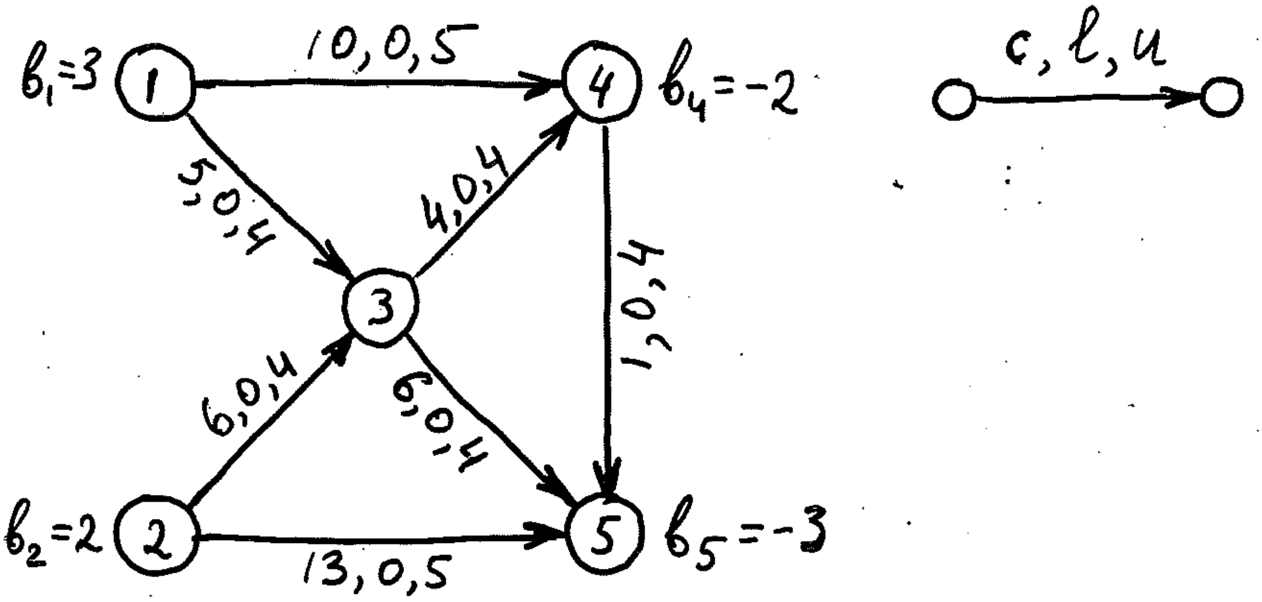
Minimum-cost network flow problem



x_{ij} = number of units of flow sent from node i to node j via arc (i, j)

b_i = net supply (outflow − inflow) at node i;
If $b_i > 0$ then node i is a source (adds flow)
If $b_i < 0$ then node i is a sink (removes flow)
If $b_i = 0$ then node i is a transshipment node

c_{ij} = cost of transporting 1 unit of flow from node i to node j via arc (i, j)

l_{ij} = lower bound on flow through arc (i, j)

u_{ij} = upper bound on flow through arc (i, j)

LP formulation

$$\min \sum_{(i,j)\in E} c_{ij} x_{ij}$$

$$\text{s.t.} \quad \sum_{j:(i,j)\in E} x_{ij} - \sum_{j:(j,i)\in E} x_{ji} = b_i \qquad \forall i \in N$$

$$l_{ij} \le x_{ij} \le u_{ij} \qquad \forall (i,j)\in E$$

Basic feasible solutions (b.f.s.)

0. It is a feasible flow.
1. Total number of basic variables $= |N| - 1$
2. The basic variables compose a spanning tree
3. Nonbasic variables are among those x_{ij} for which $x_{ij} = l_{ij}$ or $x_{ij} = u_{ij}$. If $l_{ij} < x_{ij} < u_{ij}$ then x_{ij} is a basic variable (if the number of such variables $< |N| - 1 \Rightarrow$ degeneracy $\Rightarrow$ choose some of other variables as basic to compose a spanning tree)

Ex 1. $x_{14} = x_{25} = 2$, $x_{13} = x_{35} = 1$, $x_{23} = x_{34} = x_{45} = 0$

Is it a feasible solution?

