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Matematiska institutionen
Beräkningsmatematik/Fredrik Berntsson

Exam TANA15 Numerical Linear Algebra, Y4, Mat4

Datum: Klockan 14-18, 15:e Mars, 2017.

Hjälpmedel:

1. Föreläsningsanteckningar utskrivna från kurshemsidan utan egna anteckningar.
2. Räknedosa i fickformat, med nollställt minne och utan instruktionsbok.

Examinator: Fredrik Berntsson

Maximalt antal poäng: 25 poäng. För godkänt krävs 8 poäng.

Jourhavandelärare Fredrik Berntsson - (telefon 013 282860)

Besök av jourhavande lärare sker ungefär 15.15 och 17.45.

Resultat meddelas via epost senast 23:e Mars.

Visning av tentamen sker på Examinators kontor Fredag den 24:e Mars, klockan 12.15-13.00 (Hus B, Ing. 25-27, Plan-3, A-korr).

Good luck!

(4p) **1:** Let $A \in \mathbb{R}^{m\times n}$ have the singular value decomposition $A = U\Sigma V^T$. If $\text{Rank}(A) = k < \min(m,n)$ we cannot be certain that a linear system $Ax = b$ has a unique solution. Do the following:

- **a)** Write down a basis for $\text{Range}(A)$ in terms of the singular vectors. What are the dimensions of the spaces $\text{Range}(A)$ and $\text{Range}(A)^{\perp}$?
- **b)** Write down an expression for the orthogonal projection onto $\text{Range}(A)^{\perp}$ in terms of the singular vectors. Also show that the linear system $Ax = b$ has a solution if $Pb = 0$ where P is the orthogonal projection onto $\text{Range}(A)^{\perp}$.
- **c)** Give a basis for the space $\text{Null}(A)$ in terms of the singular vectors. Also show that the solution of $Ax = b$, if it exists, is unique if $\text{Null}(A) = \{0\}$.

(4p) **2:** A Householder reflection can be written as

$$H = I - 2\frac{vv^T}{v^Tv}.$$

where v is a vector. Householder reflections are useful since, if $x \in \mathbb{R}^m$ is a vector and we chose $v = x \pm \|x\|_2 e_1$ then $Hx = \pm\|x\|_2 e_1$. Do the following:

- **a)** Show how the product of H and a vector $x \in \mathbb{R}^m$ can be computed using only $\mathcal{O}(m)$ arithmetic operations without forming H explicitly.
- **b)** Clearly show how many different Householder reflections that are needed to compute the QR decomposition of a matrix $A \in \mathbb{R}^{m\times n}$.
- **c)** Use the results from **a)** and **b)** to prove that computing the R matrix of the QR decomposition requires $\mathcal{O}(mn^2)$ arithmetic operations. Also show how many arithmetic operations that are needed to obtain also the Q matrix.

(4p) **3:** Let A be the following 4×4 matrix:

$$A = \begin{pmatrix} 10.3 & 1.2 & -0.7 & 0.4 \\ -0.7 & 4.5 & 1.2 & -0.1 \\ 0.2 & 0 & -7.6 & 0.3 \\ 0 & 0 & 1.4 & 5.8 \end{pmatrix}.$$

- **a)** Use the Gershgorin theorem to locate the eigenvalues of A as accurately as possible. Can you conclude that the matrix is non-singular? Also do the results allow you to conclude that all the eigenvalues of A are real?
- **b)** One of the eigenvalues of the matrix is $\lambda_1 = 10.1476$. Does the results from **a)** allow you to conclude that *power iteration* will converge to the eigenvector x_1 that correspond to the eigenvalue λ_1?

(4p) **4:** A *projection method* for solving a linear system $Ax = b$ is characterized by two m-dimensional subspaces $\mathcal{K}$ and $\mathcal{L}$. The method finds an approximate solution

$$x^{(m)} = x^{(0)} + x^*, \text{ where } x^* \in \mathcal{K},$$

such that

$$r^{(m)} = b - Ax^{(m)} \perp \mathcal{L}.$$

a) Introduce two sets of basis vectors $V = (v_1, \ldots, v_m)$ for $\mathcal{K}$ and $W = (w_1, \ldots, w_m)$ for $\mathcal{L}$. Derive a formula for computing the approximate solution $x^{(m)}$ in terms of A, b, V, W and $x^{(0)}$.

b) Show that if A is symmetric and positive definite then the choice $\mathcal{K} = \mathcal{L}$, i.e. $V = W$, means that the projection method is always well-defined. That is there is a unique solution $x^{(m)}$ that satisfies the above criteria.

