$. Prove the formula:

$$\|x - \widehat{x}\| \leq \|A^{-1}\|\|r\|.$$

Exercise 2.3 Suppose A, B, and C are matrices and b is a vector. How would you implement the formula

$$x = B^{-1}(2A + I)(C^{-1} + A)b.$$

without computing any matrix inverse? Aim for as few arithmetic operations as possible.

Exercise 2.4 Suppose A has a Cholesky decomposition $A = R^TR$. Prove that A is symmetric and positive definite.

Exercise 2.5 Let $PA = LU$ be the LU decomposition. Prove the formula

$$\det(A) = (-1)^k\Pi_{i=1}^n u_{ii}.$$

What is k here?

Exercise 2.6 Suppose we want to solve the upper triangular system $Rx = y$ by backwards substitution. Clearly show how many floating point operations are needed.

Exercise 2.7 How would you solve a partitioned linear system

$$\begin{pmatrix} L_1 & 0 \\ B & L_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

where L_1 and L_2 are lower triangular. Show the steps in terms of the given submatrices and vectors.

3 Least Squares Problems

Exercise 3.1 Suppose Q is an orthogonal matrix. Show that $\|Qx\|_2 = \|x\|_2$, for all vectors x, and thus $\|Q\|_2 = 1$.

Exercise 3.2 If A is both an ortgohonal matrix and an orthogonal projection. What can you conclude about A?

Exercise 3.3 Let $v \neq 0$. The Householder reflection can be written as,

$$H = I - 2\frac{vv^T}{v^Tv}.$$

Prove the following properties of Householder reflections

a) H is symmetric and orthogonal.

b) Suppose that $x = (1,2,3)^T$ and $Hx = (\alpha, 0, 0)^T$. What is the value $|\alpha|$? □

Exercise 3.4 Suppose $A \in \mathbb{R}^{m\times n}$, $m > n$, and that we have the QR decomposition

$$A = Q\begin{pmatrix} R \\ 0 \end{pmatrix} = Q_1R$$

a) Show that if $\text{Range}(A) = \text{Span}(Q_1)$ then the least squares system has a unique solution. Show that in this case R is non-singular.

b) Show that the linear system $Ax = b$ has an exact solution if $b = Q_1Q_1^Tb$.

Exercise 3.5 Consider the vector a as an $n \times 1$ matrix. Write out its QR decomposition explicitly. Also write down a formula for the solution of the least squares problem $ax \approx b$, where b is a given $n \times 1$ vector.

Exercise 3.6 Let a be any non-zero vector. If $v = a - \alpha e_1$, $\alpha = \pm\|a\|_2$, and

$$H = I - 2\frac{vv^T}{v^Tv},$$

show that $Ha = \alpha e_1$.

Exercise 3.7 Prove that the product of two lower triangular matrices is also lower triangular and that the inverse of a lower triangular matrix is also lower triangular.

Hint This is important since it leads to a uniqueness result for the QR decomposition.

Exercise 3.8 Suppose B is an $n\times n$ matrix and assume B is both orthogonal and triangular. Prove that B is a diagonal matrix and that the diagonal entries are ± 1. Use the result to prove that the decomposition $A = QR$ is "essentially unique", i.e. only the sign of the diagonal entries in R may differ.

Exercise 3.9 We are intressted in the least squares problem $\min \|Ax - b\|_2$. Suppose $A = Q_1 R$ is the reduced QR decomposition. Use Q_1 to give a formula for a orthogonal projection P, such that $Pb = r = b - Ax$, where x is the least squares solution.

Exercise 3.10 Compute the QR factorization of the matrix

$$A = \begin{pmatrix} 0 & \sqrt{2} \\ -1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Let $b = (1, 2, 3)^T$. Find the vector x that minimize $\|Ax - b\|$.

Exercise 3.11 Consider the least squares problem $\min \|Ax - b\|_2$. We compute the QR decomposition of the *augumented* matrix

$$QR = [A, b] \in \mathbb{R}^{m \times (n+1)}.$$

Clearly show how the least squares problem can be solved using *only* the R matrix. That is we don't need to save the Q matrix when computing the factorization.

Exercise 3.12 Let $W \in \mathbb{R}^{n \times n}$ be real, symmetric, positive definite, and let $\| \cdot \|_W$ be defined by,

$$\|x\|_W^2 = x^T W x.$$

a) Verify that $\|x\|_W = 0$ if and only if $x = 0$.

Hint Use the Cholesky factorization $W = R^T R$.

b) Derive an expression for the normal equations of the minimization problem,

$$\min_x \|Ax - b\|_W.$$

Exercise 3.13 Suppose A is an $m \times n$ matrix and $m > n$. Clearly demonstrate how Householder reflections can be used to compute the decomposition $A = QR$. It is sufficient to consider the 5×4 case.

Exercise 3.14 Let A be an $m \times n$ matrix, $m > n$. How many Givens rotations would be needed to transform A into upper triangular shape?

Exercise 3.15 Suppose A is an $m \times n$ matrix, $m > n$, and that we have the QR decomposition of A and also the least squares solution x. We add a new row to the previously solved least squares problem and need to transform a matrix of the form,

$$\widetilde{A} = \begin{pmatrix} x & x & x & x \\ 0 & x & x & x \\ 0 & 0 & x & x \\ x & x & x & x \end{pmatrix},$$

into upper triangular form. Clearly demonstrate how this can be accomplished using Givens rotations.

4 Eigenvalues

Exercise 4.1 What are the eigenvectors and eigenvalues of the following matrix?

$$\begin{pmatrix} 1 & 2 & -4 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$$

Exercise 4.2 Suppose that all of the row sums of an $n \times n$ matrix have the same value, e.g. α. Prove that α is an eigenvalue. What is the corresponding eigenvector?

Exercise 4.3 Show that a matrix is *singular* if and only if zero is one of its eigenvalues.

Exercise 4.4 Let A be an $n \times n$ matrix. Prove that A and A^T have the same eigenvalues. Do A and A^T also have the same eigenvectors? Prove or give a counter example.

Hint Use Matlab for the second part to avoid tedius calculations.

Exercise 4.5 Prove that all the eigenvalues of a Hermitean matrix A are real. Use the result to prove that the eigenvalues of a real symmetric matrix are real.

Exercise 4.6 Prove that for any matrix norm induced by a vector norm the spectral radius satisfies $\rho(A) \leq \|A\|$.

Exercise 4.7 A matrix A is said to be *nilpotent* if $A^k = 0$ for some positive integer k. Show that all eigenvalues of a nilpotent matrix are zero. Can you conclude that A is the zero matrix?

Exercise 4.8 Suppose $A, B \in \mathbb{R}^{n\times n}$ and in addition A is non-singular. Show that AB and BA have the same eigenvalues.

Exercise 4.9 Let A have block-triangular form, i.e.

$$\begin{pmatrix} A_{11} & A_{21} \\ 0 & A_{22} \end{pmatrix},$$

where A_{11} and A_{22} are both quadratic but not nessecarily of the same size. Show that λ is an eigenvalue of A if and only if it is an eigenvalue of either A_{11} or A_{22}.

Remark This is often called the *decoupling theorem*.

Exercise 4.10 Let $A \in \mathbb{R}^{n\times n]}$ have eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Show that

$$\det(A) = \Pi_{i=1}^{n} \lambda_i.$$

Exercise 4.11 Suppose $A \in \mathbb{R}^{n\times n}$ is real and of rank one. Show that $A = uv^T$, for some vectors u and v, and also that $u^T v$ is an eigenvalue of A. What are the other eigenvalues of A?

Exercise 4.12 Show that for any two real vectors u and v the formula

$$\det(I + uv^T) = 1 + u^T v$$

holds.

Exercise 4.13 Let $A \in \mathbb{R}^{n\times n}$ and $\rho(A) < 1$. Show that $I - A$ is non-singular and

$$(I - A)^{-1} = \sum_{k=0}^{\infty} A^k.$$

Exercise 4.14 Let A be real and symmetric with eigenvalues $\lambda_{min} \leq \lambda(A) \leq \lambda_{max}$. Show that for $x \neq 0$,

$$\lambda_{min} \leq \frac{x^T A x}{x^T x} \leq \lambda_{max}.$$

Exercise 4.15 Clearly show how Housedolder reflections can be used to reduce a matrix into Hessenberg form by a sequence of similarity transformations. It is enough to consider the 4×4 case.

Exercise 4.16 Suppose $A \in \mathbb{R}^{n\times n}$, $\mathrm{rank}(A) = n$, and $A = QR$, where Q is orthogonal and R is upper triangular.

a) Show that RQ is a Hessenberg matrix if A is a Hessenberg matrix.

b) Show that RQ is tridiagonal if A is symmetric and tridiagonal.

Hint: Write Q as a product of Givens rotations. It is enough to treat the 4×4 case to clearly see the pattern.

Exercise 4.17 The kth step of the shifted QR algorithm for computing eigenvalues is

$$A_k - s_k I = Q_k R_k, \qquad A_{k+1} = R_k Q_k + s_k I.$$

Show that A_{k+1} and A_k has the same eigenvalues.

Exercise 4.18 Given $A \in \mathcal{C}^{n\times n}$ show that for every $\varepsilon > 0$ there exists a diagonalizable matrix B such that $\|A - B\|_2 \leq \epsilon$. This shows that the set of diagonalizable matrices is dense in $\mathcal{C}^{n\times n}$.

Hint Use the Schur decomposition.

Exercise 4.19 Any matrix $A \in \mathbb{R}^{n\times n}$ can be factorized as $A = QTQ^H$, where Q is unitary and T upper triangular. This is called the *Schur decomposition* and is mainly of theoretical importance. Prove that the diagonal elements of T are eigenvalues of A and also real symmetric matrix A has orthogonal eigenvectors.

Exercise 4.20 Suppose (λ, x) is a known eigenpair for the matrix A. Give an algorithm for computing an orthogonal matrix P such that

$$P^T A P = \begin{pmatrix} \lambda & w^T \\ 0 & \widetilde{A} \end{pmatrix}$$

Write P as a product of Givens rotations.

Exercise 4.21 Let

$$A = \begin{pmatrix} 16 & 0 & 4 \\ 0 & 16 & 4 \\ 4 & 4 & 12 \end{pmatrix}.$$

The matrix has one eigenvalue $\lambda_1 = 20$ with the corresponding eigenvector

$$v_1 = (-0.5774, -0.5774, -0.5774)^T.$$

a) Use Gershgorin's theorem to prove that $\lambda_1 = 20$ is the largest eigenvalue of A and that A is non-singular.

b) If we attempt to use shifts to modify the convergence rate we replace A by $B = (A - sI)^{-1}$. Suppose (x, λ) is an eigenpair of A. What is the corresponding eigenpair of B?

c) Let s be a scalar. Show that the matrix $B = A + s v_1 v_1^T$ has the same eigenvectors as A. What are the eigenvalues of B?

