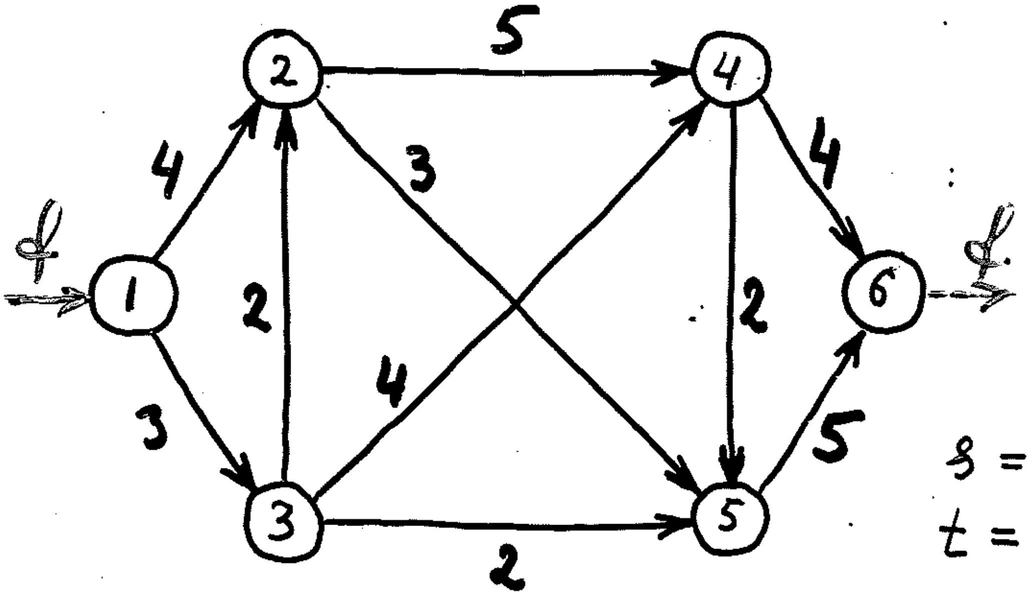
Maximum flow problem



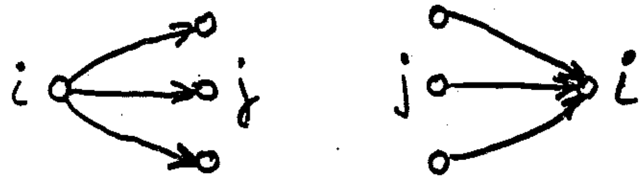
$s = 1$ (source)
$t = 6$ (sink)

LP formulation

$\max f$

s.t. conservation-flow $\forall i \in N$:

$$\sum_{j:(i,j)\in E} x_{ij} - \sum_{j:(j,i)\in E} x_{ji} = \begin{cases} f, & \text{if } i = s \\ 0, & \text{if } i \neq s, i \neq t \\ -f, & \text{if } i = t \end{cases}$$



capacity constraints $\forall (i,j) \in E$:

$$0 \le x_{ij} \le u_{ij}$$

f is free

Q1. How to find a feasible flow?

Q2. How to check if a given flow is optimal?
(path $s - \dots - t$) & (min cut)

Q3. If a feasible flow is not optimal, how to modify it to obtain a new feasible flow with a larger value of f?

Modified Dijkstra's algorithm

(Given capacities $\bar{u}_{ij}$, find a path with the largest path capacity)

1. Set $A = \emptyset$, $D = N$; $y_s = \overset{+\infty}{M}$, $p_s = -$, $y_i = 0$, $p_i = -$ $\forall i \in N \setminus \{s\}$

2. Find $k \in D$ such that $y_k = \max_{i \in D} y_i$

3. Set $A = A \cup \{k\}$, $D = D \setminus \{k\}$

4. Stop if $t \in A$ (or $D = \emptyset$)

5. For all $i \in D$ such that $(k,i) \in E$, do:

   if $\min\{\bar{u}_{ki}, y_k\} > y_i$

   then set $y_i = \min\{\bar{u}_{ki}, y_k\}$ and $p_i = k$

6. Go to Step 2

The Ford–Fulkerson algorithm

1. Set each arc's flow to zero
2. Find a flow increasing path from s to t (modified Dijkstra's algorithm). Stop if there is no such flow ($\Rightarrow$ the current flow is a maximum flow)
3. Send maximal possible flow along this path.
4. Update each arc's feasible directions.
5. Go to Step 2.