(4p) **5:** A river has been polluted by a large spill of a chemical substance that breaks down slowly with time. A theoretical model suggests that the concentration of the pollutant in the river decays with time according to a model

$$F(t) = c_0 \exp(-c_1 t) + c_2 \sin(\omega t),$$

where the last term is due to smaller naturally occuring pollution by the same chemical compund. In order to estimate the parameters of the model we measure the concetration $F(t_i)$, for times $t_1 < t_2 < \ldots < t_m$, and want to use the Gauss-Newton method to fit the parameters c_0, c_1, and c_2 to the measured data.

a) The Gauss-Newton method is based on writing down an appropriate residual vector $r := r(c)$, where $c = (c_0\,,\, c_1\,,\, c_2)^T$ are the coefficients of the model. Give the residual vector for the above situation.

b) In each step of the Guass-Newton method we need to solve a least squares problem to minimize $\|J_r(c^{(k)})s_k + r(c^{(k)})\|_2$, where $c^{(k)}$ is the approximate coefficient vector at step k. Derive an expression for the Jacobian J_r for this case.

(5p) **6:** Consider the least squares problem $\min \|Ax - b\|_2$, where $A \in \mathbb{R}^{m\times n}$, $m > n$.

a) Show that any minimizer x of the above least squares problem satisfies the *normal equations* $(A^TA)x = A^Tb$. Also show that the least squares solution x is unique if $\text{rank}(A) = n$.

b) Suppose $A = QR$ is the reduced QR decomposition, i.e. $Q \in \mathbb{R}^{m\times n}$ and $R \in \mathbb{R}^{n\times n}$. Use the decomposition to give an orthogonal projection P such that $Pb = r$, where $r = b - Ax$ is the residual for the least squares solution x.

c) Suppose $Q^T[A, b] = R$, where $Q \in \mathbb{R}^{(m+1)\times(m+1)}$ is orthogonal. Clearly demonstrate how the minimizer of $\|Ax - b\|_2$ can be computed using only the R matrix.

1: Let $A = U\Sigma V^T$, where $U = (u_1, u_2, \ldots, u_m)$ and $V = (v_1, v_2, \ldots, v_n)$. Since $\text{Rank}(A) = k$ we can also write A as

$$A = \sum_{i=1}^{k} \sigma_i u_i v_i^T.$$

a) A basis for $\text{Range}(A) = \{u_1, \ldots, u_k\}$, so the dimension of the subspace is k. It follows that $\{u_{k+1}, \ldots, u_m\}$ is a basis for $\text{Range}(A)^\perp$ and the dimension is $m - k$.

b) Since we have an orthogonal basis for the subspace we can write the projection as

$$P = U_{m-k} U_{m-k}^T, \quad U_{m-k} = (u_{k+1}, \ldots, u_m).$$

Write b in the basis given by the U matrix. We get

$$b = \sum_{i=1}^{m} c_i u_i = \sum_{i=1}^{k} c_i u_i \in \text{Range}(A),$$

since $Pb = 0$ means that the coefficients c_i for $i = k+1, \ldots, m$ are zero. Also $b \in \text{Range}(A)$ means there is an x such that $Ax = b$, i.e. there is a solution to the linear system.

c) A basis for $\text{Null}(A)$ is $\{v_{k+1}, \ldots, v_n\}$, if $k < n$. In the case $k = n$ and $\text{Null}(A) = \{0\}$ we can assume there are two different solutions $x_1 \neq x_2$. In that case we have

$$A(x_1 - x_2) = Ax_1 - Ax_2 = b - b = 0,$$

so $0 \neq x_1 - x_2 \in \text{Null}(A)$ which would contradict $\text{Null}(A) = \{0\}$. So the assumption has to be wrong and two different solutions can't exist

2: **a)** The product is evaluated as

$$Hx = (I - 2\frac{vv^T}{v^T v})x = x - 2(v^T v)^{-1}(v^T x)v.$$

so we need one scalar product $v^T v$, one scalar product $v^T x$, one scalar-times-vector (αv) and one vector-minus-vector. All these are $\mathcal{O}(m)$ operations if m is the vector length. **b)** Each Householder reflection creates zeros below the diagonal on one column of A. Thus n Householder reflections are needed in total.

c) Let H_k be the Housedolder reflection needed to create zeros in the kth column of the matrix. The reflection H_k has to be applied to the columns $k, k+1, \ldots, n$ of A and also to all columns of the matrix used to build Q. Thus the number of matrix-vector products needed to obtain R is $n\frac{n}{2}$ and the number of matrix-vector products needed to obtain Q is nm. Each matrix-vector product needs $\mathcal{O}(m)$ arithmetic operations according to **b)** so the amount of work is $\mathcal{O}(n^2 m)$ to get R and $\mathcal{O}(nm^2)$ to get Q.

3: **a)** The Gershorin theorem applied to the rows gives the circles

$$|\lambda - 10.3| \leq 2.3, \quad |\lambda - 4.5| \leq 2.0, \quad |\lambda + 7.6| \leq 0.5, \quad |\lambda - 5.8| \leq 1.4.$$

The same theorem applied to the columns gives

$$|\lambda - 10.3| \leq 0.9, \quad |\lambda - 4.5| \leq 1.2, \quad |\lambda + 7.6| \leq 3.3, \quad |\lambda - 5.8| \leq 0.8.$$

Generally we get better bounds by looking at the columns in this case. We see that $\lambda = 0$ is not included in any of the discs so zero is not an eigenvalue. Thus the matrix is non-singular. Since the the circles for row/column 2 and 4 overlapp we could potentially have a complex conjugate pair of eigenvalues.

b) If the eigenvalue is $\lambda_1 = 10.1476$ then it belongs to the disc $|\lambda - 10.3| \leq 0.9$. That disc does not overlapp any of the others so there is no other positive eigenvalue of equal size. Also -10.1476 does not belong to any of the discs so the matrix has one eigenvalue such that $|\lambda_1| > |\lambda_k|$, $k = 1, 2, 3$. That one eigenvalue is strictly larger than the rest is the criteria for the power method to be convergent. So yes we can conclucde that the power method will converge to x_1.