Exercise 4.22 Consider the matrix

$$A = \begin{pmatrix} 14 & -2 & -1 \\ -2 & 9 & 0.5 \\ -1 & 0.5 & 4 \end{pmatrix}$$

a) Use Gershgorin's theorem to estimate the eigenvalues as accurately as possible. Can you conclude that the matrix is non-singular?

b) Let s be a scalar and v_1 an eigenvector. Show that the matrix $B = A + s v_1 v_1^T$ has the same eigenvectors as A. What are the eigenvalues of B?

Exercise 4.23 Consider the matrix

$$A = \begin{pmatrix} 100 & 0 & 1 \\ 5 & -10 & 1 \\ -1 & 2 & 4 \end{pmatrix}$$

a) Use the Gershgorin theorem to estimate the eigenvalues of the matrix A as accurately as possible.

b) Show that A is singular if and only if $\lambda = 0$ is an eigen value of A. Can you conclude that A is non-singular?

c) Suppose we apply inverse iteration to the matrix, i.e.

$$w^{(k)} = A^{-1} x^{(k)}, \quad x^{(k+1)} = w^{(k)} / \|w^{(k)}\|_2, \qquad k = 0, 1, 2, \ldots$$

Does the results from **a)** imply that the iteration will converge?

5 Non-linear Equations and Least Squares

Exercise 5.1 Determine if the following functions are *coercive* on $\mathbb{R}^2$.

(a) $f(x, y) = x + y + 2$.

(b) $f(x, y) = x^2 + y^2 + 2$.

(c) $f(x, y) = x^2 - 2xy + y^2$.

(d) $f(x, y) = x^4 - 2xy + y^4$.

Exercise 5.2 Formulate the Newton method for solving the system of equations

$$x_1^2 + x_1 x_2^3 = 9, \text{ and } 3x_1^2 x_2 - x_2^3 = 4.$$

Exercise 5.3 Let $x^{(0)} = (0\,,\,1)^T$. Perform one iteration in Newtons method applied to the system of equations

$$x_1^2 - x_2^2 = 0, \text{ and } 2x_1 x_2 = 1.$$

Exercise 5.4 Prove that if x^* is a fixed point of a smooth function $g : \mathbb{R} \mapsto \mathbb{R}$ and $g'(x^*) = 0$ then the convergence rate of the fixed point iteration $x_{k+1} = g(x_k)$ is at least quadratic.

Exercise 5.5 Let,

$$f(x) = \begin{pmatrix} x_1^2 + \sin(x_2) \\ 1 + \cos(x_2^2) + x_2 \end{pmatrix}.$$

Do the follwing: Compute the Jacobian matrix $J_f(x)$ of the function $f(x)$. Also perform one Newton step for solving the non–linear equation $f(x) = 0$ using the starting value $x^{(0)} = (1, 0)^T$.

Exercise 5.6 Consider the over determined system of equations

$$\begin{aligned}(x+1)^2 + y^2 &= 0.25,\\ x^2 + (y-1)^2 &= 0.25,\\ (x-1)^2 + y^2 &= 0.25.\end{aligned}$$

a) Describe how the Gauss-Newton Method can be used for finding an approximate solution the above problem. What is the residual vector $r(x)$ and the Jacobian $J_r(x)$?

b) Perform one Gauss-Newton step using the starting vector $x^{(0)} = (0, 0)^T$.

Exercise 5.7 Let $f(x) : \mathbb{R}^n \mapsto \mathbb{R}$ be given by

$$f(x) = \frac{1}{2} x^T A x - x^T b + c,$$

where $A \in \mathbb{R}^{n \times n}$ is symmetric and b is a vector. Formulate Newtons method for finding the minimum of $f(x)$ and show that the method converges in one iteration.

Exercise 5.8 In non-linear least squares we seek to a minimizer of

$$\min_{x\in\mathbb{R}^n} \frac{1}{2}\|r(x)\|_2^2 = \min_{x\in\mathbb{R}^n} f(x), \qquad f(x) = \frac{1}{2}r(x)^T r(x),$$

where $r(x) : \mathbb{R}^n \mapsto \mathbb{R}^m$, $m > n$. Using Newtons method we attempt to find a root x^* of the equation $\nabla f(x) = 0$. In each step we compute the next approximation $x^{(k+1)}$ by,

$$H_f(x^{(k)})s^{(k)} = -\nabla f(x^{(k)}), \qquad x^{(k+1)} = x^{(k)} + s^{(k)}.$$

where H_f is the *Hessian* matrix and ∇f is the *gradient*.

a) What are the dimensions of H_f and ∇f?

b) Typically the Hessian matrix is difficult to compute. The approximation,

$$H_f \approx J_r^T J_r,$$

where J_r is the Jacobian of the function $r(x)$, leads to the Gauss-Newton method. Show that using this approximation the above Newton step reduces to the least squares problem,

$$\min \|J_r(x^{(k)})s^{(k)} + r(x^{(k)})\|_2.$$

c) Suppose $r(x) = b - Ax$, where $A \in \mathbb{R}^{m\times n}$ is a matrix. Show that the Gauss-Newton method produces the exact solution after just one step.

Exercise 5.9 A river has been polluted by a large spill of a chemical substance that breaks down slowly with time. A theoretical model suggests that the concentration of the pollutant in the river decays with time according to a model

$$F(t) = c_0 \exp(-c_1 t) + c_2 \sin(\omega t),$$

where the last term is due to smaller naturally occuring pollution by the same chemical compund. In order to estimate the parameters of the model we measure the concetration $F(t_i)$, for times $t_1 < t_2 < \ldots < t_m$, and want to use the Gauss-Newton method to fit the parameters c_0, c_1, and c_2 to the measured data.

a) The Gauss-Newton method is based on writing down an appropriate residual vector $r := r(c)$, where $c = (c_0\,, c_1\,, c_2)^T$ are the coefficients of the model. Give the residual vector for the above situation.

b) In each step of the Guass-Newton method we need to solve a least squares problem to minimize $\|J_r(c^{(k)})s_k + r(c^{(k)})\|_2$, where $c^{(k)}$ is the approximate coefficient vector at step k. Derive an expression for the Jacobian J_r for this case.

6 The Singular Value Decomposition

Exercise 6.1 Let $a = (a_1, a_2, \ldots, a_n)^T$ be a column vector. What is the singular value decomposition of a considered as a $n \times 1$ matrix? Similarily what is the singular value decomposition of a^T?

Exercise 6.2 Let A^T be an $m \times n$ matrix of rank $k < \min(m, n)$. Use the decomposition $A = U\Sigma V^T$ to give an orthogonal basis for $\text{null}(A^T)$.

Exercise 6.3 Show that the eigenvalues of the symmetric matrix

$$\begin{pmatrix} 0 & A^T \\ A & 0 \end{pmatrix}$$

are precisely $\{\pm\sigma_i\}$. What are the corresponding eigenvectors?

Exercise 6.4 Show that if $A \in \mathbb{R}^{m\times n}$ has rank n, then $\|A(A^TA)^{-1}A^T\|_2 = 1$.

Exercise 6.5 Suppose the matrix $B \in \mathbb{R}^{m\times n}$ has full column rank. Use the decomposition $B = U\Sigma V^T$ to give a formula for the solution to the the problem

$$\min_x \|Bx\|_2, \text{ subject to } \|x\|_2 = 1.$$

Exercise 6.6 Suppose $A \in \mathbb{R}^{m\times n}$, $m > n$, $\text{rank}(A) = n$, and that we have a factorization $A = U\Sigma V^T$. Clearly demonstrate how the matrices U and V provides basis vectors for the spaces $\text{Range}(A)$ and $\text{null}(A)$. What are the dimension of the range and null space respectively.

Exercise 6.7 Let $A \in \mathbb{R}^{m\times n}$, $m > n$, and $\text{rank}(A) = n$. Demonstrate how the decomposition $A = U\Sigma V^T$ can be used for solving the least squares problem

$$\min_{x\in\mathbb{R}^n} \|Ax - b\|_2.$$

Give formulas for both the solution x and the redisual $r = b - Ax$.

Exercise 6.8 Let $A \in \mathbb{R}^{m\times n}$, $m < n$, and $\text{rank}(A) = m$. Let $b \in \mathbb{R}^m$. Show that the formula

$$x = \sum_{i=1}^{m} \frac{u_i^T b}{\sigma_i} v_i$$

provides a solution to $Ax = b$. Is the solution unique? If not what is the property that characterize the solution x provided by the above formula?

Exercise 6.9 Suppose we want to find the solution to a linear system $Ax = b$, where $\text{rank}(A) = k < n$ so that the solution x is not unique. Demonstrate how the solution x can be split into two parts,

$$x = x_1 + x_2, \qquad x_1 \in \text{null}(A)^\perp, \quad \text{and}, \quad x_2 \in \text{null}(A),$$

and how the SVD of A can be used to write expressions for the solution components x_1 and x_2.

Exercise 6.10 Consider the Least Squares problem with linear constraints,

$$\min \|Ax - b\|_2, \qquad \text{for all } x \in \mathbb{R}^n \text{ such that } Bx = 0,$$

where A is $m \times n$, $m > n$, and B is $n \times n$.

a) Suppose $\text{rank}(B) = n$. What is the solution of the least squares problem?

b) Suppose $\text{rank}(B) = k < n$. Show how the SVD can be used to derive a formula for the solution of the least squares problem.

Exercise 6.11 Let $A \in \mathbb{R}^{m \times n}$, where $m >> n$, have full column rank. Use the decomposition $A = U\Sigma V^T$ to develop a criteria that ensures that the linear system $Ax = b$ has a solution. Try and make the criteria as inexpensive as possible to check.

Exercise 6.12 Demonstrate how a 4×4 matrix A can be reduced to upper bidiagonal form using Householder reflections. That is $U^T AV = B$, where U and V are products of Householder reflections and B is upper bidiagonal.

Exercise 6.13 Tikhonov regularization means replacing an ill-conditioned linear system $Ax = b$ by the more stable problem,

$$\min_x \|Ax - b\|_2^2 + \lambda^2 \|x\|_2^2,$$

where λ is the regularization parameter. Show that the normal equations of the above least squares problem are

$$(A^T A + \lambda^2 I)x = A^T b.$$

Also derive a formula for the singular values of the matrix $(A^T A + \lambda^2 I)$ and use the result to show that the normal equations are not ill-conditioned (provided λ is selected appropriately). Finally derive a formula for the solution x_λ.

Exercise 6.14 Show that

$$\sigma_{max}(A) = \max_{y \in \mathbb{R}^m, x \in \mathbb{R}^n} \frac{y^T Ax}{\|y\|_2 \|x\|_2}.$$

7 Sparse Matrices and Iterative Methods

Exercise 7.1 Show that the Jacobi iteration can be written in the form $x^{(k+1)} = x^{(k)} + Hr^{(k)}$, where $r^{(k)}$ is the residual.