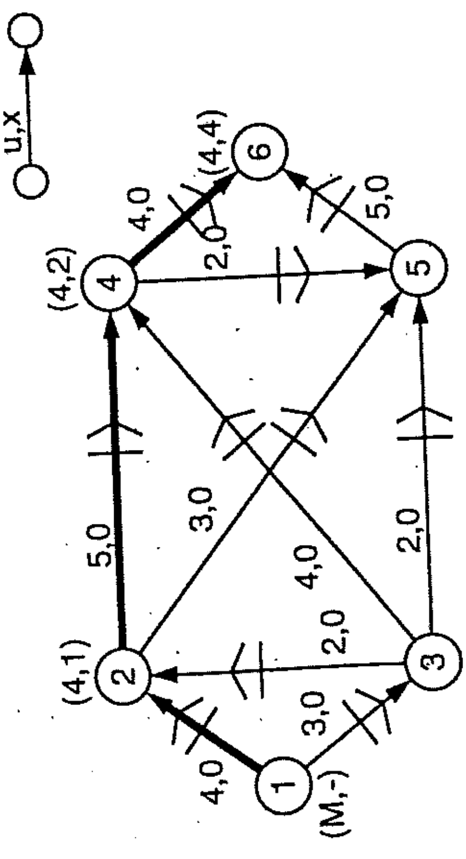
How to update arc's feasible directions

flow	feasible directions	notations	u_{ij}^{+}	u_{ij}^{-}
$x_{ij}=0$	forward		u_{ij}	0
$0<x_{ij}<u_{ij}$	forward and backward		$u_{ij}-x_{ij}$	x_{ij}
$x_{ij}=u_{ij}$	backward		0	u_{ij}

