TEKNISKA HÖGSKOLAN I LINKÖPING
Matematiska institutionen
Beräkningsmatematik/Fredrik Berntsson

Exam TANA15 Numerical Linear Algebra, Y4, Mat4

Datum: Klockan 14-18, 23:e Mars, 2016.

Hjälpmedel:

1. Föreläsningsanteckningar utskrivna från kurshemsidan utan egna anteckningar.
2. Räknedosa i fickformat, med nollställt minne och utan instruktionsbok.

Examinator: Fredrik Berntsson

Maximalt antal poäng: 25 poäng. För godkänt krävs 8 poäng.

Jourhavandelärare Fredrik Berntsson - (telefon 013 282860)

Besök av jourhavande lärare sker ungefär 15.15 och 17.45.

Resultat meddelas via epost senast 31:e Mars.

Visning av tentamen sker på Examinators kontor Fredag den 1:a April, klockan 12.15-13.00 (Hus B, Ing. 25-27, Plan-3, A-korr).

Good luck!

(4p) **1:** Prove the following:

a) A matrix A is positive definite if $x^T A x > 0$, for all $x \neq 0$. Show that any symmetric and positive definite matrix is non-singular and also show that A^{-1} is positive definite.

b) Is it true that $\|A^{-1}\|_2 = \|A\|_2^{-1}$? Either prove or give a counter example.

c) Prove that $(A^{-1})^T = (A^T)^{-1}$. Thus the notation A^{-T} makes sense.

(4p) **2:** Suppose we implement matrix-vector multiplication by a loop:

```
y=zeros(n,1);
for i=1:n
   for j=1:n
      y(i)=y(i)+A(i,j)*x(j);
   end;
end;
```

on a machine where matrices are stored by column in main memory.

a) Suppose one memory block corresponds exactly to the size of one column `A(:,i)` or the vectors x and y. Further assume that only a couple of memory blocks fit in Cache memory. Clearly explain why the above code is inefficient. Also check the ratio between the number of memory blocks loaded into Cache memory and the number of floating point operations needed.

c) Propose an alternative implementation of matrix-vector multiply and clearly explain why it is better.

(4p) **3:** Suppose A is an $m \times n$ matrix $m > n$. Reduction to upper triangular form can be done using either reflections or rotations.

a) Suppose we want to compute the QR decomposition of an $m \times n$ matrix $(m > n)$. If rotations were to be used how many Givens rotations would be needed? If instead reflections were to be used how many Householder reflections would be needed?

b) After adding a new row to a previously solved least squares problem we need to transform a matrix of the form,

$$\widetilde{A} = \begin{pmatrix} x & x & x & x \\ 0 & x & x & x \\ 0 & 0 & x & x \\ x & x & x & x \end{pmatrix},$$

into upper triangular form. Clearly demonstrate how this can be accomplished using Givens rotations. $\square$

(4p) **4:** We want to find the solution to a linear system $Ax = b$, where $\text{rank}(A) = k < n$ so the solution x is not unique.

a) Suppose $b \in \text{Range}(A)$ so that the solution exists. Demonstrate how the solution x can be split into two parts,

$$x = x_1 + x_2, \qquad x_1 \in \text{null}(A)^{\perp}, \quad \text{and}, \quad x_2 \in \text{null}(A),$$

and how the SVD of A can be used to write expressions for the solution components x_1 and x_2. Clearly demonstrate how the condition $b \in \text{Range}(A)$ can be expressed in terms of the singular vectors of A.

b) Instead suppose that the system $Ax = b$ is inconsistent, i.e. $b \notin \text{Range}(A)$, and no solution exists. The solution formula derived in **a)** still works. Clearly explain what happens if we apply the above solution formula in the case of an rank deficient and inconsistent system $Ax = b$. Which problem does the computed solution actually solve?

c) We want to compute the distance from b and the subspace $\text{Range}(A)$. Explain how the SVD provides a basis for $\text{Range}(A)^{\perp}$ and write down a formula for the residual $r = b - Pb$, where P is the orthogonal projection onto $\text{Range}(A)$, based on this basis.

(4p) **5:** A river has been polluted by a large spill of a chemical substance that breaks down slowly with time. A theoretical model suggests that the concentration of the pollutant in the river decays with time according to a model

$$F(t) = c_0 \exp(-c_1 t) + c_2 \sin(\omega t),$$

where the last term is due to smaller naturally occuring pollution by the same chemical compund. In order to estimate the parameters of the model we measure the concetration $F(t_i)$, for times $t_1 < t_2 < \ldots < t_m$, and want to use the Gauss-Newton method to fit the parameters c_0, c_1, and c_2 to the measured data.

a) The Gauss-Newton method is based on writing down an appropriate residual vector $r := r(c)$, where $c = (c_0\,,\, c_1\,,\, c_2)^T$ are the coefficients of the model. Give the residual vector for the above situation.

b) In each step of the Guass-Newton method we need to solve a least squares problem to minimize $\|J_r(c^{(k)})s_k + r(c^{(k)})\|_2$, where $c^{(k)}$ is the approximate coefficient vector at step k. Derive an expression for the Jacobian J_r for this case.

(5p) **6:** Any matrix $A \in \mathbb{R}^{n\times n}$ can be factorized as $A = QTQ^H$, where Q is unitary and T upper triangular. This is called the *Schur decomposition* and is mainly of theoretical importance. Do the following:

a) Prove that the diagonal elements of T are eigenvalues of A.

b) Use the Schur decomposition to prove that any real symmetric matrix A has orthogonal eigenvectors.

c) A matrix B is called non-defective if it has a full set of eigenvectors, i.e. the decomposition $B = XDX^{-1}$ exists. Use the Shur decomposition to prove that if A is defective then for any $\varepsilon > 0$ there is a non-defective matrix B such that $\|A - B\|_2 \leq \varepsilon$.

Remark From **c)** we conclude that if a matrix is supposed to be defective and we compute a numerical approximation it is likely that the matrix turns out to be non-defective due to round-off errors.