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Beräkningsmatematik/Fredrik Berntsson

Exam TANA15 Numerical Linear Algebra, Y4, Mat4

Datum: 22:e Mars, 2023.

Hjälpmedel:

1. Föreläsningsanteckningar utskrivna från kurshemsidan utan egna anteckningar.
2. Räknedosa i fickformat, med nollställt minne och utan instruktionsbok.

Examinator: Fredrik Berntsson

Maximalt antal poäng: 25 poäng. För godkänt krävs 10 poäng.

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Good luck!

(4p) **1:** Do the following

a) What does it mean that a matrix Q is orthogonal? Give the precise definition. Also show that $\|Qx\|_2 = \|x\|_2$, for all vectors x, if Q is orthogonal.

b) State the definition of the matrix norm which is *induced* from the vector norm $\|\cdot\|_2$. Also show that $\|Q\|_2 = 1$ for this particular norm if Q is orthogonal.

c) Let $A = Q_1R$ be the *reduced* QR decomposition of a full rank matrix of dimension $m \times n$, where $m > n$. Show that $P = I - Q_1Q_1^T$ is an orthogonal projection such that $Pb = r$, where $r = b - Ax$ is the residual and x is the solution to the least squares problem $\min \|b - Ax\|_2$.

(4p) **2:** Let,

$$f(x) = \begin{pmatrix} 2x_1 + x_2 + (1+x_2)^2 - 1 \\ (3x_1+1)x_2 - 1 \end{pmatrix}.$$

a) Compute the Jacobian J_f and formulate the Newton method for finding a root of the equation $f(x) = 0$.

b) Let $x^{(0)} = (0\,,\,0)^T$ and perform one step of the Newton metod and compute the next iterate $x^{(1)}$.

(4p) **3:** Let

$$A = \begin{pmatrix} 4.3 & 0.7 & -0.3 \\ -1.2 & 7.8 & -0.2 \\ -0.7 & 0.4 & -4.2 \end{pmatrix}.$$

Do the following

a) Use the Gershgorin theorem to find as good approximations of the eigenvalues as possible.

b) Determine if the matrix A is non-singular. Also is the Gerschgorin theorem sufficient to prove that the eigenvalues are real?

c) Let v_1 be one of the eigenvectors of A and let $B = A + sv_1v_1^T$. Can you prove that v_1 is also an eigenvector of B? Also let v_2 be another eigenvector of A. Is v_2 also an eigenvector of B? Motivate your answer.

(4p) **4:** Any matrix $A \in \mathbb{R}^{m\times n}$, $m > n$, has a *singular value decomposition* $A = U\Sigma V^T$. Do the following:

a) Consider a linear system $Ax = b$, $m > n$, where $\text{rank}(A) = k < n$. Use the SVD to a basis for the both the range Range(A) and its orthogonal complement. Also give a criteria that guarantees that a solution to the linear system exists. Your criteria should be expressen in terms of the basis vectors and the vector b. Also the criteria should be efficient to check for the case when $k \approx n \approx m$.

b) Consider the linear system $A^T x = b$, where as before $\text{rank}(A) = k < n$. Provide a criteria for existance of a solution to the linear system expressed in terms of b and the singular vectors. Also write down the formula for the solution x. Is the solution unique? Motivate clearly.

(4p) **5:** Do the following:

a) Clearly demonstrate how the Hessenberg decomposition $H = QAQ^T$ can be computed using Householder reflections. You have to specify which elements of the matrix are used to create each reflection. It is enough to consider the 4×4 case.

b) Let $H = QAQ^T$ be a Hessenberg decomposition. Show that A and H have the same eigenvalues.

(5p) **6:** Any matrix $A \in \mathbb{R}^{n\times n}$ can be factorized as $A = QTQ^H$, where Q is unitary and T upper triangular. This is called the *Schur decomposition* and is mainly of theoretical importance. Do the following:

a) Let (x, λ) be an eigenpair of A. The first step in the existence proof for the Shur decomposition consists of finding an orthogonal matrix V_1 such that

$$V_1^T A V_1 = \begin{pmatrix} \lambda & w^T \\ 0 & B \end{pmatrix}.$$

Clearly explain how to construct such a matrix V_1 and show that it the product $V_1^T A V_1$ has the desired structure.

b) Use the Schur decomposition to prove that any real symmetric matrix A has orthogonal eigenvectors.

c) A matrix B is called non-defective if it has a full set of eigenvectors, i.e. the decomposition $B = XDX^{-1}$ exists. Use the Shur decomposition to prove that if A is defective then for any $\varepsilon > 0$ there is a non-defective matrix B such that $\|A - B\|_2 \leq \varepsilon$.