Exercise 7.2 Show that the iterative method,

$$x^{(k+1)} = Gx^{(k)} + c,$$

is convergent if the spectral radius $\rho(G) < 1$.

Exercise 7.3 Show that if $A = M - N$ is singular then we can never have $\rho(M^{-1}N) < 1$ even if M is non-singular.

Exercise 7.4 Let $A \in \mathbb{R}^{n\times n}$ be symmetric and positive definite. Prove that the Conjugate Gradient iteration converges to the exact solution of $Ax = b$ within at most n steps.

Exercise 7.5 Let A be symmetric and positive definite. Write down the formulas for a general projection method in the case $\mathcal{K} = \mathcal{L} = \text{span}(e_i)$.

Exercise 7.6 A general projection method is defined by solving: Find $x^{(m)} \in x^{(0)} + \mathcal{K}_m$ such that $r^{(m)}$ is orthogonal to $\mathcal{L}_m$, where $\mathcal{K}_m$ and $\mathcal{L}_m$ are two m dimensional subspaces. Introduce basis sets for the two subspaces and derive an explicit formula for the approximate solution $x^{(m)}$.

Exercise 7.7 Let A be symmetric and positive definite, and suppose that $V \in \mathbb{R}^{n\times k}$ is a basis for a k dimensional subspace. Prove the following

a) The matrix A^{-1} exists.

b) The matrix V^TAV is non-singular.

Exercise 7.8 Let A be symmetric and positive definite. Consider a projection method where at each step $\mathcal{K} = \mathcal{L} = \text{span}(r, Ar)$, and $r = b - Ax$ is the current residual. Do the following:

a) As basis for $\mathcal{K}$ we use r and a vector p obtained by orthogonalizing Ar against r with respecy to the A-inner product. Derive a formula for computing p.

b) Write down the algorithm for performing the projection step using the subspace $\mathcal{K}$.

Exercise 7.9 Suppose the Arnoldi method is used to create an orthogonal basis for the Krylov subspace $\mathcal{K}_m(A, r^{(0)})$; and that break-down occurs after k steps. Show the following

a) The space $\mathcal{K}_k(A, r^{(0)})$ is an *invariant subspace*, i.e. $x \in \mathcal{K}_k$ then $Ax \in \mathcal{K}_k$.

b) The solution of the equation $Ax = b$ belong to the space $x^{(0)} + \mathcal{K}_k$. Hence a projection method should produce an exact solution.

c) Suppose V_k is the basis computed by the Arnoldi process and $H_k = V_k^TAV_k$. Show that the eigenvalues of H_k are also eigenvalues of A in this case. What are the corresponding eigenvectors?

1 Basic Matrix Computations

Exercise 1.1 First look at an element of $(AB)^T$. We have

$$(AB)^T_{ij} = (AB)_{ji} = \sum_{k=1}^{n} A_{jk}B_{ki} = \sum_{k=1}^{n} A^T_{kj}B^T_{ik} = \sum_{k=1}^{n} B^T_{ik}A^T_{kj} = (B^T A^T)_{ij}.$$

To prove that $(AB)^{-1} = B^{-1}A^{-1}$ we use

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AA^{-1} = I.$$

Exercise 1.2 To demonstrate that $(A^T)^{-1} = (A^{-1})^T$ we use

$$A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I.$$

Exercise 1.3 Suppose the dimension of the null space is k. Then there is a basis $\{x_1, x_2, \dots, x_k\}$ for the nullspace. Add $n-k$ linearly independent vectors $\{\widetilde{x}_{k+1}, \dots, \widetilde{x}_n\}$ so that we have a basis for $\mathbb{R}^n$. Now take a vector y that belongs to the subspace Range(A), i.e. $y = Ax$ for some $x \in \mathbb{R}^n$. We can express x using the above basis and since $Ax_i = 0$, for $i = 1, \dots, k$, we find that y is a linear combination of the vectors $\{A\widetilde{x}_{k+1}, \dots, A\widetilde{x}\}$. So the dimension is at most $n-k$. To show that the dimension is exactly $n-k$ we assume that there is a linear combination so that

$$0 = \sum_{i=k+1}^{n} c_i A\widetilde{x}_i = A(\sum_{i=k+1}^{n} c_i A\widetilde{x}_i) = Az,$$

so z belongs to the nullspace which contradicts the assumption that the set of vectors

$$\{x_1, x_2, \dots, x_k, \widetilde{x}_{k+1}, \dots, \widetilde{x}_n\}$$

was a basis. So the dimension of the range is exactly $n-k$.

Exercise 1.4 We evaluate the expression using the following operations

$$z = (A+I)Bx + y = (A+I)x_1 + y = Ax_1 + x_1 + y = x_2 + x_1 + y = x_3 + y = x_4$$

Computing the matrix vector product $x_1 = Bx$ requires mn multiplications and additions so a total of $2mn$ operations and the product $x_2 = Ax_1$ requires $2m^2$ mult/adds. The remaining two vector additons require m additions (as $y, x_1 \in \mathbb{R}^m$). So the operation count is $m(2m+2n+2)$ or $2(m^2+mn)$ if just the leading terms are kept.

Exercise 1.5 $A = I - uv^T$ is non-singular if $Ax \neq 0$ for each $x \neq 0$. We calculate

$$0 = Ax = x - uv^T x = x - (v^T x)u \text{ or } x = \alpha u \text{ and } v^T x = \alpha \neq 0.$$

Thus it is sufficient to check that

$$Au = u - (v^T u)u = (1 - v^T u)u \neq 0,$$

which is true if $v^T u \neq 1$. Check that A^{-1} is elementary by

$$A^{-1}A = (I - \alpha uv^T)(I - uv^T) = I + (-\alpha - 1 + \alpha v^T u)uv^T.$$

so we have an inverse if we select α so that $-\alpha - 1 + \alpha v^T u = 0$ or if $\alpha = (v^T u - 1)^{-1}$.

Exercise 1.6 Define a scalar $\alpha = v^T A^{-1} u$. Verify the formula by

$$(A - uv^T)(A^{-1} + A^{-1}\frac{uv^T}{1-\alpha}A^{-1}) = AA^{-1} + AA^{-1}\frac{uv^T}{1-\alpha}A^{-1} - uv^T A^{-1} - uv^T A^{-1}\frac{uv^T}{1-\alpha}A^{-1} = (*).$$

Now replace $v^T A^{-1} u$ by α to obtain

$$(*) = I + (\frac{1}{1-\alpha} - 1 - \frac{\alpha}{1-\alpha})uv^T A^{-1} = I.$$

Exercise 1.7 Demonstrate the first inequality by

$$\|x\|_\infty^2 = \max_{1\le i\le n} |x_i|^2 \le \sum_{i=1}^n |x_i|^2 = \|x\|_2^2.$$

Also, since $|x_i| \le \|x\|_\infty$, we have

$$\|x\|_2^2 = \sum_{i=1}^n |x_i|^2 \le \sum_{i=1}^n \|x\|_\infty^2 = n\|x\|_\infty.$$

Exercise 1.8 Recall the definition

$$\|uv^T\|_2 = \max_{x\in\mathbb{R}^n} \frac{\|uv^T x\|_2}{\|x\|_2} = \max_{x\in\mathbb{R}^n} \frac{|v^T x|\|u\|_2}{\|x\|_2}.$$

The Cauchy-Schwarz inequality is $|v^T x| \le \|v\|_2\|x\|_2$ with equality for $x = v$. So

$$\|uv^T\|_2 = \frac{|v^T v|\|u\|_2}{\|v\|_2} = \|v\|_2\|u\|_2.$$

Exercise 1.9 First from the definition of the matrix norm, and since $Ix = x$ we have

$$\|I\| = \max_{x\neq 0} \frac{\|Ix\|}{\|x\|} = \max_{x\neq 0} \frac{\|x\|}{\|x\|} = 1, \text{ so } 1 = \|I\| = \|AA^{-1}\| \le \|A\|\|A^{-1}\|.$$

Exercise 1.10 **a)**First during the inner loop $y(i)$ and x can be kept in Cache memory. But the elements $A(i,j)$, for $j = 1, \ldots, n$, all belong to different blocks. Thus a new block needs to be loaded for each multiply `A(i,j)*x(j)`. So the ratio memory loads to multiplies is $1-1$.

b) To fix the issue is is enough to change the order of the loops. So the inner loop copmputes `y(i)=y(i)+A(i,j)*x(j)`, for $i = 1, \ldots, n$. Now the column $A(:,j)$ can be loaded into Cache and n multiplications can be performed until the next vector load is needed.

2 Linear Systems of Equations

Exercise 2.1 Use the matrix

$$L_1 = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{pmatrix}.$$

Exercise 2.2 We have that

$$x - \widehat{x} = A^{-1}(Ax - A\widehat{x}) = A^{-1}(b - A\widehat{x}) = A^{-1}r.$$

Thus $\|x - \widehat{x}\| \leq \|A^{-1}\|\|r\|$.

Exercise 2.3 We aim to keep intermediate results small. Multiplication by an inverse is dealt with by solving the corresponding linear system, i.e. compute $z = A^{-1}x$ by solving $Az = x$. The order of computation is

$$z_1 = Ab, \quad Cz_2 = b, \quad z_3 = z_1 + z_2, \quad z_4 = Az_3, \quad z_5 = 2z_4 + z_3, \text{ and finally } Bx = z_5.$$

All intermediate results are vectors.

Exercise 2.4 That A is symmetric follows from $A^T = (R^TR)^T = R^T(R^T)^T = R^TR = A$. For $x \neq 0$ we have $x^TAx = x^T(R^TR)x = (Rx)^T(Rx) = \|Rx\|_2^2 \geq 0$. If in addition A is non singular then so is R and $Rx \neq 0$ so A is strictly positive definite.

Exercise 2.5 Use the LU decomposition of A to obtain

$$A = P^TLU, \text{ so } \det(A) = \det(P^T)\det(L)\det(U).$$

Here both L and U are triangular so the determinant is the product of the diagonal elements. Also P is a permutation matrix. If we exchange two rows in a matrix then the determinant changes sign. So k is the number of row exchanges that actually occured during the Guassian elimination when computing the LU decomposition.

Exercise 2.6 Simplest is to note that there are $n^2/2 - n$ subdiagonal elements in a triangular matrix. Every element should appear in one multiply and one addition. The n diagonal elements should appear in one division each. Thus the number of operations is approximately n^2.

Exercise 2.7 First solve $L_1x_1 = b_1$. Then compute $b_3 = b_2 - Bx_1$. Finally solve $L_2x_2 = b_2$.

