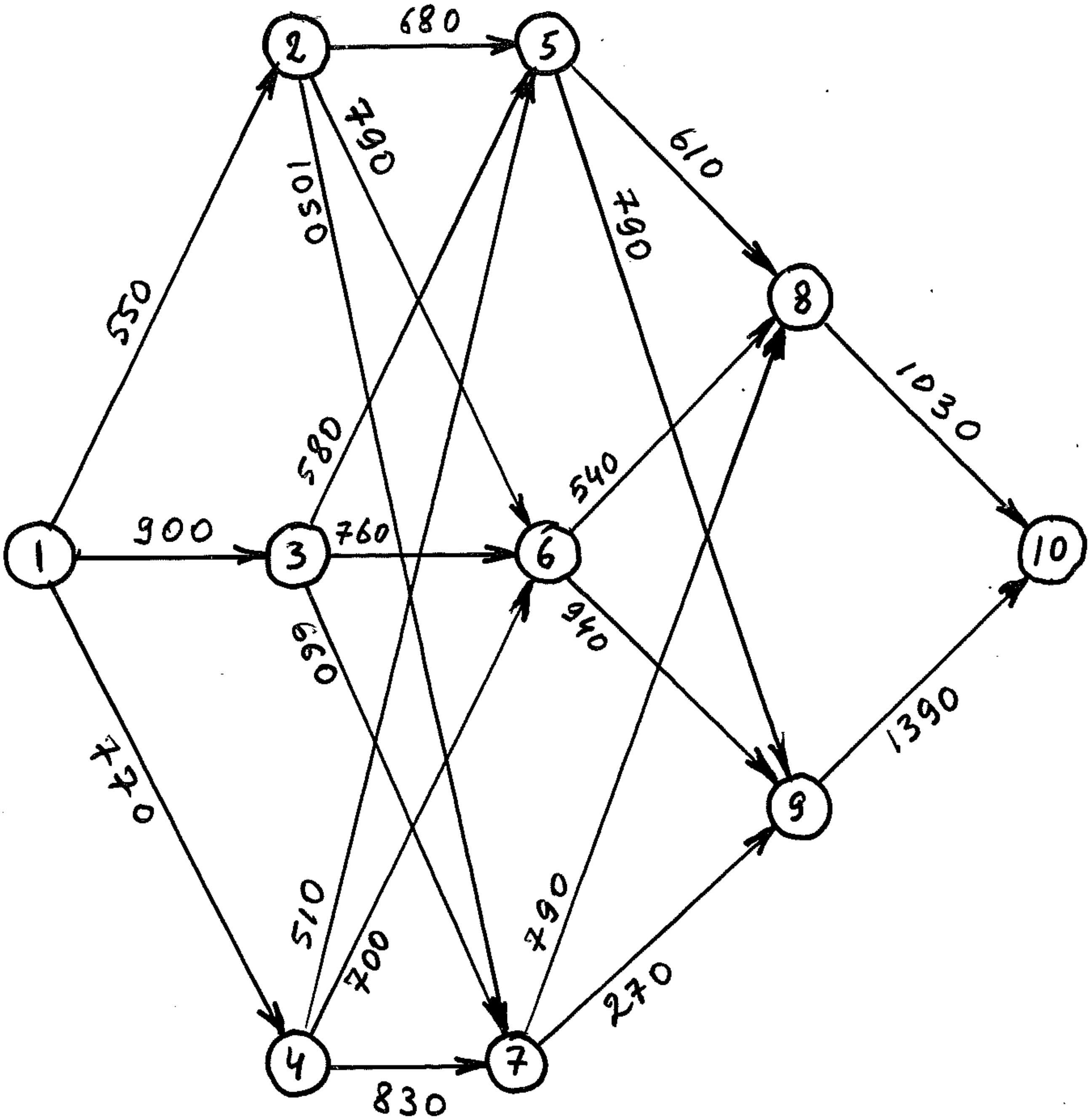
Lecture 4 on Dynamic programming

- Shortest path problem
- Knapsack problem, resource allocation problem
- Inventory problem

Shortest path problem



1 1 1 1 1
1 2 3 4 5

Stages: $t = 1, 2, 3, 4, 5$ (day)

Decision variable: x_t = city to go on stage t

State: s_t = city from which to go on stage t

State transition: $s_{t+1} = x_t$

Optimal value function: $f_t(s_t)$ = the length of shortest path from city s_t to 10

Recursion: $f_t(s_t) = \min_{x_t}\left(f_{t+1}(s_{t+1}) + c_t(s_t, x_t)\right)$, where $c_t(s_t, x_t)$ is the distance between s_t and s_{t+1} $(= x_t)$. The objective is to find $f_1(s_1)$

Boundary values: $s_1 = 1$, $f_5(s_5) = 0$

Restrictions:

$x_1, s_2 \in \{2, 3, 4\}$

$x_2, s_3 \in \{5, 6, 7\}$

$x_3, s_4 \in \{8, 9\}$

$x_4, s_5 \in \{10\}$

Stage 4:

$x_4 \backslash s_4$	8	9
10	1030	1390
$f_4(s_4)$	1030	1390
$x_4^*(s_4)$	10	10

Stage 3:

$x_3 \backslash s_3$	5	6	7
8	1640	1570	1820
9	2180	2330	1660
$f_3(x_3)$	1640	1570	1660
$x_3^*(s_3)$	8	8	9

