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# Exam TANA15 Numerical Linear Algebra, Y4, Mat4

**Datum:** 23:e Mars, 2022.

**Hjälpmedel:**

1. Föreläsningsanteckningar utskrivna från kurshemsidan utan egna anteckningar.
2. Räknedosa i fickformat, med nollställt minne och utan instruktionsbok.

**Examinator:** Fredrik Berntsson

**Maximalt antal poäng:** 25 poäng. För godkänt krävs 10 poäng.

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**Good luck!**

*(4p)* **1:** Let $A$ and $B$ be matrices. Do the following:

**a)** Prove that $(AB)^T = B^T A^T$.

**b)** Is it true that $\|A^{-1}\|_2 = \|A\|_2^{-1}$? Either prove or give a counter example.

**c)** Prove that $(A^{-1})^T = (A^T)^{-1}$. Thus the notation $A^{-T}$ makes sense.

*(4p)* **2:** Let,

$$r(x) = \begin{pmatrix} x_1 + x_2^2 - 2 \\ 2x_1 + x_1 x_2 - 2 \\ x_1 + \sqrt{x_1} x_2 - 3 \end{pmatrix}.$$

**a)** Compute the Jacobian matrix $J_r(x)$ of the residual function $r(x)$.

**b)** Perform one Gauss-Newton step for minimizing $\|r(x)\|_2^2$, using the starting value $x^{(0)} = (1, 0)^T$.

*(4p)* **3:** Let $A$ be an $m \times n$ matrix, where $m >> n$. Do the following

**a)** A Householder reflection can be written as

$$H = I - 2uu^T,$$

where $\|u\|_2 = 1$. Demonstrate how the product of $HA$ can be computed as efficiently as possible and estimate the amount of arithmetic work needed.

**b)** Use the result from **a)** to estimate the total amount of arithmetic work required for computing both the $R$ and the $Q$ matrices in the *reduced* $QR$ decomposition of the matrix $A$.

**Hint** Use $m >> n$ to simplify the expression for the amount of work required.

*(4p)* **4:** Let

$$A = \begin{pmatrix} 7.2 & 0.5 & -0.2 \\ -0.2 & 4.8 & -0.3 \\ -0.6 & 0.4 & -2.2 \end{pmatrix}.$$

Do the following

**a)** Use the Gershgorin theorem to show that the matrix $A$ is non-singular and has real eigenvalues. Clearly motivate your answer.

**b)** Will the power method converge if it is used to find one of the eigenvalues of the matrix $A$? Motivate your answer by making use the Gershgorin theorem.

**c)** Suppose we pick a shift $s = 6.1$ and use inverse iteration, i.e. apply power iteration to $(A - sI)^{-1}$. Can the Gershgorin theorem be used to prove that the inverse iteration with shift $s$ will converge? Again motivate your answer.

*(4p)* **5:** Let $A$ be an $n \times n$ matrix. If $A$ has a high condition number an approximate solution to $Ax = b$ can be found by selecting a parameter $\lambda > 0$ and solving the least squares problem

$$\min_{x \in \mathbb{R}^n} \|Ax - b\|_2^2 + \lambda^2 \|x\|_2^2.$$

Do the following:

**a)** Show that the normal equations of the above least squares problem are $(A^T A + \lambda^2 I)x = A^T b$.

**b)** Derive a formula for the singular values of the matrix $(A^T A + \lambda^2 I)$ and use the result to show that the normal equations are not ill-conditioned (provided $\lambda$ is selected appropriately).

**c)** Consider the case $n = 3$. If the $QR$ decomposition of $A$ is known we need to solve a least squares problem with a matrix that has the structure

$$\begin{pmatrix} x & x & x \\ 0 & x & x \\ 0 & 0 & x \\ x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{pmatrix}$$

for each $\lambda$. Show that the above matrix can be transformed into an upper triangular matrix using exactly 6 Givens rotations.

*(5p)* **6:** A general projection method is defined by solving: Find $x^{(m)} \in x^{(0)} + \mathcal{K}_m$ such that $r^{(m)}$ is orthogonal to $\mathcal{L}_m$, where $\mathcal{K}_m$ and $\mathcal{L}_m$ are two $m$ dimensional subspaces.

**a)** Introduce basis sets for the two subspaces and derive an explicit formula for the approximate solution $x^{(m)}$.

**b)** Consider the case when $A$ is non-singular and we make the choice $\mathcal{L}_m = A\mathcal{K}_m$. Show that the general projection method is always well-defined for this particular case.

**c)** Show that the general projection method, with the choice $\mathcal{L}_m = A\mathcal{K}_m$, finds the approximate solution that solves the least squares problem

$$\min_{x \in x^{(0)} + \mathcal{K}_m} \|b - Ax\|_2.$$

Thus the residual is minimized.

