TEKNISKA HÖGSKOLAN I LINKÖPING
Matematiska institutionen
Beräkningsmatematik/Fredrik Berntsson

Exam TANA15 Numerical Linear Algebra, Y4, Mat4

Datum: Klockan 14-18, 21:e Mars, 2015.

Hjälpmedel:

1. Föreläsningsanteckningar utskrivna från kurshemsidan utan egna anteckningar.
2. Räknedosa i fickformat, med nollställt minne och utan instruktionsbok.

Examinator: Fredrik Berntsson

Maximalt antal poäng: 25 poäng. För godkänt krävs 8 poäng.

Jourhavandelärare Fredrik Berntsson - (telefon 013 282860)

Besök av jourhavande lärare sker ungefär 15.15.

Resultat meddelas via epost senast 1:a April. Lösningsförslag finns på kurshemsidan efter tentans slut.

Visning av tentamen sker på Examinators kontor Torsdag den 2:a April, klockan 12.15-13.00 (Hus B, Ing. 23, Plan-2, A-korr).

Good luck!

(3p) **1:** Consider the matrix

$$A = \begin{pmatrix} 11 & -1.3 & -0.5 \\ -1.3 & 9 & 0.3 \\ -0.5 & 0.3 & 5 \end{pmatrix}$$

a) Use Gershgorin's theorem to estimate the eigenvalues as accurately as possible. Can you conclude that the matrix is non-singular?

b) Let s be a scalar and v_1 be an eigenvector. Show that the matrix $B = A + sv_1v_1^T$ has the same eigenvectors as A. What are the eigenvalues of B?

(4p) **2:** Let,

$$f(x) = \begin{pmatrix} x_1 + x_2^2 - 2 \\ 2x_1 + x_1x_2 - 2 \end{pmatrix}.$$

a) Compute the Jacobian matrix $J_f(x)$ of the function $f(x)$.

b) Perform one Newton step for solving the non–linear equation $f(x) = 0$ using the starting value $x^{(0)} = (1, 0)^T$.

(5p) **3:** Two important parts of the QR algorithm for computing eigenvalues is the initial Hessenberg reduction and the deflation step.

a) The Hessenberg decomposition can be written as $A = VHV^T$ where H is has Hessenberg structure and V is orthogonal. Clearly give state what Hessenberg structure means and also prove that A and H have the same eigenvalues.

b) Suppose that after k steps in the QR algorithm the matrix A_k has the following structure

$$A_k = \begin{pmatrix} A_{11} & A_{21} \\ 0 & A_{22} \end{pmatrix},$$

where both A_{11} and A_{22} are quadratic matrices. Show that if λ is an eigenvalue of either A_{11} or A_{22} then λ is also an eigenvalue of A_k. Thus we can reduce the dimension of the problem and compute the eigenvalues of the two smaller problems A_{11} and A_{22}.

(3p) **4:** Suppose we want to QR decomposition the *structured matrix*

$$A = \begin{pmatrix} x & x & x & x \\ 0 & x & x & x \\ 0 & 0 & x & x \\ x & x & x & x \end{pmatrix},$$

where the x denotes a non-zero element. The matrix is already "almost" triangular. Clearly demonstrate how a sequence of Givens rotations can be used to compute the R part of the QR factorization. Present the sparsity pattern after each step. How many Givens rotations are needed?

(5p) **5:** Suppose $A \in \mathbb{R}^{m \times n}$, $m > n$, $\text{rank}(A) = k < n$, and that we have the decomposition $A = U\Sigma V^T$.

a) Use the singular value decomposition to write down an orthogonal basis for the subspaces $\text{Range}(A)$ and $\text{Range}(A)^\perp$. What are the dimensions of the two subspaces?

b) Suppose $b \in \text{Range}(A)$ so the solution to the linear system $Ax = b$ exists. Demonstrate that the formula

$$x = \sum_{i=1}^{k} \frac{u_i^T b}{\sigma_i} v_i,$$

gives an x that satisfies the linear system $Ax = b$. Is the solution unique? If not then use the SVD to construct at least one additional solution.

c) Suppose that instead we are intrested in solving $A^T x = b$, where as previously $A \in \mathbb{R}^{m \times n}$, $m > n$, $\text{rank}(A) = k < n$, and the singular value decomposition of A is known. Give a basis for $\text{Range}(A^T)$ and give a criteria that guarantees the existance of a solution to $A^T x = b$ and also derive a formula that gives a solution.

(5p) **6:** We want to solve $Ax = b$, where A is a large sparse and non-singular matrix, and have computed an orthogonal basis V_k for the Krylov subspace $\mathcal{K}_k(A, r^{(0)})$ of dimension k, where $r^{(0)} = b - Ax^{(0)}$, and $x^{(0)}$ is the inital guess.

The following projection method is used: Find $x^{(k)} = x^{(0)} + V_k y$, $y \in \mathbb{R}^k$, such that the resuidual is orthogonal to the space $\text{span} AV_k$.

a) Derive a formula for the solution $x^{(k)}$ in terms of the initial guess $x^{(0)}$ and the matrices A and V_k.

b) For the projection step to be well-defined it is essential that the matrix $V_k^T(A^T A)V_k$ is non-singular. Prove that this is indeed the case here.

c) Let the starting residual be $r^{(0)}$ and normalize so $v_1 = r^{(0)}/\|r^{(0)}\|_2$ is the first basis vector in the Krylov subspace. Suppose we compute $w_2 = Av_1$. Clearly demonstrate how to use Gram-Schmidt ortogonalization to obtain an orthonormal basis $\{v_1, v_2\}$ for the Krylov subspace of dimension 2.