3 Least Squares Problems

Exercise 3.1 Since Q is orthogonal $Q^TQ = I$. So

$$\|Qx\|_2^2 = (Qx)^T(Qx) = x^TQ^TQx = x^Tx = \|x\|_2^2.$$

Exercise 3.2 First Range$(A) = \mathbb{R}^n$ since A is orthogonal and thus has linearly independent columns. So A is an orthogonal projection on the whole of $\mathbb{R}^n$. So $Ax = x$ for every $x \in \mathbb{R}^n$ so $A = I$ is the identity matrix.

Exercise 3.3 That H is symmetric follows from the observation that

$$H^T = (I - 2\frac{vv^T}{v^Tv})^T = I^T - 2\frac{(vv^T)^T}{v^Tv} = I - 2\frac{vv^T}{v^Tv} = H.$$

The orthogonality can be seen by computing

$$H^TH = HH = (I - 2\frac{vv^T}{v^Tv})(I - 2\frac{vv^T}{v^Tv}) = I - 2\frac{vv^T}{v^Tv} - 2\frac{vv^T}{v^Tv} + 4\frac{v(v^Tv)v^T}{(v^Tv)^2} = I.$$

Since H is orthogonal $\|Hx\|_2 = \|x\|_2$ so $|\alpha| = \sqrt{14}$.

Exercise 3.4 **a)** A Vector $y \in$ Range(A) if $y = Ax$ for some $x \in \mathbb{R}^n$.

$$y = Ax = Q_1Rx.$$

Since Q_1 has orthogonal columns dim(span$\{Q_1\}$) $= n$. So, if Range$(A) =$ span$\{Q_1\}$ then Rank$(A) = n$. This can only hold if Rank$(R) = n$ so R is non-singular. The solution is unique since if Rank$(A) = n$ then A doesn't have a non-trivial null space.
b) Since Q_1 is an orthogonal basis for Range(A) then $Q_1Q_1^T$ is an orthogonal projection onto Range(A). The criteria $b = Q_1Q_1^Tb$ simply means that b belongs to the range which means that a solution exists.

Exercise 3.5 The vector a can be seen as a matrix in $\mathbb{R}^{n\times 1}$. This means that

$$a = (a/\|a\|_2)\|a\|_2 = QR$$

where $Q \in \mathbb{R}^{n\times 1}$ and $R \in \mathbb{R}^{1\times 1}$. Since in this case R is a scalar we can write the solutuion formula as $x = R^{-1}Q^Tb = (a/\|a\|_2)^Tb/\|a\|_2 = (a^Tb)/\|a\|_2^2$. The same formula is obtained by forming the normal equations $a^Tax = a^Tb$.

Exercise 3.6 If $v = a - \alpha e_1$ and $\alpha^2 = \pm a^Ta$. Then

$$v^Tv = (a - \alpha e_1)^T(a - \alpha e_1) = a^Ta - 2\alpha(e_1^Ta) + \alpha^2 = 2a^Ta - 2\alpha(e_1^Ta).$$

and

$$v^Ta = (a - \alpha e_1)^Ta = a^Ta - \alpha(e_1^Ta).$$

This means that

$$Ha = (I - 2\frac{vv^T}{v^Tv})a = a - 2\frac{v(v^Ta)}{v^Tv} = a - 2\frac{a^Ta - \alpha(e_1^Ta)}{2a^Ta - 2\alpha(e_1^Ta)}(a - \alpha e_1) = a - (1)(a - \alpha e_1) = \alpha e_1.$$

Exercise 3.7 If both A and B are lower triangular then $a_{ik} = 0$ for $i < k$ and $b_{kj} = 0$ for $k < j$. So if $C = AB$ and $i < j$ then

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = \sum_{k=j}^{i} a_{ik} b_{kj} = 0.$$

For the second part we write $AB = I$ and eliminate elements in B. Consider the 3×3 case:

$$AB = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I.$$

First note thar $a_{11} b_{11} = 1$ and $a_{11} \neq 0$ since A^{-1} exists. This leads to $b_{11} \neq 0$. Then $a_{11} b_{12} = 0$ and $a_{11} b_{13} = 0$ means $b_{12} = b_{13} = 0$. Second, use $a_{21} b_{12} + a_{22} b_{22} = a_{21} 0 + a_{22} b_{22} = 1$ to conclude that $b_{22} \neq 0$. Also $a_{22} \neq 0$ and since $a_{21} b_{13} + a_{22} b_{23} = a_{22} b_{23} = 0$ we find that $b_{23} = 0$ so B is lower triangular. In both cases a similar proof works for the upper triangular case.

Exercise 3.8 Assume B is upper triangular and orthogonal. The 3×3 case can then be written

$$B^T B = \begin{pmatrix} b_{11} & 0 & 0 \\ b_{12} & b_{22} & 0 \\ b_{13} & b_{23} & b_{33} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & 0 & b_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I.$$

First look at the first column of B^T. We obtain $b_{11}^2 = 1$ which means $b_{11} = \pm 1$. Also $b_{12} b_{11} = b_{13} b_{11} = 0$ which means $b_{12} = b_{13} = 0$. Now multiply second row of B^T with second column of B to obtain $b_{22}^2 = 1$ and $b_{22} = \pm 1$. Second row of B^T multiplied with third column of B now gives $b_{22} b_{23} = 0$ so $b_{23} = 0$. Finally $b_{33}^2 = 1$ means $b_{33} = \pm 1$ so B is a diagonal matrix with entries ± 1.

Now assume $A = Q_1 R_1 = Q_2 R_2$ are two different reduced QR decompositions. Then

$$B = Q_1^T Q_2 = R_1 R_2^{-1}$$

is both orthogonal and upper triangular. Thus B is a diagonal matrix. This means that $R_1 = B R_2$. So we obtain R_1 by changing signs on the rows of R_2.

Exercise 3.9 Let x be the least squares solution. Since the orthogonal projection onto Range(A) is $Q_1 Q_1^T$ we have $Ax = Q_1 Q_1^T b$. So $r = b - Ax = b - Q_1 Q_1^T b = (I - Q_1 Q_1^T) b = Pb$.

Exercise 3.10 The matrix has orthogonal columns, i.e. if $A = (a_1, a_1)$ then $(a_1, a_2) = 0$. Thus the QR decomposition is

$$A = (a_1 / \|a_1\|_2 \, , \, \|a_2 / \|a_2\|_2) \begin{pmatrix} \|a_1\|_2 & 0 \\ 0 & \|a_2\|_2 \end{pmatrix} = Q_1 R.$$

The numbers are not very important.

Exercise 3.11 We want to minimize $\|Ax - b\|_2$. We rewrite the problem as

$$\|Ax - b\|_2 = \|[A, b] \begin{pmatrix} x \\ -1 \end{pmatrix} \|_2 = \|Q \begin{pmatrix} \widetilde{R} \\ 0 \end{pmatrix} \begin{pmatrix} x \\ -1 \end{pmatrix} \|_2 = \| \begin{pmatrix} \widetilde{R} \\ 0 \end{pmatrix} \begin{pmatrix} x \\ -1 \end{pmatrix} \|_2 = (*).$$

where $\widetilde{R} \in \mathbb{R}^{(n+1)\times(n+1)}$. We find the minimum by noting that

$$(*)^2 = \| \begin{pmatrix} R & \gamma \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} x \\ -1 \end{pmatrix} \|_2^2 = \| \begin{pmatrix} Rx - \gamma \\ \alpha \end{pmatrix} \|_2^2 = \|Rx - \gamma\|_2^2 + |\alpha|^2,$$

where $R \in \mathbb{R}^{n\times n}$ and $\gamma \in \mathbb{R}^n$, so the minimum is achived for $x = R^{-1}\gamma$. Note that the last column of $\widetilde{R}$ is $\begin{pmatrix} \gamma \\ \alpha \end{pmatrix} \in \mathbb{R}^{n+1}$.

Exercise 3.12 **a)** Let $W = R^T R$ be the Cholesky decomposition ans rewrite

$$\|x\|_W^2 = x^T W x = x^T R^T R x = (Rx)^T (Rx) = \|Rx\|_2^2.$$

Since R is non-singular $Rx = 0$ if and only if $x = 0$.
b) Use the same reformulation as above to obtain

$$\|Ax - b\|_W = \|R(Ax - b)\|_2.$$

The normal equations are now $(RA)^T(RA)x = (RA)^T(Rb)$ or $A^T W A x = A^T W b$.

Exercise 3.13 We start with a full matrix and eliminate columns using reflections H_1, H_2, H_3 and H_4:

$$\begin{pmatrix} x & x & x & x \\ x & x & x & x \\ x & x & x & x \\ x & x & x & x \\ x & x & x & x \end{pmatrix} \sim \begin{pmatrix} + & + & + & + \\ 0 & + & + & + \\ 0 & + & + & + \\ 0 & + & + & + \\ 0 & + & + & + \end{pmatrix} \sim \begin{pmatrix} x & x & x & x \\ 0 & + & + & + \\ 0 & 0 & + & + \\ 0 & 0 & + & + \\ 0 & 0 & + & + \end{pmatrix} \sim \begin{pmatrix} x & x & x & x \\ 0 & x & x & x \\ 0 & 0 & + & + \\ 0 & 0 & 0 & + \\ 0 & 0 & 0 & + \end{pmatrix} \sim \begin{pmatrix} x & x & x & x \\ 0 & x & x & x \\ 0 & 0 & x & x \\ 0 & 0 & 0 & + \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We need one reflection to zero out each column.

Exercise 3.14 Each element below the diagonal requires one givens rotation to zero out. Thus the number of rotations is $mn - n^2/2$.

Exercise 3.15 We use rotations R_{14}, R_{24} and R_{34} to obtain

$$\begin{pmatrix} x & x & x & x \\ 0 & x & x & x \\ 0 & 0 & x & x \\ x & x & x & x \end{pmatrix} \sim \begin{pmatrix} + & + & + & + \\ 0 & x & x & x \\ 0 & 0 & x & x \\ 0 & + & + & + \end{pmatrix} \sim \begin{pmatrix} x & x & x & x \\ 0 & + & + & + \\ 0 & 0 & x & x \\ 0 & 0 & + & + \end{pmatrix} \sim \begin{pmatrix} x & x & x & x \\ 0 & x & x & x \\ 0 & 0 & + & + \\ 0 & 0 & 0 & + \end{pmatrix}.$$

4 Eigenvalues

Exercise 4.1 The eigenvalues are the diagonal elements so $\lambda_k = 1, 2, 3$. The eigenvectors are obtaied by solving $(A - \lambda_k I)x_k = 0$, or for $\lambda = 2$,

$$\left(\begin{pmatrix} 1 & 2 & -4 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) x_k = \begin{pmatrix} -1 & 2 & -4 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} x_k = 0.$$

So for instance $x_k^T = (2\,,\,1\,,\,0)$ is an eigenvector. Can find the other eigenvectors the same way.