**1:** For **a)** we look at an element of $(AB)^T$. We have

$$(AB)^T_{ij} = (AB)_{ji} = \sum_{k=1}^{n} A_{jk}B_{ki} = \sum_{k=1}^{n} A^T_{kj}B^T_{ik} = \sum_{k=1}^{n} B^T_{ik}A^T_{kj} = (B^TA^T)_{ij}.$$

For **b)** a counter example is given by the diagonal matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

For **c)** we demonstrate that $(A^T)^{-1} = (A^{-1})^T$ by

$$A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I.$$

**2:** For **a)** we recall that $(J_r)_{ij}(x) = (\partial_{x_j} r_i(x))$. Thus

$$J_r(x) = \begin{pmatrix} 1 & 2x_2 \\ 2 + x_2 & x_1 \\ 1 + \frac{1}{2}{x_1}^{-1/2}x_2 & \sqrt{x_1} \end{pmatrix},$$

where $x = (x_1, x_2)^T$. For **b)** we evaluate $r(x^{(0)}) = r((1,0)^T) = (-1, 0, -2)^T$, and

$$J_r((1,0)^T) = \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

In the Gauss-Newton step we first solve the least squares problem $J_r^T J_r s^{(0)} = -J_r^T r$, or

$$\begin{pmatrix} 6 & 3 \\ 3 & 2 \end{pmatrix} s^{(0)} = \begin{pmatrix} 3 \\ 2 \end{pmatrix},$$

which gives $s^{(0} = (0, 1)$. Thus $x^{(1)} = x^{(0)} + s^{(0)} = (1, 1)^T$.

**3:** For **a)** we need to compute

$$HA = (I - 2uu^T)A = A - 2u(u^TA).$$

First $y = u^TA$ is a matrix-vector multiply that requires $2mn$ floating point operations. Second we have to compute the outer product $B = uy^T$. This again requires $mn$ multiplications (we ignore the 2 as that could be included in the $y$ matrix using $n$ operations). Finally $A - B$ is computed using $mn$ subtractions. The total is this $4mn$ floating point operations.

For **b)** we just need to recall that in step $k$ of the Householder algorithm we need to apply a reflection $H_k$ to the block $A(k:m, k:n)$, of size $(m-k+1)\times(n-k+1)$, and will get $R$ after $n$ steps. This means that the total amount of work is

$$\sum_{k=1}^{n} 4(m-k+1)(n-k+1) \approx 4m\sum_{k=1}^{n}(n-k+1) \approx 4mn(n/2).$$

where we used the asumption $m >> n$ to obtain $m-k+1 \approx m$. Otherwise we need to look up the sum in a table.

Similarily, to get the first $n$ columns of the full $Q$ we need to start with the corresponding columns of the identity matrix $I$, i.e. with the $I_n = I(:,1:n)$ of size $m \times n$, and apply $H_k$ to the block $Q(k:m, 1:n)$, which is of size $(m-k+1)\times n$. The work is thus

$$\sum_{k=1}^{n} 4(m-k+1)n \approx 4mn^2,$$

where again $m >> n$ was used. To conclude $2mn^2$ operations needed for $R$ and $4mn^2$ needed for the reduced $Q$ matrix.

**4:** For **a)** we compute the Gershgorin circles as

$$|\lambda - 7.2| \leq 0.7,\ \ |\lambda - 4.8| \leq 0.5 \text{ and } |\lambda - (-2.2)| \leq 1.0.$$

First 0 is not inside any of the discs. Thus $\lambda = 0$ is not an eigenvalue of $A$. This means $A$ is non-singular. Next since the discs don't overlap there is exactly one eigenvalue in each disc. Any complex eigenvalues has to occur in complex conjugate pairs since the matrix has real elements. Thus there can only be real eigenvalues.

For **b)** the power method converges if one of the eigenvalues is strictly larger than the others in magnitude. The eigenvalue from the first disc $\{|\lambda - 7.2| \leq 0.7\}$ can be as small as $\lambda_1 = 7.2 - 0.7 = 6.5$. This is larger than the possible eigenvalues from the second disc. From the third disc the eigenvalues can be as small as $\lambda_3 = -2.2 - 1.0 = -3.3$ which is also strictly smaller in magnitude. Thus the power method will converge.

For **c)** $s = 6.1$ is located in between the eigenvalues $\lambda_1$ and $\lambda_2$. It is not possible to exclude the case when $s = 6.1$ is exactly in between $\lambda_1$ and $\lambda_2$. This would mean $B = (A - sI)^{-1}$ has two eigenvalues of equal magnitude and the power method does not nessecarily converge.