4: **a)** First $x^{(m)} = x^{(0)} + x^*$, where $x^* \in \mathcal{K}$, means that $x^{(m)} = x^{(0)} + Vy$, $y \in \mathbb{R}^m$. Next calculate the residual

$$r^{(m)} = b - Ax^{(m)} = b - A(x^{(0)} + Vy) = r^{(0)} - AVy.$$

The criteria $r^{(m)} \perp \mathcal{L}$ means

$$0 = W^T r^{(m)} = W^T r^{(0)} - W^T AVy \Longrightarrow y = (W^T AV)^{-1} W^T r^{(0)}.$$

We obtain $x^{(m)} = x^{(0)} + V(W^T AV)^{-1} W^T r^{(0)}$. **b)** With $W = V$ int he formula from **a)** we need to prove that $(V^T AV)^{-1}$ exists. Take $x \in \mathbb{R}^m \neq 0$ then

$$x^T V^T AVx = (Vx)^T A(Vx) = y^T Ay > 0,$$

since A is positive definite and V having full column rank,since its columns are basis for $\mathcal{K}$, means $y \neq 0$ whenever $x \neq 0$. So $V^T AV$ is positive definite and therefore also non-singular.

(4p) **5:** **a)** The residual vector is $r \in \mathbb{R}^m$, where the kth component is

$$r_k(c) = (c_0 \exp(-c_1 t_k) + c_2 \sin(\omega t_k) - F_k^\delta),$$

where F_k^δ is the measured concentration at time t_k.

b) The Jacobian is computed by differentiating the residual vector with respect to the parameters. Thus the kth row is

$$(J_r(c))_{k,:} = (\partial_{c_0} r_k \,,\, \partial_{c_1} r_k \,,\, \partial_{c_2} r_k) = (\exp(-c_1 t_k) \,,\, -c_0 t_k \exp(-c_1 t_k) \,,\, \sin(\omega t_k)).$$

(5p) **6:** Let $A \in \mathbb{R}^{m\times n}$, where $m > n$.

a) If x minimize $\|Ax - b\|_2$ then $Ax \in \text{Range}(A)$ is orthogonal to the residual $r = b - Ax$. This means that $A^T(b - Ax) = 0$ or $(A^TA)x = A^Tb$. If $\text{Rank}(A) = n$ then $\text{Null}(A) = \{0\}$ so there can not be two different $x_1 \neq x_2$ such that $Ax_1 = Ax_2 = P_{\text{Range}(A)}b$. Hence the solution is unique. Another argument would be that (A^TA) is $n \times n$ and has full rank so $(A^TA)^{-1}$ exists and the normal equations has a unique solution.

b) Since A has rank n we have $A = QR$ where $R \in \mathbb{R}^{n\times n}$ is non-singular. Thus $b \in \text{Range}(A)$ if $b = QRy$, for some $y \in \mathbb{R}^n$, or $b = Qz = q_1z_1 + \ldots + q_nz_n$. Thus $(q_1, \ldots, q_n)$ is an orthogonal basis for $\text{Range}(A)$ and since Ax is the orthogonal projection of b onto $\text{Range}(A)$ the residual can be written $r = b - Ax = b - QQ^Tb = (I - QQ^T)b$. This means that $P = I - QQ^T$.

c) First we observe that

$$\|Ax - b\|_2^2 = \|(A, b)\begin{pmatrix} x \\ -1 \end{pmatrix}\|_2 = \|QR\begin{pmatrix} x \\ -1 \end{pmatrix}\|_2 = \|R\begin{pmatrix} x \\ -1 \end{pmatrix}\|_2 = \|\begin{pmatrix} R_{n+1} \\ 0 \end{pmatrix}\begin{pmatrix} x \\ -1 \end{pmatrix}\|_2,$$

where R_{n+1} is $(n+1) \times (n+1)$ and upper triangular. We remove the zero rows to obtain

$$\|\begin{pmatrix} R_{n+1} \\ 0 \end{pmatrix}\begin{pmatrix} x \\ -1 \end{pmatrix}\|_2^2 = \|\begin{pmatrix} \widetilde{R} & \gamma \\ 0 & \alpha \end{pmatrix}\begin{pmatrix} x \\ -1 \end{pmatrix}\|_2^2 = \|\widetilde{R}x - \gamma\|_2^2 + |\alpha|^2,$$

where the minimum $|\alpha|$ is obtained for $x = \widetilde{R}^{-1}\gamma$. Thus we do not need Q to find the least squares solution.