Is it a b.f.s.?

If so, find the basic tree.

Dual variables (node prices)

$y_i, \quad i \in N$

Given a basic tree, set $y_1 = 0$ and compute the rest of the node prices from the relations $y_j - y_i = c_{ij}$ valid for the basic variables x_{ij} (arcs of tree).

Ex 2.

For b.f.s. in Ex 1, find the node prices

$y_1 = 0$

$y_4 = y_1 + c_{14} = 0 + 10 = 10$

$y_3 = y_1 + c_{13} = 0 + 5 = 5$

$y_5 = y_3 + c_{35} = 5 + 6 = 11$

$y_5 - y_2 = c_{25} \Rightarrow y_2 = y_5 - c_{25} = 11 - 13 = -2$

Reduced costs

$\bar{c}_{ij} = c_{ij} + y_i - y_j$ for each nonbasic variable x_{ij}

Ex 3. In Ex 2, find the reduced costs.

$\bar{c}_{23} = 6 + (-2) - 5 = -1$

$\bar{c}_{34} = -1$

$\bar{c}_{45} = 0$

Optimality conditions

1. $x^*_{ij} = l_{ij} \Rightarrow$ increase in x_{ij} cannot decrease the total cost $(z) \Rightarrow \bar{c}_{ij} \geq 0$
2. $x^*_{ij} = u_{ij} \Rightarrow$ decrease in x_{ij} cannot decrease the total cost $\Rightarrow \bar{c}_{ij} \leq 0$

Ex. 4. Is the given b.f.s. optimal?

$\bar{c}_{23} = -1 \not\geq 0 \qquad (x_{23} = 0 = l_{23})$

$\bar{c}_{34} = -1 \not\geq 0 \qquad (x_{34} = 0 = l_{34})$

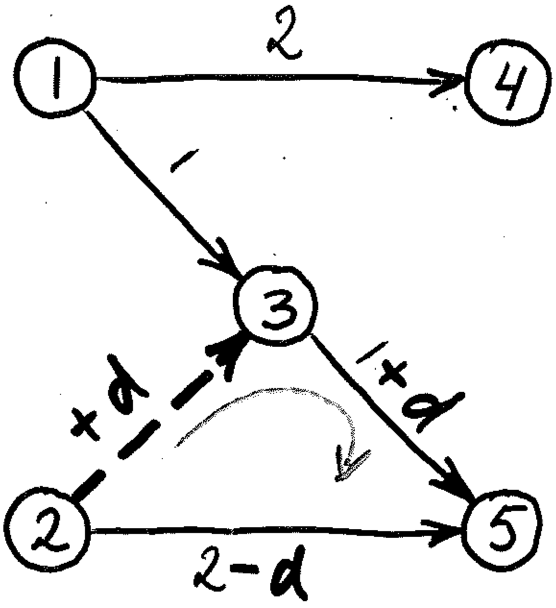
$\bar{c}_{45} = 0 \geq 0 \qquad (x_{45} = 0 = l_{45})$