Exercise 4.2 Let $x = (1\,,\,1\,,\,\ldots\,,\,1)^T$. Then $Ax = (\alpha\,,\,\alpha\,,\,\ldots\,,\,\alpha)^T = \alpha x$. So x is the corresponding eigenvector.

Exercise 4.3 Singular means there is a non trivial null space. So there is an $x \neq 0$ such that $Ax = 0 = 0x$ which means that zero is an eigenvalue.

Exercise 4.4 This follows from $\det(A) = \det(A^T)$. Thus the Characteristic polynomials are

$$p_A(\lambda) = \det(A - \lambda I) = \det((A - \lambda I)^T) = \det(A^T - \lambda I) = p_{A^T}(\lambda).$$

So both A and A^T have the same characteristic polynomial. To see that the eigenvectors are different just pick a random matrix and check in Matlab. Its rare that A and A^T has the same eigenvectors.

Exercise 4.5 Pick an eigenpair (λ, x). Then $x^H Ax = \lambda x^H x$. So

$$\bar{\lambda} x^H x = (\lambda x^H x)^H = (x^H Ax)^H = x^H A^H x = \{\text{use } A = A^H\} = x^H Ax = \lambda x^H x.$$

So $\bar{\lambda} = \lambda$. For a real matrix $A^T = A^H$ so this is just a special case.

Exercise 4.6 The spectral radius is the largest eigenvalue. Thus

$$\rho(A) = \max|\lambda_k| = \{x_k \text{ eigenvector }\} = \frac{x_k^T A x_k}{x_k^T x_k} \leq \frac{\|x_k\|\|Ax_k\|}{\|x_k\|^2} \leq \max_{x\neq 0} \frac{\|Ax_k\|}{\|x_k\|} = \|A\|.$$

Exercise 4.7 If $A^k = 0$ and $Ax = \lambda x$ for $x \neq 0$ then $A^k x = \lambda^k x = 0$ means $\lambda = 0$. The matrix does not have to be zero. Easiest is to look at

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Exercise 4.8 Since A^{-1} exists we have

$$A(BA)A^{-1} = AB(AA^{-1}) = AB.$$

So BA and AB are *similar*.

Exercise 4.9 If $x = (x_x, x_2)^T$ is an eigenvector and λ is an eigenvalue of A then we have

$$\begin{pmatrix} A_{11} & A_{21} \\ 0 & A_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

If we first assume that λ is an eigenvalue of A then there is an $x = (x_1, x_2)^T$ such that the above relation holds. If $x_2 \neq 0$ then the second row is $A_{22}x_2 = \lambda x_2$ so λ is an eigenvalue of A_{22}. If $x_2 = 0$ then the first row is $A_{11}x_1 = \lambda x_1$ so in this case λ is an eigenvalue of A_{11}. This means that $\lambda(A) \subset \lambda(A_{11}) \cup \lambda(A_{22})$. Secondly we assume that λ is either an eigenvalue of A_{11} or of A_{22} and want to show that then λ is also an eigenbvalue of A. If λ is an eigenvalue of A_{11} then there is an $x_1 \neq 0$ such that $A_{11}x_1 = \lambda x_1$. This means that $x = (x_1\,,\,0)^T$ is an eigenvector to A. So $\lambda \in \lambda(A)$. If, intead λ is an eigenvalue of A_{22} but not of A_{11}, we find an $x_2 \neq 0$ such that $A_{22}x_2 = \lambda x_2$. Let $x = (x_1\,,\,x_2)^T$ and chose x_1 so that the first row is satisfied, i.e.

$$A_{11}x_1 + A_{21}x_2 = \lambda x_1 \Longrightarrow x_1 = -(A_{11} - \lambda I)^{-1}A_{21}x_2.$$

The inverse exists since λ is not an eigenvalue of A_{11} by the assumption. Thus we can construct an eigenvector for A. So $\lambda \in \lambda(A)$ again. The conclusion is that $\lambda(A_{11}) \cup \lambda(A_{22}) \subset \lambda(A)$.

Exercise 4.10 The eigenvalues are the roots of the characteristic polynomial. Thus

$$p(\lambda) = \det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_n - \lambda) \Longrightarrow p(0) = \lambda_1 \lambda_2 \cdots \lambda_n.$$

Exercise 4.11 Since A is rank one we have $\text{Range}(A) = \text{span}(u)$, for a non-zero vector u. Every column of A must be a multiple of u. So

$$A = (v_1 u\,,\, v_2 u\,,\, \ldots\,,\, v_n u) = uv^T.$$

Also $Au = u(v^T u) = (v^T u)u$ so $v^T u$ is an eigenvalue and the corresponding eigenvector is u. If y is orthogonal to v then $Ay = (v^T y)u = 0 = 0y$ so zero is also an eigenvalue. The number of lineary independent eigenvectors is the same as the dimension of the space $\text{span}(v)^\perp$ which is $n-1$.

Exercise 4.12 The determinant of A is the product of the eigenvalues. We have

$$\lambda(I + uv^T) = \{1 + u^T v, 1, 1, \ldots, 1\} \Longrightarrow \det(I + uv^T) = (1 + u^T v).$$

Exercise 4.13 Since $\rho(A) < 1$ we have $|\lambda_i| < 1$ so zero is not an eigenvalue of $I - A$. To verify the formula for the inverse we use

$$(I - A)^{-1}(I - A) = (I + A + A^2 + A^3 + \ldots)(I - A) =$$

$$I - A + A(I - A) + A^2(I - A) + A^3(I - A) = I + (-A + AI) + (-A^2 + A^2 I) + \ldots = I.$$

Strictly we need to show that we can change the order of summation. This holds since the series is *absolute convergent* for $\rho(A) < 1$.

Exercise 4.14 If A is real and symmetric it has a full set of eigenpairs (λ_i, x_i) such that $X = (x_1, x_2, \ldots, x_n)$ is an orthogonal matrix. Any $x \neq 0$ can be written using the basis $\{x_i\}$ so . We obtain

$$x = \sum_{i=1}^{n} c_i x_i, \text{ and } \frac{x^T A x}{x^T x} = \frac{\sum_{i=1}^{n} c_i^2 \lambda_i}{\sum_{i=1}^{n} c_i^2}.$$

The lower bound follows from $\lambda_{min}(A) \leq \lambda_i$ and the upper bound from $\lambda_i \leq \lambda_{max}(A)$.

Exercise 4.15 First we use the same reflection H_1 applied from the left and from the right. The reflection is selected so the elements $A(3:4,1)$ are set to zero. We get

$$H_1 \begin{pmatrix} x & x & x & x \\ x & x & x & x \\ x & x & x & x \\ x & x & x & x \end{pmatrix} H_1^T = \begin{pmatrix} x & x & x & x \\ + & + & + & + \\ 0 & + & + & + \\ 0 & + & + & + \end{pmatrix} H_1^T = \begin{pmatrix} x & + & + & + \\ x & + & + & + \\ 0 & + & + & + \\ 0 & + & + & + \end{pmatrix}.$$

Second we find a reflection H_2 that zeroes out the element $A(4,2)$. We get

$$H_2 \begin{pmatrix} x & x & x & x \\ x & x & x & x \\ 0 & x & x & x \\ 0 & x & x & x \end{pmatrix} H_2^T = \begin{pmatrix} x & x & x & x \\ x & x & x & x \\ 0 & + & + & + \\ 0 & 0 & + & + \end{pmatrix} H_2^T = \begin{pmatrix} x & x & + & + \\ x & x & + & + \\ 0 & x & + & + \\ 0 & 0 & + & + \end{pmatrix},$$

which is Hessenberg.

Exercise 4.16 **a)** In the 4×4 case we need 3 rotations $G_{34}G_{23}G_{12}$ to compute the QR decomposition of A. We have

$$G_{34}G_{23}G_{12} \begin{pmatrix} x & x & x & x \\ x & x & x & x \\ 0 & x & x & x \\ 0 & 0 & x & x \end{pmatrix} = G_{34}G_{23} \begin{pmatrix} + & + & + & + \\ 0 & + & + & + \\ 0 & x & x & x \\ 0 & 0 & x & x \end{pmatrix} = G_{34} \begin{pmatrix} x & x & x & x \\ 0 & + & + & + \\ 0 & 0 & + & + \\ 0 & 0 & x & x \end{pmatrix} = \begin{pmatrix} x & x & x & x \\ 0 & x & x & x \\ 0 & 0 & + & + \\ 0 & 0 & 0 & + \end{pmatrix}.$$

Now compute RQ by multiplying the rotations in the same order from the left.

$$\begin{pmatrix} x & x & x & x \\ 0 & x & x & x \\ 0 & 0 & x & x \\ 0 & 0 & 0 & x \end{pmatrix} G_{12}^T G_{23}^T G_{34}^T = \begin{pmatrix} + & + & x & x \\ + & + & x & x \\ 0 & 0 & x & x \\ 0 & 0 & 0 & x \end{pmatrix} G_{23}^T G_{34}^T = \begin{pmatrix} x & + & + & x \\ x & + & + & x \\ 0 & + & + & x \\ 0 & 0 & 0 & x \end{pmatrix} G_{34}^T = \begin{pmatrix} x & x & + & + \\ x & x & + & + \\ 0 & x & + & + \\ 0 & 0 & + & + \end{pmatrix}.$$

Which is again Hessenberg. The part **b)** follows from the fact that if RQ is Hessenberg and Symmetric then it is trigdiagonal. No new proof is needed.

Exercise 4.17 We show that A_{k+1} is similar to A_k by

$$Q_k^T A_k Q_k = Q_k^T (Q_k R_k + sI) Q_k = R_k Q_k + s Q_k^T Q_k = R_k Q_k + sI = A_{k+1}.$$

Exercise 4.18 If B have distinct eigenvalues then B is diagonalizable. Use the *Schur decomposition* $T = QAQ^T$. The eigenvalues are the diagonal elements of T. Pick a diagonal matrix D, with $|d_{ii}| < \varepsilon$, such that $T + D$ has distinct diagonal elements. Then $B = Q^T(T + D)Q$ is diagonalizable and $\|A - B\|_2 = \|Q^T D Q\|_2 = \|D\|_2 < \varepsilon$.

Exercise 4.19 First T and A are similar and have the same eigenvalues. Since T is upper triangular we have $p_T(\lambda) = (T_{11} - \lambda) \cdots (T_{nn} - \lambda)$. So the diagonal elements are the roots of the characteristic polynomial. If A is symmetrix then $A = QTQ^T$ and $A^T = (QTQ^T)^T = QT^TQ^T$ so $T = T^T$. So T is upper triangular and symmetric and thus T is actually a diagonal matrix. So $A = QTQ^T$ is an eigenvalue decomposition with the columns of Q as eigenvectors.