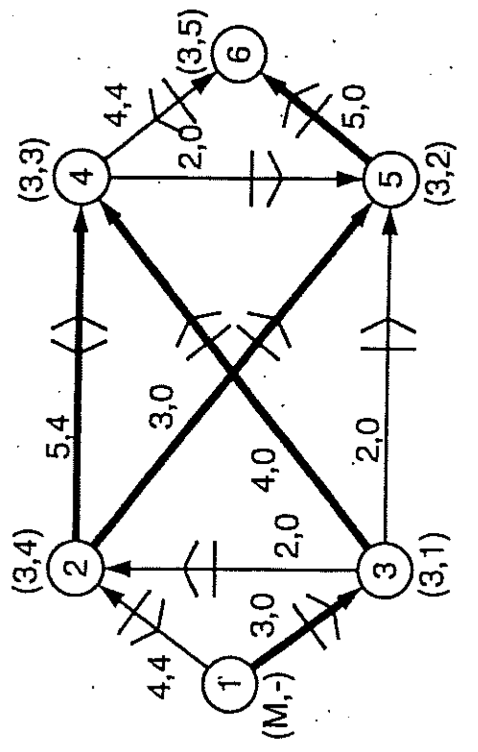
u_{ij}^{+} = max extra flow forward

u_{ij}^{-} = max extra flow backward

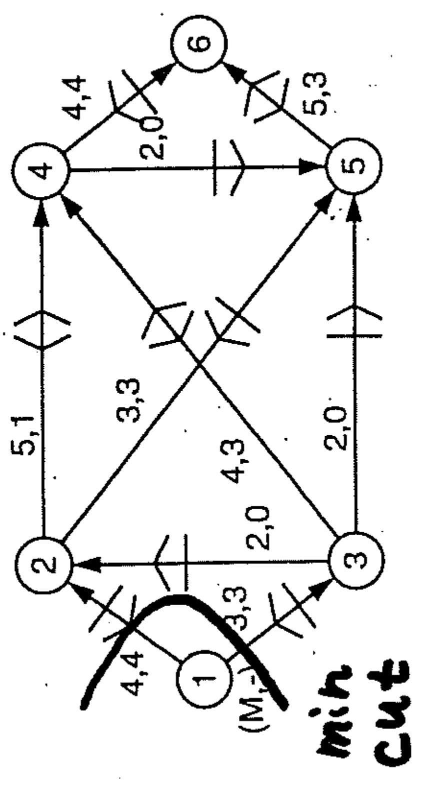
Example for Ford-Fulkerson algorithm



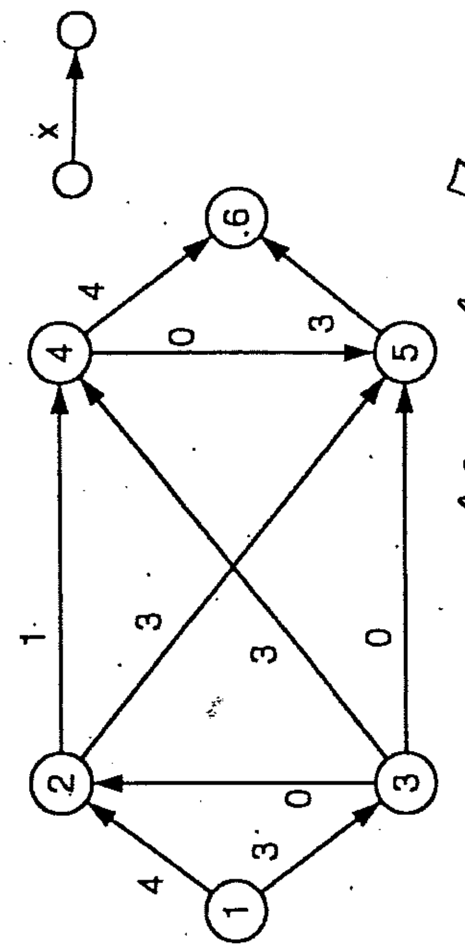
Capacity of 1-2-4-6 is 4.



Capacity of 1-3-4-2-5-6 is 3.



Stop. Optimal solution.



Maximal flow $f_* = 7$

Cut, minimum-capacity cut

Def. A division of N into two disjoint subsets N_s and N_t is called a **cut** if $s \in N_s$ and $t \in N_t$.

Remark. $N_s \cup N_t = N$, $N_s \cap N_t = \emptyset$.

Def. An edge $(i,j) \in E$ is called a **forward edge** if $i \in N_s$ and $j \in N_t$.

An edge $(i,j) \in E$ is called a **backward edge** if $i \in N_t$ and $j \in N_s$.

Let x be a feasible flow. Then the **flow across the cut** (N_s, N_t):

$$F_x(N_s, N_t) = \sum_{\substack{(i,j) \in \text{forward} \\ \text{edges}}} x_{ij} - \sum_{\substack{(i,j) \in \text{backward} \\ \text{edges}}} x_{ij}$$

The **capacity of a cut** (N_s, N_t):

$$C(N_s, N_t) = \sum_{\substack{(i,j) \in \text{forward} \\ \text{edges}}} u_{ij}$$

Theorem 1 (weak duality)
The flow across any cut equals the value of the flow and does not exceed the cut capacity.

Proof. Sum up the conservation-flow constraints for N_s.

If $i, j \in N_s$, then x_{ij} in the equation for node j cancels $-x_{ij}$ in the equation for node i. Thus,

$$f = \sum_{(i,j)\in \text{forward edges}} \underset{\substack{\uparrow \\ x_{ij} \le u_{ij}}}{x_{ij}} - \sum_{(i,j)\in \text{backward edges}} \underset{\substack{\uparrow \\ 0 \le x_{ij}}}{x_{ij}} = F_x(N_s, N_t)$$

$$\Downarrow$$

$$F_x(N_s, N_t) \le \sum_{(i,j)\in \text{forward edges}} u_{ij} = C(N_s, N_t)$$

Theorem 2 (strong duality, max-flow min-cut theorem)
The maximum value of flow from s to t equals the minimum capacity of all cuts.