$\Rightarrow x_{23}$ (or x_{34}) can be used as an entering variable

$\Rightarrow$ the total flow cost can be decreased

Network simplex method

1. Given a starting b.f.s., find the basic tree.
2. Compute the node prices $y_1, \ldots, y_{|N|}$ from the relations $y_1 = 0$ and $y_j - y_i = c_{ij}$ valid for the basic variables.
3. Compute the reduced costs $\bar{c}_{ij} = c_{ij} + y_i - y_j$ for all nonbasic variables x_{ij}
4. Stop if $\bar{c}_{ij} \geq 0$ $\forall x_{ij} = l_{ij}$ and $\bar{c}_{ij} \leq 0$ $\forall x_{ij} = u_{ij}$ (optimal solution). Otherwise, choose the nonbasic variable that most violates the optimality conditions as the entering basic variable.
5. Identify the cycle created by adding the arc corresponding to the entering variable to the spanning tree of the b.f.s. Change the entering x_{ij} $\begin{pmatrix} x_{ij} = l_{ij} \Rightarrow x_{ij} \uparrow \\ x_{ij} = u_{ij} \Rightarrow x_{ij} \downarrow \end{pmatrix}$ as much as possible. The variable that exits the basis will be the variable that first hits its upper or lower bound as the entering variable x_{ij} changes.
6. Find the new b.f.s. by changing the flows of the arcs in the cycle. Go to Step 2.



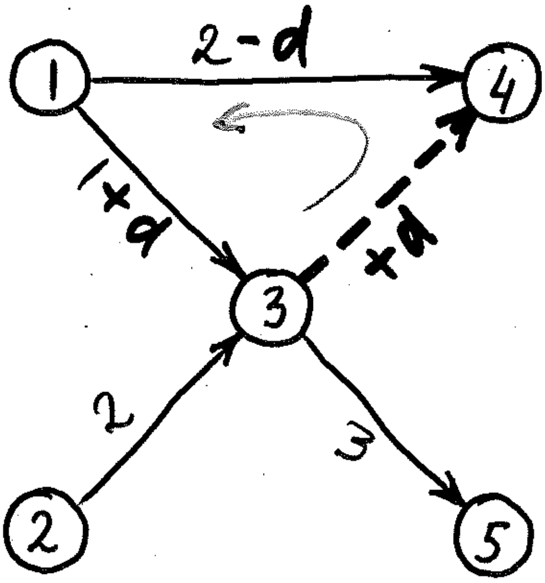
x_{23} enters $\Rightarrow$

cycle: 2-3-5-2

$x_{23} = 0 + d$

$$\left.\begin{array}{l} x_{23} \leq 4 \Rightarrow d \leq 4 \\ x_{35} \leq 4 \Rightarrow d \leq 3 \\ x_{25} \geqslant 0 \Rightarrow d \leq 2 \end{array}\right\} \Rightarrow d = 2 \Rightarrow x_{25} \text{ exits}$$

New values: $x_{23} = 2$, $x_{35} = 3$, $x_{25} = 0$



$y_1 = 0$, $y_2 = -1$, $y_3 = 5$,

$y_4 = 10$, $y_5 = 11$

$\bar{c}_{34} = -1 \not\geqslant 0 \Rightarrow x_{34}$ enters

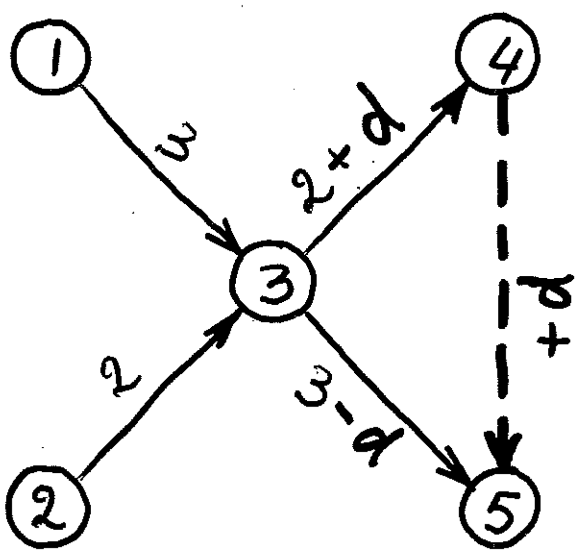
$\bar{c}_{25} = 1 \geqslant 0$

$\bar{c}_{45} = 0 \geqslant 0$

cycle: 3-4-1-3

$$\left.\begin{array}{l} x_{34} \leq 4 \Rightarrow d \leq 4 \\ x_{14} \geqslant 0 \Rightarrow d \leq 2 \\ x_{13} \leq 4 \Rightarrow d \leq 3 \end{array}\right\} \Rightarrow d = 2, \ x_{14} \text{ exits}$$