Exercise 4.20 Compute a QR decomposition

$$x = Q \begin{pmatrix} r_{11} \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

where Q is $n \times n$ orthogonal. This means that $x = r_{11}q_1$ where q_1 is the first column of Q. So q_1 is also an eigenvector. We get

$$Q^T AQ = (q_1, Q_2)^T A(q_1, Q_2) = \begin{pmatrix} q_1^T Aq_1 & q_1^T AQ_2 \\ Q_2^T Aq_1 & Q_2^T AQ_2 \end{pmatrix} = \begin{pmatrix} \lambda & w^T \\ 0 & \widetilde{A} \end{pmatrix}$$

where $Q_2^T Aq_1 = \lambda Q_2^T q_1 = 0$ since Q is orthogonal. We can thus obtain $P = Q$ by using rotations to compute the QR decomposition of x. Thus $P = G_{12}G_{13}G_{14}\cdots G_{1n}$.

Exercise 4.21 a) The Gershgorin discs are

$$|\lambda - 16| \le 4, \quad |\lambda - 16| \le 4, \text{ and } |\lambda - 12| \le 8.$$

Since any eigenvalue λ has to belong to one of the discs we see that its not possible for an eigenvalue to be larger than 20.
b) We have $Ax = \lambda x$ so $(A - sI)x = (\lambda - s)x$ and finally $Bx = (A - sI)^{-1}x = (\lambda - s)^{-1}x$ so $(x, (\lambda - s)^{-1})$ is the eigenpair of B.
c) The matrix A is symmetric so the eigenvectors v_1, v_2, and v_3 are orthogonal. So

$$Bv_1 = Av_1 + sv_1v_1^T v_1 = \lambda_1 v_1 + sv_1 = (\lambda_1 + s)v_1,$$

so $(\lambda_1 + s, v_1)$ is an eigenpair. Since $v_1^T v_2 = v_1^T v_3 = 0$ we see that $Bv_2 = \lambda_2 v_2$ and $Bv_3 = \lambda_3 v_3$ so the two eigenpairs remain the same.

Exercise 4.22 a The Gershgorin discs are

$$|\lambda - 14| \le 3, \quad |\lambda - 9| \le 2.5, \text{ and } |\lambda - 4| \le 1.5.$$

Since the matrix is symmetric there is no need to consider A^T. Zero is not included in any of the discs so zero cannot be an eigenvalue. Thus the matrix is non-singular.
b) The matrix is symmetric so the eigenvectors v_1, v_2 and v_3 are orthogonal. We get

$$Bv_1 = Av_1 + sv_1v_1^T v_1 = (\lambda_1 + s)v_1, \text{ and } Bv_k = Av_k + sv_1v_1^T v_k = \lambda_k v_k, \text{ for } k = 2, 3.$$

Thus the eigenvevtors remain the same and the eigenvalues are $\{\lambda_1 + s, \lambda_2, \lambda_3\}$.

Exercise 4.23 a) The Gershgorin discs are

$$|\lambda - 100| \le 1, \quad |\lambda + 10| \le 6, \text{ and } |\lambda - 4| \le 3.$$

We can also consider the discs obtained by looking at A^T. We get

$$|\lambda - 100| \le 6, \quad |\lambda + 10| \le 2, \text{ and } |\lambda - 4| \le 2.$$

Since the discs are disjoint each disc has exactly one eigenvalue and we can pick the best disc for each. Since the matrix is real and each disc contains one eigenvalue we cannot have complex eigenvalues.

b) If A is non-singular then there is an $x \neq 0$ such that $Ax = 0 = 0x$ so zero is en eigenbalue. None of the above discs contain zero so the matrix is non-singular.

c) The iteration will converge since all the eigenvalues are different as the discs are disjoint. This means that one eigenvalue will have a strictly smaller absolute value than the others. Which in turn means one of the eigenvalues of A^{-1} will be strictly larger than the other two.

5 Non-linear Equations and Least Squares

Exercise 5.1 Determine if the following functions are *coercive* on $\mathbb{R}^2$.

(a) No, since $x = -2 - y$ gives $f(x, y) = 0$.

(b) Yes. We have $f(x, y) \geq \| \begin{pmatrix} x \\ y \end{pmatrix} \|_2$.

(c) No. Since $f(x, y) = x^2 - 2xy + y^2 = (x - y)^2$. So $x = y$ gives $f(x, y) = 0$.

(d) Yes. We see that $f(x, y) = x^4 - 2xy + y^4 = x^2(x^2 - 1) + y^2(y^2 - 1) + (x + y)^2$. If $(x\ ,\ y)^T \to \infty$ then $f(x, y) \to \infty$.

Exercise 5.2 Let $f(x) = (x_1^2 + x_1 x_2^3 - 9\,,\, 3x_1^2 x_2 - x_2^3 - 4)^T$. Then the Jacobian is

$$J_f(x) = \begin{pmatrix} 2x_1 + x_2^3 & 3x_1 x_2^2 \\ 6x_1 x_2 & 3x_1^2 - 3x_2^2 \end{pmatrix}.$$

The Newton iteration is

$$x^{(k+1)} = x^{(k)} - J_f^{-1}(x^{(k)}) f(x^{(k)}).$$

There is no reason to write the iteration in more detail.

Exercise 5.3 We have

$$f(x) = \begin{pmatrix} x_1^2 - x_2^2 \\ 2x_1 x_2 - 1 \end{pmatrix} \text{ and } J_f(x) = \begin{pmatrix} 2x_1 & -2x_2 \\ 2x_2 & 2x_1 \end{pmatrix}.$$

In the first step we solve $J_f(\begin{pmatrix} 0 \\ 1 \end{pmatrix}) s^{(0)} = -f(\begin{pmatrix} 0 \\ 1 \end{pmatrix})$ or

$$\begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} s^{(0)} = -\begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

So we obtain

$$x^{(1)} = x^{(0)} + s^{(0)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0.5 \\ -0.5 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}.$$

Exercise 5.4 We use the Taylor series expansion of the iteraton function $g(x)$, i.e.

$$g(x^{(k)}) = g(x^* + (x^{(k)} - x^*)) = g(x^*) + g'(x^*)(x^{(k)} - x^*) + \frac{g''(x^*)}{2}(x^{(k)} - x^*)^2 + \mathcal{O}(|x^{(k)} - x^*|^3).$$

Since $g'(x^*) = 0$ the error can be written

$$|x^{(k+1)} - x^*| = |g(x^{(k)}) - g(x^*)| \leq |\frac{g''(x^*)}{2}|(x^{(k)} - x^*)^2 + \mathcal{O}(|x^{(k)} - x^*|^3).$$

This is quadratic convergence.

Exercise 5.5 We have

$$f(x) = \begin{pmatrix} x_1^2 + \sin(x_2) \\ 1 + \cos(x_2^2) + x_2 \end{pmatrix} \text{ so } J_f(x) = \begin{pmatrix} 2x_1 & \cos(x_2) \\ 0 & -2x_2\sin(x_2^2) + 1 \end{pmatrix}.$$

In the first step we solve $J_f(\begin{pmatrix} 1 \\ 0 \end{pmatrix})s^{(0)} = -f(\begin{pmatrix} 1 \\ 0 \end{pmatrix})$ or

$$\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} s^{(0)} = -\begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

So we obtain

$$x^{(1)} = x^{(0)} + s^{(0)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0.5 \\ -2 \end{pmatrix} = \begin{pmatrix} 1.5 \\ -2 \end{pmatrix}.$$

Exercise 5.6 a) Use Gauss-Newton to minimize $r^T r$ where

$$r((x\,,\,y)^T) = \begin{pmatrix} (x+1)^2 + y^2 - 0.25 \\ x^2 + (y-1)^2 - 0.25 \\ (x-1)^2 + y^2 - 0.25 \end{pmatrix} \text{ and } J_r((x\,,\,y)^T) = \begin{pmatrix} 2(x+1) & 2y \\ 2x & 2(y-1) \\ 2(x-1) & 2y \end{pmatrix}.$$

b) In the first Gauss-Newton step we minimize $\|J_r(x^{(0)})s^{(0)} + r(x^{(0)})\|_2$. If $x^{(0)} = (0,0)^T$ we obtain

$$r((0\,,\,0)^T) = \begin{pmatrix} 0.75 \\ 0.75 \\ 0.75 \end{pmatrix} \text{ and } J_r((0\,,\,0)^T) = \begin{pmatrix} 2 & 0 \\ 0 & -2 \\ -2 & 0 \end{pmatrix}.$$

Easiest is to solve using the normal equations which gives $s^{(0)} = (0\,,\,0.375)^T$. So $x^{(1)} = x^{(0)} + s^{(0)} = (0\,,\,0.375)^T$.

Exercise 5.7 The problem is to prove that the Hessian and gradient are the expected ones. In order to do this we write $f(x)$ in component form, i.e.

$$f(x_1, x_2, \ldots, x_n) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_i x_j - \sum_{i=1}^{n} b_i x_i + c$$

In order to find the gradient ∇f. we differentiate with respect to a particular x_k to obtain

$$\partial_{x_k} f(x) = \sum_{j=1}^{n} a_{kj}x_j - b_k.$$

This is one row of the residual vector $Ax - b$. Since the gradient is supposed to be a column vector we find that $\nabla f = Ax - b$. Differentiate again to obtain

$$\partial_{x_i}\partial_{x_j} f(x) = a_{ij}.$$

Thus $H_f = A$. One Newton step would then consist of solving $H_f s^{(0)} = -\nabla f$ or $As^{(0)} = -(Ax^{(0)} - b)$. So $x^{(1)} = x^{(0)} + s^{(0)}$ will be a solution to $Ax = b$. For the next step $\nabla f(x^{(1)}) = Ax^{(1)} - b = 0$ so the method terminates.

Exercise 5.8 a) The vector $r(x)$ is of length m and the number of variables in the vector x is n. Thus $f(x)$ is simply a function of n variables so its gradient is $n \times 1$ and its Hessian, or the matrix consisting of all second derivatives, H_f is $n \times n$.
b) If $H_f \approx J_r^T J_r$ and we use $\nabla f = J_r^T r(x)$ then we get $J_r^T J_r s^{(k)} = -J_r^T r$ which is the normal equations for the least squares problem $\min \|J_r(x^{(k)})s^{(k)} + r(x^{(k)})\|_2$.
c) If $r^{(0)} = b - Ax^{(0)}$ then $J_r(x) = -A$ so Gauss-Newton minimize $\| - As^{(0)} + b - Ax^{(0)}\|_2 = \|A(x^{(0)} + s^{(0)}) - b\|_2$. So $x^{(1)}$ is the minimum to $r^T r = \|b - Ax\|_2^2$.