Remark

$$x_{ij}^* = \begin{cases} u_{ij}, & (i,j) \in \text{forward edges} \\ 0, & (i,j) \in \text{backward edges} \end{cases}$$

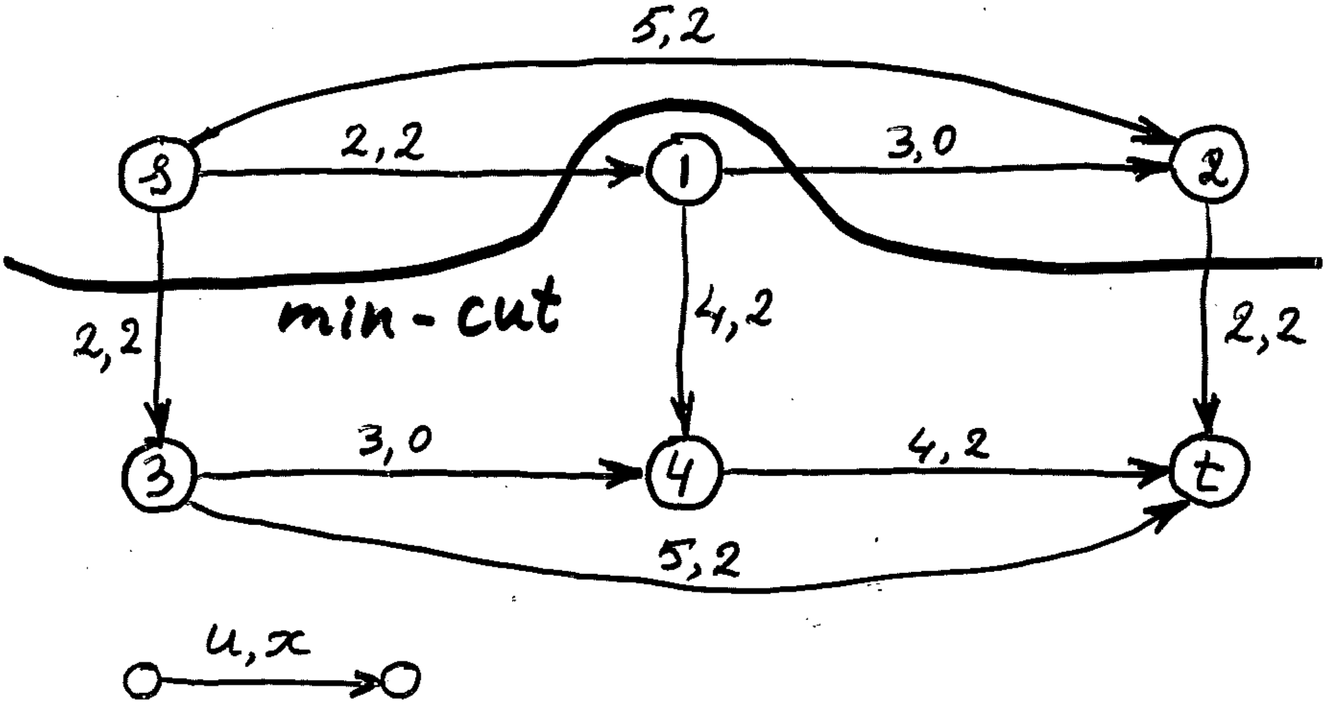
How to find min-cut?

Use x^* and the following properties of min-cut:

$$x^*_{ij} = u_{ij} \quad \forall \text{ forward edges } (i,j)$$

$$x^*_{ij} = 0 \quad \forall \text{ backward edges } (i,j)$$

Ex. The flow is optimal (maximal)



$N_s = \{s, 2\}$

min-cut = 6 = max-flow