Remark From **c)** we conclude that if a matrix is supposed to be defective and we compute a numerical approximation it is likely that the matrix turns out to be non-defective due to round-off errors.

Lösningsförslag till tentan 22:a Mars 2023.

1: For **a)** we state that a matrix Q is orthogonal if it is quadratic, i.e. the dimension is $n \times n$, and if $Q^T Q = I$, where I is the identity matrix. The second part follows from $VertQx\|_2^2 = (Qx)^T(Qx) = x^T Q^T Q x = x^T x = \|x\|_2^2$.

For **b)** we state that the induced matrix norm is given by

$$\|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|},$$

where $\|\cdot\|$, in the righthandside, is any vector norm. This means

$$\|Q\|_2 = \max_{x \neq 0} \frac{\|Qx\|_2}{\|x\|_2} = \max_{x \neq 0} \frac{\|x\|_2}{\|x\|_2} = 1.$$

For **c)** we note that the solution of the least squares problem $\min \|Ax - b\|_2$ is given by $x = R^{-1} Q_1^T b$. This means that

$$r = b - Ax = b - (Q_1 R)(R^{-1} Q_1^T b) = b - Q_1 Q_1^T b = (I - Q_1 Q_1^T) b = Pb.$$

2: For **a)** we recall that $(J_f)_{ij}(x) = (\partial_{x_j} f_i(x))$. Thus

$$J_f(x) = \begin{pmatrix} 2 & 1 + 2(1 + x_2) \\ 3x_2 & 3x_1 + 1 \end{pmatrix},$$

where $x = (x_1, x_2)^T$. For **b)** we evaluate $f(x^{(0)}) = f((0,0)^T) = (0,-1)^T$, and

$$J_f((0,0)^T) = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}.$$

In the Newton step we first solve the linear system $J_f s^{(0)} = -f(x^{(0)}$, or

$$\begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix} s^{(0)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

which gives $s^{(0} = (-1.5, 1)$. Thus $x^{(1)} = x^{(0)} + s^{(0)} = (-1.5, 1)^T$.

3: For **a)** we need to find the Gershgorin discs

$$|\lambda_1 - 4.3| \leq 1, \quad |\lambda_2 - 7.8| \leq 1.4 \text{ and } |\lambda_3 + 4.2| \leq 1.1.$$

For **b)** we note that the discs are disjoint which means all eigenvalues are real since if an eigenvalue λ were complex also its complex conjugate $\bar{\lambda}$ would be an eigenvalue. This holds for real matrices. This is not consistent with one eigenvalue in each disc. Also 0 is not in any of the discs so is not an eigenvalue. Thus the matrix is non-singular. For **c)** we have

$$Bv_1 = (A + s v_1 v_1^T) v_1 = A v_1 + s(v_1^T v_1) v_1 = \lambda_1 v_1 + s v_1 = (\lambda_1 + s) v_1.$$

If we try another eigenvecvtor v_2 this does not work since

$$Bv_2 = (A + sv_1v_1^T)v_2 = \lambda_2 v_2 + s(v_1^T v_2)v_1 = \alpha v_1,$$

only for the case $v_1^T v_2 = 0$. However since A is not symmetric there is no guarantee that the eigenvectors are orthogonal.

4: For **a)** we remark that we can write A in the form

$$A = \sum_{i=1}^{k} \sigma_i u_i v_i^T.$$

Here we clearly see that $\mathrm{Range}(A) = \mathrm{span}(u_1, \ldots, u_k)$. The orthogonal complement is $\mathrm{Range}(A)^\perp = \mathrm{span}(u_{k+1}, \ldots, u_m)$. Existance of solution means that $b \in \mathrm{Range}(A)$ which means b doesn't have a component in $\mathrm{Range}(A)^\perp$. For large k the easiest way to check this is $u_i^T b = 0$, for $i = k+1, \ldots, m$.

