TEKNISKA HÖGSKOLAN I LINKÖPING
Matematiska institutionen
Beräkningsmatematik/Fredrik Berntsson

Exam TANA15 Numerical Linear Algebra, Y4, Mat4

Datum: Klockan 8-12, 11:e Juni, 2015.

Hjälpmedel:

1. Föreläsningsanteckningar utskrivna från kurshemsidan utan egna anteckningar.
2. Räknedosa i fickformat, med nollställt minne och utan instruktionsbok.

Examinator: Fredrik Berntsson

Maximalt antal poäng: 25 poäng. För godkänt krävs 8 poäng.

Jourhavandelärare Fredrik Berntsson - (telefon 013 282860)

Besök av jourhavande lärare sker ungefär 10.15.

Resultat meddelas via epost senast 17:e Juni. Lösningsförslag finns på kurshemsidan efter tentans slut.

Visning

Good luck!

(4p) **1:** Let $v \neq 0$. The Householder reflection can be written as,

$$H = I - 2\frac{vv^T}{v^Tv}.$$

Prove the following properties of Householder reflections

a) H is symmetric and orthogonal.

b) If we form the matrix H explicitly and then perform a matrix vector multiply Hx then $\mathcal{O}(n^2)$ floating point operations are required. Also n^2 slots of memory are needed to store A. Show how the product Hx can be evaluated more efficiently and estimate the number of floating point operations and memory needed.

c) The eigenvalues of H are $\lambda_1 = -1$ and $\lambda_2 = \ldots = \lambda_n = 1$.

(3p) **2:** Consider the matrix

$$A = \begin{pmatrix} 12.3 & -0.7 & 1.2 \\ 0.3 & -4.2 & -0.15 \\ -0.1 & 0.2 & 5.2 \end{pmatrix}$$

a) Use Gershgorin's theorem to estimate the eigenvalues as accurately as possible.

b) Suppose that we want to use *Power iteration* to compute an eigen value and an eigenvector to the above matrix. Using the result from **a)** can you guarantee that the power iteration will converge? Explain your conclusion.

(5p) **3:** The singular value decomposition of the matrix is $A = U\Sigma V^T$, where U and V are orthogonal and Σ is diagonal.

a) Suppose $A \in \mathbb{R}^{m\times n}$, $m > n$, and A has rank $k < n$. Use the SVD to give a basis for the spaces range$(A)^\perp$ and null(A). Also what are the dimensions of these two subspaces?

b) Show how the SVD can be used for solving,

$$\min_x \|Ax\|_2, \quad \text{subject to } \|x\|_2 = 1.$$

Also give a criteria that guarantees that the minimum is positive.

c) Bidiagonalization is an important part of the algorithm for computing the SVD. Clearly demonstrate how a 5×4 matrix A can be reduced to upper bidiagonal form using Householder reflections. That is $U^TAV = B$, where U and V are products of Householder reflections and B is upper bidiagonal.

(3p) **4:** Prove the following

a) Let A be a symmetric and positive definite matrix. Show that all its eigen values are positive.

b) Show that $\|PAQ\|_2 = \|A\|_2$, when P and Q are orthogonal matrices.

c) Suppose (x, λ) is an eigen pair of a matrix A. Put $B = Q^T AQ$, with Q orthogonal. Derive the corresponding eigen pair of B.

(4p) **5:** Consider the over determined system of equations

$$
\begin{aligned}
(x+1)^2 + y^2 &= 0.25, \\
x^2 + (y-1)^2 &= 0.25, \\
(x-1)^2 + y^2 &= 0.25.
\end{aligned}
$$

a) Describe how the Gauss-Newton Method can be used for solving the above problem. What is the residual vector $r(x)$ and the Jacobian $J_r(x)$?

b) Perform one Gauss-Newton step using the starting vector $x^{(0)} = (0, 0)^T$.

(6p) **6:** Consider the least squares problem $\min \|Ax - b\|_2$, where $A \in \mathbb{R}^{m \times n}$, $m > n$.

a) Show that any minimizer x of the above least squares problem satisfies the *normal equations* $(A^T A)x = A^T b$. Also show that the least squares solution x is unique if $\text{rank}(A) = n$.

b) Suppose $A = QR$ is the reduced QR decomposition, i.e. $Q \in \mathbb{R}^{m \times n}$ and $R \in \mathbb{R}^{n \times n}$. Use the QR decomposition to give an orthogonal projection P such that $Pb = r$, where $r = b - Ax$ is the residual for the least squares solution x.

c) Suppose $Q^T[A, b] = R$, where $Q \in \mathbb{R}^{(m+1) \times (m+1)}$ is orthogonal. Clearly demonstrate how the minimizer of $\|Ax - b\|_2$ can be computed using only the R matrix.