Exercise 5.9 a) The residual vector would be

$$r(c) = \left(c_0 \mathrm{e}^{-c_1 t_1} + c_2 \sin(\omega t_1) - F_1, \, , \, \ldots \, , \, c_0 \mathrm{e}^{-c_1 t_m} + c_2 \sin(\omega t_m) - F_m\right)^T .$$

b) The Jacobian is obtained by differentiation with respect to the parameters c_0, c_1 and c_2. A row of the matrix is given by

$$(J_r)_{i,1:3} = (\mathrm{e}^{-c_1 t_i} \, , \, -c_0 t_i \mathrm{e}^{-c_1 t_i} \, , \, \sin(\omega t_i)).$$

The whole matrix is $m \times 3$.

6 The Singular Value Decomposition

Exercise 6.1 Since a is $n\times 1$ the dimensions of the factors are $U \in \mathbb{R}^{n\times n}$, $\Sigma \in \mathbb{R}^{n\times 1}$ and $V \in \mathbb{R}^{1\times 1}$. The decomposition is

$$a = U\Sigma V^T = \begin{pmatrix} \frac{a}{\|a\|_2} A_2 \end{pmatrix} \begin{pmatrix} \|a\|_2 \\ 0 \end{pmatrix} (1),$$

where $A_2 \in \mathbb{R}^{n\times n-1}$ has columns that are orthogonal to a. To obtain the SVD of a^T we simply use $A^T = V\Sigma^T U^T$.

Exercise 6.2 $A^T = V\Sigma^T U^T$. If $y \in \operatorname{span}(u_{k+1},\dots,u_m)$ then $u_i^T y = 0$ for $i = 1,\dots,k$. This is the null space of A^T.

Exercise 6.3 Let $A = U\Sigma V^T$ be an $m\times n$ matrix. First note that the whole matrix is $(m+n)\times(m+n)$. We obtain

$$\begin{pmatrix} 0 & A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} \pm v_i \\ u_i \end{pmatrix} = \begin{pmatrix} A^T u_i \\ \pm A v_i \end{pmatrix} = \begin{pmatrix} \sigma_i v_i \\ \pm\sigma_i u_i \end{pmatrix} = \pm\sigma_i \begin{pmatrix} \pm v_i \\ u_i \end{pmatrix}.$$

There is also the posibility of zero eigenvalues corresponding to eigenvevtors of the type $(0, u_i^T)^T$ or $(v_i^T, 0)^T$ if A or A^T has a null space (i.e. if $m > n$ or $m < n$).

Exercise 6.4 First compute $(A^TA)^{-1} = (V\Sigma^T U^T U\Sigma V^T)^{-1} = V(\Sigma^T\Sigma)^{-1}V^T$. Here $\Sigma^T\Sigma = \operatorname{diag}(\sigma_i^2) \in \mathbb{R}^{n\times n}$. Thus $A(A^TA)^{-1}A^T = U\Sigma V^T V(\Sigma^T\Sigma)^{-1}V^T V\Sigma^T U^T = U\Sigma(\Sigma^T\Sigma)^{-1}\Sigma^T U^T$. Since U is orthogonal $\|A(A^TA)^{-1}A\|_2 = \|\Sigma(\Sigma^T\Sigma)^{-1}\Sigma^T\|_2$. Evaluate the product of the diagonal matrices to obtain

$$\Sigma(\Sigma^T\Sigma)^{-1}\Sigma^T = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{R}^{m\times m}, \quad I \in \mathbb{R}^{n\times n}.$$

The norm is the largest diagonal entry, i.e. 1.

Exercise 6.5 Let $B = U\Sigma V^T$. Since $V = (v_1,\dots,v_n)$ provides a basis for $\mathbb{R}^n$ any x can be written

$$x = \sum_{i=1}^n c_i v_i \Longrightarrow Bx = \sum_{i=1}^n c_i\sigma_i u_i.$$

If $\|x\|_2 = 1$ then $\sum c_i^2 = 1$. So

$$\|Bx\|_2^2 = \sum_{i=1}^n \sigma_i^2 c_i^2 \ge \sigma_n \sum_{i=1}^n c_i^2 = \sigma_n^2,$$

with equality for $c = e_n$. So the minimum is σ_n and it is obtained for $x = \pm v_n$.

Exercise 6.6 The decomposition $A = U\Sigma V^T$ can be written

$$A = \sum_{i=1}^n \sigma_i u_i v_i^T,$$

where $\sigma_n > 0$ as $\text{rank}(A) = n$. This means that $Av_i = \sigma_i u_i \neq 0$ for $i = 1, \ldots, n$. So the null space is only the trivial one $\text{null}(A) = \{0\}$ with dimension 0. Similarily, if y belongs to the range then there is an x such that $y = Ax$, or

$$y = Ax = \sum_{i=1}^{n} \sigma_i (v_i^T x) u_i,$$

so the y is a linear combination of $\{u_1, \ldots, u_n\}$. Thus $\text{range}(A) = \text{span}(u_1, \ldots, u_n)$ and the dimension of the range is n.

Exercise 6.7 Let $A = U\Sigma V^T$. Since $U = (u_1, \ldots, u_m)$ is a basis for $\mathbb{R}^m$ we can write

$$b = \sum_{i=1}^{m} (u_i^T b) u_i,$$

Similarily, $V = (v_1, \ldots, v_n)$ is a basis for $\mathbb{R}^n$ so

$$Ax = A(\sum_{i=1}^{n} (v_i^T x) v_i) = \sum_{i=1}^{n} \sigma_i (v_i^T x) u_i.$$

We obtain

$$\|Ax - b\|_2^2 = \| \sum_{i=1}^{n} (\sigma_i (v_i^T x) - (u_i^T b)) u_i - \sum_{i=n+1}^{m} (u_i^T b)) u_i \|_2^2 = \sum_{i=1}^{n} |\sigma_i (v_i^T x) - (u_i^T b)|^2 + \sum_{i=n+1}^{m} |u_i^T b|^2.$$

The minimum is obtained for $\sigma_i (v_i^T x) - u_i^T b$ for $i = 1, \ldots, n$, so

$$x = \sum_{i=1}^{n} \frac{u_i^T b}{\sigma_i} v_i.$$

For this particular x we get

$$r = b - Ax = \sum_{i=n+1}^{m} (u_i^T b) u_i.$$

Exercise 6.8 Compute Ax to obtain

$$Ax = A(\sum_{i=1}^{m} \frac{u_i^T b}{\sigma_i} v_i) = \sum_{i=1}^{m} \frac{u_i^T b}{\sigma_i} A v_i = \sum_{i=1}^{m} \frac{u_i^T b}{\sigma_i} \sigma_i u_i = \sum_{i=1}^{m} (u_i^T b) u_i = b,$$

where the last equality holds since $U = (u_1, \ldots, u_m)$ provides an orthogonal basis for $\mathbb{R}^m$ which is the space b belongs to.

Since $m < n$ the matrix has a null space $\text{null}(A) = \text{span}(v_{m+1}, \ldots, v_n)$. If x_2 belongs to the nullspace then $A(x + x_2) = Ax = b$ so the solution is not unique. Since the above formula for x does not include a component from the null space it can be characterized as

$$\min \|x\|_2 \text{ such that } Ax = b,$$

that is the *minimum norm* solution of the linear system $Ax = b$.

Exercise 6.9 Since rank$(A) = k$ we note that $\{v_{k+1}, \ldots, v_n\}$ is a basis for null(A) and $\{v_1, \ldots, v_k\}$ is a basis for its orthogonal complement $(\text{null}(A))^\perp$. Thus for evey x we can write

$$x = x_1 + x_2 = (\sum_{i=1}^{k} c_i v_i) + (\sum_{i=k+1}^{n} c_i v_i).$$

In order to determine x_1 we compute

$$Ax = A(x_1 + x_2) = Ax_1 + 0 = \sum_{i=1}^{k} c_i \sigma_i u_i = b = \sum_{i=1}^{m} (u_i^T b) u_i.$$

Where $(u_i^T b) = 0$, for $i = k+1, \ldots, m$, or a solution doesn't exist. Thus

$$x_1 = \sum_{i=1}^{k} \frac{u_i^T b}{\sigma_i} v_i \text{ and } x_2 = \sum_{i=k+1}^{n} c_i v_i,$$

where $c_i, i = k+1, \ldots, n$, are undetermined parameters.

Exercise 6.10 a) If B has full rank then $Bx = 0$ if and only if $x = 0$ so the unique, and only feasible, solution is precisely $x = 0$.

b) If rank$(B) = k < n$ then B has a non-trivial null space and write $V = (V_k, V_{n-k})$ so that the null space is given by V_{n-k} then the feasible solutions are $x = V_{n-k}c$, $c \in \mathbb{R}^{n-k}$. So in fact we have a regular least squares problem

$$\min_{c \in \mathbb{R}^{n-k}} \|(AV_{n-k})c - b\|_2 \text{ and } x = V_{n-k}c.$$

The above qualifies as a formula. Otherwise continmue and write the normal equations for the above least squares problem.

Exercise 6.11 Let $A = U \Sigma V^T$ and $U = (u_1, \ldots, u_m)$. A solution exists if $b \in \text{range}(A) = \text{span}(u_1, \ldots, u_n)$. We can check this by, for instance, verifying that $u_i^T b = 0$, for $i = n+1, \ldots, m$. If $m >> n$ it is cheaper to instead check if

$$b - \sum_{i=1}^{n} (u_i^T b) u_i = 0.$$

If we split the matrix $U = (U_1, U_2)$ then the same criteria can be written as $U_2^T b = 0$ or $b - U_1 U_1^T b = 0$.

Exercise 6.12 First we use a reflection H_1 applied from the left. The reflection is selected so the elements $A(2:4,1)$ are set to zero. Second we apply a reflection H_2 from the right to zero out the elements $\widetilde{A}(1, 3:4)$. We get

$$H_1 \begin{pmatrix} x & x & x & x \\ x & x & x & x \\ x & x & x & x \\ x & x & x & x \end{pmatrix} . \begin{pmatrix} + & + & + & + \\ 0 & + & + & + \\ 0 & + & + & + \\ 0 & + & + & + \end{pmatrix} H_2^T = \begin{pmatrix} x & + & 0 & 0 \\ 0 & + & + & + \\ 0 & + & + & + \\ 0 & + & + & + \end{pmatrix}.$$

Now we continue with reflections H_3 and H_4 that zero out $A(3:4,2)$ and $A(2,4)$. We get

$$H_3\begin{pmatrix} x & x & 0 & 0\\ 0 & x & x & x\\ 0 & x & x & x\\ 0 & x & x & x\end{pmatrix}H_4^T = \begin{pmatrix} x & x & 0 & 0\\ 0 & + & + & +\\ 0 & 0 & + & +\\ 0 & 0 & + & +\end{pmatrix}H_4^T = \begin{pmatrix} x & x & 0 & 0\\ 0 & x & + & 0\\ 0 & 0 & + & +\\ 0 & 0 & + & +\end{pmatrix}.$$

Finally we apply one reflection H_5 from the left to zero out the element $A(4,3)$. We get

$$H_5\begin{pmatrix} x & x & 0 & 0\\ 0 & x & x & 0\\ 0 & 0 & x & x\\ 0 & 0 & x & x\end{pmatrix} = \begin{pmatrix} x & x & 0 & 0\\ 0 & x & x & 0\\ 0 & 0 & + & +\\ 0 & 0 & 0 & +\end{pmatrix},$$

which is bidiagonal.