New values: $x_{13} = 3$, $x_{14} = 0$, $x_{34} = 2$



$y_1=0,\ y_2=-1,\ y_3=5$

$y_4=9,\ y_5=11$

$\bar{c}_{14}=1 \geqslant 0$

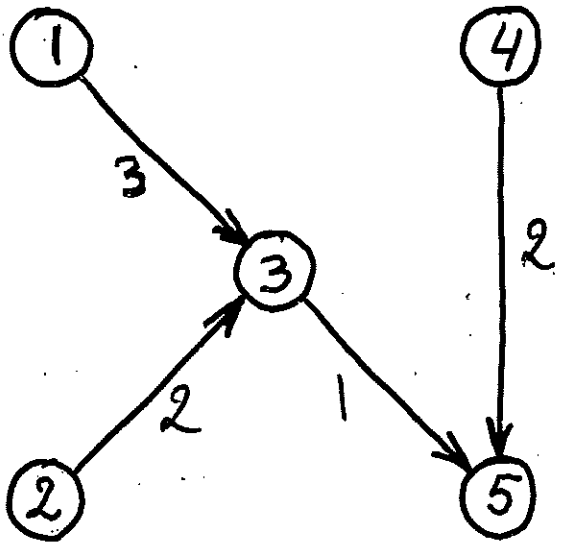
$\bar{c}_{25}=1 \geqslant 0$

$\bar{c}_{45}=-1 \not\geqslant 0 \Rightarrow x_{45}$ enters

cycle: $4-5-3-4$

$$\left.\begin{array}{l} x_{34}\leq 4 \Rightarrow d\leq 2 \\ x_{45}\leq 4 \Rightarrow d\leq 4 \\ x_{35}\geqslant 0 \Rightarrow d\leq 3 \end{array}\right\} \Rightarrow d=2,\ x_{34} \text{ exits}$$

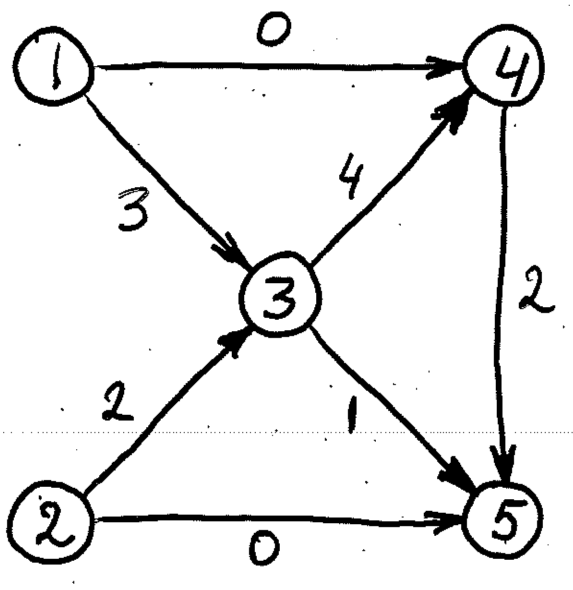
New values: $x_{34}=4,\ x_{45}=2,\ x_{35}=1$



$y_1=0,\ y_2=-1,\ y_3=5,$

$y_4=10,\ y_5=11$

$$\left.\begin{array}{l} \bar{c}_{14}=0 \geqslant 0 \\ \bar{c}_{25}=1 \geqslant 0 \\ \bar{c}_{34}=-1 \leq 0 \end{array}\right\} \Rightarrow \text{optimal flow}$$



Total flow cost = 51