For **b)** we simply apply the transpose to the above formula for A to obtain

$$A^T = \sum_{i=1}^{k} \sigma_i v_i u_i^T.$$

This means that now we have $\mathrm{Range}(A^T) = \mathrm{span}(v_1, \ldots, v_k)$. A criteria for existance is thus $v_i^T b = 0$, for $i = k+1, \ldots, n$. If this criteria is satisfied we can write

$$b = \sum_{i=1}^{k} (v_i^T b) v_i = A^T x = \sum_{i=1}^{k} \sigma_i (u_i^T x) v_i.$$

Identifying coefficients gives us $v_i^T b = \sigma_i (u_i^T x)$, for $i = 1, \ldots, k$. We can express x in the basis $\{u_1, \ldots, u_m\}$ so

$$x = \sum_{i=1}^{m} (u_i^T x) u_i = \sum_{i=1}^{k} \frac{v_i^T b}{\sigma_i} u_i + \sum_{i=k+1}^{m} c_i u_i$$

where c_i are free parameters. The solution is not unique.

5: For **a)** we illustrate the algorithm as follows: First we use the same reflection H_1 applied from the left and from the right. The reflection is selected so the elements $A(3:4, 1)$ are set to zero. We get

$$H_1 \begin{pmatrix} x & x & x & x \\ x & x & x & x \\ x & x & x & x \\ x & x & x & x \end{pmatrix} H_1^T = \begin{pmatrix} x & x & x & x \\ + & + & + & + \\ 0 & + & + & + \\ 0 & + & + & + \end{pmatrix} H_1^T = \begin{pmatrix} x & + & + & + \\ x & + & + & + \\ 0 & + & + & + \\ 0 & + & + & + \end{pmatrix}.$$

Second we find a reflection H_2 that zeroes out the element $A(4, 2)$. We get

$$H_2 \begin{pmatrix} x & x & x & x \\ x & x & x & x \\ 0 & x & x & x \\ 0 & x & x & x \end{pmatrix} H_2^T = \begin{pmatrix} x & x & x & x \\ x & x & x & x \\ 0 & + & + & + \\ 0 & 0 & + & + \end{pmatrix} H_2^T = \begin{pmatrix} x & x & + & + \\ x & x & + & + \\ 0 & x & + & + \\ 0 & 0 & + & + \end{pmatrix},$$

which is Hessenberg. For **b)** we assume that (x, λ) is an eigen pair of A and $A = QHQ^T$. Then $QHQ^Tx = \lambda x$ or $H(Q^Tx) = \lambda(Q^Tx)$. Thus λ is also an eigen value of H. The new eigen vector is $y = Q^Tx$. Similarily an eigenvalue of H is also an eigenvalue of A.

6: For **a)** We have the eigenpair (λ, x). If we compute the full QR decomposition of $x \in \mathbb{R}^{n\times 1}$ we obtain an orthogonal matrix suxch that $Q = (x, Q_2)$, where $Q_2^H x = 0$. This is assuming that $\|x_1\|_2 = 1$. We find that

$$Q^H AQ = (x, Q_2)^T A(x, Q_2) = (x, Q_2)^H (Ax, AQ_2) = (x, Q_2)^H (\lambda x, AQ_2) =$$

$$\left(\begin{array}{cc} \lambda x^H x & x^H AQ_2 \\ \lambda Q_2^H x & Q_2^H AQ_2 \end{array} \right) = \left(\begin{array}{cc} \lambda & w^H \\ 0 & B \end{array} \right),$$

where we have the correct structure. For **b)** we simply note that $A^H = (QTQ^H)^H = QT^HQ^H$. For symmetric matrices, i.e. A real and $A^T = A$, we thus get $A^T = A^H = QT^HQ^H = A = QTQ^H$. Thus $T^H = T$ which means that T is a diagonal since we already knew that T is upper triangular. Also the diagonal elements satisfy $(T)_{ii} = (\bar{T})_{ii}$ which means the elements on the diagonal are real. Since the diagonal elements of T are also the eigenvalues of A this shows that the eigenvalues are real.

For **c)** we assume that A is defective and compute its Shur decomposition $A = QTQ^H$. For A to be defective it has to have at least one eigenvalue λ_1 with an algebraic multiplicity $\gamma_1(\lambda_1)$ strictly larger than the geometric multiplicity $\gamma_2(\lambda_1)$. Thus, if all diagonal elements of T were different then the matrix A would be non-defective. Thus we pick a diagonal matrix $D = \text{diag}(\epsilon_1, \ldots, \epsilon_n)$ so that $T + D$ has unique diagonal elements. Then $B = Q(T + D)Q^H$ is non-defective and $\|A - B\|_2 = \|D\|_2 \leq \max |\epsilon_i| = \epsilon$.