Exercise 6.13 The normal equations can be derived by the identity

$$\min_x \left(\|Ax-b\|_2^2 + \lambda^2\|x\|_2^2\right) = \min_x \left\|\begin{pmatrix} Ax-b\\ \lambda x\end{pmatrix}\right\|_2^2 = \min_x \left\|\begin{pmatrix} A\\ \lambda I\end{pmatrix}x - \begin{pmatrix} b\\ 0\end{pmatrix}\right\|_2^2.$$

The last is a regular least squares problem with an extended matrix. The normal equations are

$$\begin{pmatrix} A^T & \lambda I\end{pmatrix}\begin{pmatrix} A\\ \lambda I\end{pmatrix}x = \begin{pmatrix} A^T & I\end{pmatrix}\begin{pmatrix} b\\ 0\end{pmatrix} \quad \text{or } (A^TA+\lambda^2 I)x = A^Tb.$$

Now we can derive the solution formula using the decomposition $A = U\Sigma V^T$. Since $A^TA + \lambda^2 I = V\Sigma^T\Sigma V^T + \lambda^2 VV^T = V(\Sigma^T\Sigma + \lambda^2 I)V^T$ and $A^Tb = V\Sigma U^Tb$ we obtain the solution

$$x_\lambda = V(\Sigma^T\Sigma + \lambda^2 I)^{-1}\Sigma U^T b = \sum_{i=1}^n \frac{\sigma_i}{\sigma_i^2+\lambda^2}(u_i^Tb)v_i.$$

Too see that the normal equations are not ill-conditioned we look at A^TA which has singular values $\sigma_i^2 + \lambda^2 \geq \lambda^2$. So the addition of the regularization parameter removes the small singular values and makes the condition number smaller.

Exercise 6.14 Use $A = U\Sigma V^T$ to obtain

$$\frac{y^TAx}{\|y\|_2\|x\|_2} = \frac{(U^Ty)^T\Sigma(V^Tx)}{\|y\|_2\|x\|_2} = \widetilde{y}^T\Sigma\widetilde{x} = \sum_{i=1}^n \sigma_i\widetilde{y}_i\widetilde{x}_i \leq \sigma_1|(\widetilde{y})^T(\widetilde{x})| \leq \sigma_1\|\widetilde{y}\|_2\|\widetilde{x}\|_2 = \sigma_1$$

where $\widetilde{y} = U^Ty/\|y\|_2$ and $\widetilde{x} = V^Tx/\|x\|_2$ both are normalized and equality is achived for $x = Ve_1$ and $y = Ue_1$ (or $\widetilde{x} = \widetilde{y} = e_1$).

7 Sparse Matrices and Iterative Methods

Exercise 7.1 Start from $Ax = b$ and use the splitting $A = D + (A - D)$. The Jacobi method is

$$x^{(k+1)} = D^{-1}((D-A)x^{(k)}) + D^{-1}b = x^{(k)} - D^{-1}Ax^{(k)} + D^{-1}b = x^{(k)} + D^{-1}(b - Ax^{(k)}) = x^{(k)} + D^{-1}r^{(k)}.$$

Thus $H = D^{-1}$.

Exercise 7.2 The error in step k can be written

$$x^{(k+1)} - x^* = G(x^{(k)} - x^*) = G^k((x^{(0)} - x^*).$$

Since $G^k \to 0$ as $k \to \infty$ if $\rho(G) < 1$ we have convergene in this case.

Exercise 7.3 If A is singular then zero is an eigenvalue so there is an $x \neq 0$ such that $0 = Ax = (M - N)x$ and we obtain $M^{-1}Nx = x$. So x is an eigenvector of $M^{-1}N$ corresponding to the eigenvalue $\lambda = 1$. Thus $\rho(M^{-1}N) \geq 1$.

Exercise 7.4 The Conjugate gradient method minimizes the error over solutions of the type $x = x^{(0)} + \mathcal{K}_m(A, r^{(0)})$. After n steps either the dimension of $\mathcal{K}_n(A, r^{(0)})$ is exactly n and we minimize the error in the whole space $\mathbb{R}^n$ and get the exact solution; or we had *breakdown* and found an invariant subspace containing the exact solution earlier.

Exercise 7.5 Let $x^{(k)}$ be the current iterate. Since $\mathcal{K} = \text{span}(e_i)$ the next iterate will be $x^{(k+1)} = x^{(k)} + \alpha e_i$. The requirement that $r^{(k+1)} \perp \mathcal{L}$ leads to

$$0 = e_i^T(b - Ax^{(k+1)}) = e_i^T(b - A(x^{(k)} + \alpha e_i)) = e_i^T r^{(k)} - \alpha e_i^T A e_i = e_i^T r^{(k)} - \alpha a_{ii}.$$

So one formula would be $x^{(k+1)} = x^{(k)} + (e_i^T r^{(k)}/a_{ii})e_i$.

Exercise 7.6 Let $V = (v_1, v_2, \ldots, v_m) \in \mathbb{R}^{n \times m}$ be a basis for $\mathcal{K}_m$. Then then projection step produces an $x^{(m)} = x^{(0)} + Vy$, $y \in \mathbb{R}^m$. The orthogonality condition $r^{(m)} \perp \mathcal{L}$ can be rewritten using a basis $W = (w_1, w_2, \ldots, w_m) \in \mathbb{R}^{n \times m}$ for $\mathcal{L}_m$. We obtain

$$0 = W^T r^{(m)} = W^T(b - Ax^{(m)}) = W^T(b - A(x^{(0)} + Vy)) = W^T(r^{(0)} - AVy) = W^T r^{(0)} - W^T AVy.$$

So $y = (W^T AV)^{-1} W^T r^{(0)}$ and $x^{(m)} = x^{(0)} + V(W^T AV)^{-1} W^T r^{(0)}$.

Exercise 7.7 **a)** Positive definite means that $x^T Ax > 0$ for all $x \neq 0$. This means that zero cannot be an eigenvalue so A^{-1} exists.

b) Let $x \neq 0$. Since V has orthogonal columns we have $y = Vx \neq 0$ if $x \neq 0$ and

$$x^T V^T AVx = (Vx)^T A(Vx) = y^T Ay > 0,$$

A is positive definite. $V^T AV$ is thus positive definite and therefore also non-singular.

Exercise 7.8 **a)** We make Ar orthogonal to r by Gram-Schmidt orthogonalisation: Set $p = Ar - \alpha r$ and select α so

$$0 = (p, r)_A = p^T Ar = (Ar - \alpha r)^T Ar = (Ar)^T (Ar) - \alpha r^T Ar,$$

and $p = Ar - \|Ar\|_2^2/(r^T Ar)r$. Note that p is always well-defined unless $r = 0$ and we already have an exact solution.

b) The next iterate will be of the form $x^{(k+1)} = x^{(k)} + \beta_1 r^{(k)} + \beta_2 p$. We get

$$r^{(k+1)} = b - Ax^{(k+1)} = r^{(k)} - \beta_1 Ar^{(k)} - \beta_2 Ap.$$

We need to select β_1 and β_2 so that $r^{(k+1)}$ is orthogonal (not A-orthogonal) to both $r^{(k)}$ and p. We obtain, with $r = r^{(k)}$,

$$0 = r^T(r - \beta_1 Ar - \beta_2 Ap) = \|r\|_2^2 - \beta_1 r^T Ar - \beta_2 r^T Ap = \|r\|_2^2 - \beta_1 r^T Ar \Longrightarrow \beta_1 = \|r\|_2^2/(r^T Ar).$$

and

$$0 = p^T(r - \beta_1 Ar - \beta_2 Ap) = p^T r - \beta_1 p^T Ar - \beta_2 p^T Ap = p^T r - \beta_2 p^T Ap \Longrightarrow \beta_2 = p^T r/(p^T Ap).$$

Now we have everything needed to compute $x^{(k+1)}$.

Exercise 7.9 **a)** Suppose $x \in \mathcal{K}_k(A, r^{(0)})$. Then

$$x = c_1 r^{(0)} + c_2 Ar^{(0)} + \ldots + c_k A^{k-1} r^{(0)},$$

Break-down means $A^k r^{(0)}$ belongs to $\mathcal{K}_k(A, r^{(0)})$ so

$$Ax = c_1 Ar^{(0)} + c_2 A^2 r^{(0)} + \ldots + c_{k-1} A^{k-1} r^{(0)} + c_k A^k r^{(0)},$$

also belongs to $\mathcal{K}_k(A, r^{(0)})$ so it is invariant under multiplication by A.

b) Let $x = x^{(0)} + \widetilde{x}$, where $\widetilde{x} \in \mathcal{K}_{k+1} = \mathcal{K}_k$. Then

$$r = b - Ax = b - A(x^{(0)} + \widetilde{x}) = r^{(0)} - A\widetilde{x}.$$

Thus

$$r = r^{(0)} + A(c_0 r^{(0)} + \ldots + c_{k-1} A^{k-1} r^{(0)}) = r^{(0)} + c_0 Ar^{(0)} + \ldots + c_{k-1} A^k r^{(0)} \in \mathcal{K}_{k+1} = \mathcal{K}_k.$$

Since the vectors $r^{(0)}, \ldots, A^k r^{(0)}$ are linearly dependent (due to break down having occured) it is possible to pick $c_0, \ldots, c_{k-1}$ so that $r = 0$. This means that the exact solution belongs to the subspace $x^{(0)} + \mathcal{K}_k$.

c) Let V_k be the basis computed by the Arnoldi process. Find V_{n-k} so $V = (V_k, V_{n-k})$ is orthogonal. Then $AV_k \in \text{Range}(V_k) \perp V_{n-k}$ due to the break down. Thus

$$H = V^T AV = (V_k, V_{n-k})^T A(V_k, V_{n-k}) = \begin{pmatrix} V_k^T AV_k & V_k^T AV_{n-k} \\ V_{n-k}^T AV_k & V_{n-k}^T AV_{n-k} \end{pmatrix} = \begin{pmatrix} H_k & W \\ 0 & H_{n-k} \end{pmatrix}.$$

So the decoupling theorem says $\lambda(H_k) \subset \lambda(A)$. The eigenvectors are obtained from

$$(V_k^T AV_k)y = \lambda y \Longrightarrow A(V_k y) = \lambda V_k y$$

so $(V_k y)$ are the eigenvectors of A.

8 Sparse Matrices and Iterative Methods