**5:** The normal equations can be derived by the identity

$$\min_x \|Ax - b\|_2^2 + \lambda^2\|x\|_2^2 = \min_x \left\| \begin{pmatrix} Ax - b \\ \lambda x \end{pmatrix} \right\|_2 = \min_x \left\| \begin{pmatrix} A \\ \lambda I \end{pmatrix} x - \begin{pmatrix} b \\ 0 \end{pmatrix} \right\|_2.$$

The last is a regular least squares problem with an extended matrix. The normal equations are

$$\begin{pmatrix} A^T & \lambda I \end{pmatrix} \begin{pmatrix} A \\ \lambda I \end{pmatrix} x = \begin{pmatrix} A^T & I \end{pmatrix} \begin{pmatrix} b \\ 0 \end{pmatrix} \text{ or } (A^TA + \lambda^2 I)x = A^Tb.$$

In **b)** we derive the solution formula using the decomposition $A = U\Sigma V^T$. Since $A^TA + \lambda I = V\Sigma^T\Sigma V^T + \lambda^2 VV^T = V(\Sigma^T\Sigma + \lambda^2 I)V^T$ and $A^Tb = V\Sigma U^T b$ we obtain the solution

$$x_\lambda = V(\Sigma^T\Sigma + \lambda I)^{-1}\Sigma U^T b = \sum_{i=1}^{n} \frac{\sigma_i}{\sigma_i^2 + \lambda^2}(u_i^T b)v_i.$$

Too see that the normal equations are not ill-conditioned we look at $A^TA + \lambda^2 I$ which has singular values $\sigma_i^2 + \lambda^2 \geq \lambda^2$. So the addition of the regularization parameter removes the small singular values and makes the condition number smaller.

In **c** we use rotations in the order

$$R_{25}R_{36}\begin{pmatrix} x & x & x \\ 0 & x & x \\ 0 & 0 & x \\ x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{pmatrix} = R_{25}\begin{pmatrix} x & x & x \\ 0 & x & x \\ 0 & 0 & + \\ x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} x & x & x \\ 0 & + & + \\ 0 & 0 & x \\ x & 0 & 0 \\ 0 & 0 & + \\ 0 & 0 & 0 \end{pmatrix}.$$

Now we need to use the element $(3, 3)$ to eliminate the one at $(5, 3)$ before we continue. We get

$$R_{14}R_{35}\begin{pmatrix} x & x & x \\ 0 & x & x \\ 0 & 0 & x \\ x & 0 & 0 \\ 0 & 0 & x \\ 0 & 0 & 0 \end{pmatrix} = R_{14}\begin{pmatrix} x & x & x \\ 0 & x & x \\ 0 & 0 & + \\ x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} + & + & + \\ 0 & x & x \\ 0 & 0 & x \\ 0 & + & + \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Now we have two unwanted elements $(4, 2)$ and $(4, 3)$ which can be removed by two rotations $R_{24}$ and $R_{34}$. Thats a total of 6 rotations.

**6:** For **a)** Let $V = (v_1, v_2, \ldots, v_m) \in \mathbb{R}^{n\times m}$ be a basis for $\mathcal{K}_m$. Then then projection step produces an $x^{(m)} = x^{(0)} + Vy$, $y \in \mathbb{R}^m$. The orthogonality condition $r^{(m)} \perp \mathcal{L}_m$ can be rewritten using a basis $W = (w_1, w_2, \ldots, w_m) \in \mathbb{R}^{n\times m}$ for $\mathcal{L}_m$. We obtain

$$0 = W^T r^{(m)} = W^T(b - Ax^{(m)}) = W^T(b - A(x^{(0)} + Vy)) = W^T(r^{(0)} - AVy) = W^T r^{(0)} - W^T AVy.$$

So $y = (W^TAV)^{-1}W^T r^{(0)}$ and $x^{(m)} = x^{(0)} + V(W^TAV)^{-1}W^T r^{(0)}$.

For **b)** we note that the formula obtained in **a)** is well-defined if $W^TAV$ is non singular, since for that case $(W^TAV)^{-1}$ and the formula gives a unique $x^{(m)}$. If $\mathcal{L}_m = A\mathcal{K}_m$ then $W = AV$ is a basis for $\mathcal{L}_m$ and we need to prove that $V^TA^TAV$ is non-singular. Let $x \neq 0$ be a vector. Since $V$ has orthogonal columns we have $y = Vx \neq 0$ if $x \neq 0$ and

$$x^TV^TA^TAVx = (AVx)^T(AVx) = (Ay)^T(Ay) = \|Ay\|_2^2 > 0,$$

since $Ay \neq 0$ is $y \neq 0$ due to non-singular. Thus $V^TA^TAV$ is positive definite and hence non-singular.

For **c)** we again let $V = (v_1, \ldots, v_m)$ be a basis for $\mathcal{K}_m$. Then $AV$ isa basis for $\mathcal{L}_m$. The solution can be written in the form $x^{(m)} = x^{(0)} + Vy$, where $y \in \mathbb{R}^m$ are the unknown coordinates. We get that the residual $r^{(m)} = r^{(0)} - AVy$ is orthogonal to $\mathcal{L}_m$ if $0 = (AV)^T(r^{(0)} - AVy)$ or $(AV)^T(AV)y = (AV)^T r^{(0)}$. This is the normal equations for

$$\min_{y \in \mathbb{R}^m} \|AVy - r^{(0)}\|_2 = \min_{y \in \mathbb{R}^m} \|A(x^{(0)} + Vy) - b\|_2 = \min_{x = x^{(0)} + Vy} \|Ax - b\|_2,$$

which is the correct minimization problem.