Stage 2:

$x_2 \backslash s_2$	2	3	4
5	2320	2220	2150
6	2360	2330	2270
7	2710	2320	2490
$f_2(s_2)$	2320	2220	2150
$x_2^*(s_2)$	5	5	5

Stage 1:

$x_1 \backslash s_1$	1
2	2870
3	3120
4	2920
$f_1(s_1)$	2870
$x_1^*(s_1)$	2

Optimal solution: 1 – 2 – 5 – 8 – 10

total length = 2870

Bellmans principle of optimality:

An optimal policy has the property that whatever the initial state and the initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision

Knapsack problem

$$\max \ z = 11x_1 + 7x_2 + 12x_3$$

s.t. $\quad 4x_1 + 3x_2 + 5x_3 \le 10$

$x_1 \ge 0, \ x_2 \ge 0, \ x_3 \ge 0$, **integer**

$t = 1, 2, 3$

x_t

s_t = the knapsack volume (resources) left for stages from t to 3

$s_{t+1} = s_t - a_t x_t$, where $a_1 = 4$, $a_2 = 3$, $a_3 = 5$

$f_t(s_t)$ = maximum benefit earned from a s_t-volume knapsack filled with items from t to 3.

$$f_t(s_t) = \max_{x_t} \{ f_{t+1}(s_{t+1}) + c_t x_t \},$$

the objective is to find $f_1(s_1)$

$s_1 = 10$, $s_4 \ge 0$, $f_4(s_4) = 0$ (boundary values)

$x_t \ge 0$, integer, $t = 1, 2, 3$

$0 \le s_t \le 10$, $t = 1, 2, 3$

Stage 3: $s_4 = s_3 - 5x_3 \geq 0$

$x_3 \backslash s_3$	0	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0	0
1	–	–	–	–	–	12	12	12	12	12	12
2	–	–	–	–	–	–	–	–	–	–	24
$f_3(s_3)$	0	0	0	0	0	12	12	12	12	12	24
$x_3^*(s_3)$	0	0	0	0	0	1	1	1	1	1	2

Stage 2: $s_3 = s_2 - 3x_2 \geq 0$

$x_2 \backslash s_2$	0	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	12	12	12	12	12	24
1	–	–	–	7	7	7	7	7	19	19	19
2	–	–	–	–	–	–	14	14	14	14	14
3	–	–	–	–	–	–	–	–	–	21	21
$f_2(s_2)$	0	0	0	7	7	12	14	14	19	21	24
$x_2^*(s_2)$	0	0	0	1	1	0	2	2	1	3	0

Stage 1: $s_2 = s_1 - 4x_1 = 10 - 4x_1 \geq 0$

$x_1 \backslash s_1$	10
0	24
1	25
2	22
$f_1(s_1)$	25
$x_1^*(s_1)$	1

$s_1 = 10 \Rightarrow x_1^* = 1$

$\Rightarrow s_2 = 6 \Rightarrow x_2^* = 2$

$\Rightarrow s_3 = 0 \Rightarrow x_3^* = 0$

$z^* = 25$

General resource allocation problem

benefit

$$\max \sum_{t=1}^{T} r_t(x_t)$$

$$\text{s.t.} \quad \sum_{t=1}^{T} g_t(x_t) \leq W$$

units of resource used by activity t

An inventory problem

Example
A company knows that the demand of one of its most important products are 1, 3, 2 and 4 units respectively over the next four months. The company must plan the production of the ten units. If any production appears in a period, there is a setup cost of 3000 kr. In addition there is a production cost of 1000 kr for each produced unit. If a unit is put in inventory, there is an inventory cost of 500 kr per unit. Five units at most can be produced in a month and at most four units can be put in inventory. How should the company plan their production to satisfy the demand and minimize production and inventory costs? There are no units in the inventory at the beginning of month 1.

Formulation (backward recursion):

Stages: 4 months $\Rightarrow t = 1, 2, 3, 4$

Control variables: x_t = number of units produced in month t

States: s_t = number of units in inventory in the beginning of month t

Transition function: $s_{t+1} = s_t + x_t - d_t$ (d_t = demand month t)

Recursion function: $f_t(s_t)$ = minimum cost for production and inventory from month t until month 4 given that there are s_t units in inventory in the beginning of month t

Recursion relation: $f_t(s_t) = \min_{x_t}\{f_{t+1}(s_{t+1}) + c_t(s_t, x_t)\}$, where

$$c_t(s_t, x_t) = \begin{cases} 3 + x_t + \frac{1}{2}(s_t + x_t - d_t), & \text{if } x_t > 0 \\ \frac{1}{2}(s_t - d_t), & \text{if } x_t = 0 \end{cases}$$

Search $f_1(s_1)$.

Initial conditions: $s_1 = 0$, $s_5 = 0$, $f_5(s_5) = 0$

Restrictions: $x_t \leq 5$, $t = 1, \ldots, 4$, $s_t \leq 4$, $t = 1, \ldots, 4$

11.9

Solution:

Stage 4: $s_5 = s_4 + x_4 - 4 = 0$

$x_4 \backslash s_4$	0	1	2	3	4
0	–	–	–	–	0
1	–	–	–	4	–
2	–	–	5	–	–
3	–	6	–	–	–
4	7	–	–	–	–
$f_4(s_4)$	7	6	5	4	0
$x_4^*(s_4)$	4	3	2	1	0

Stage 3: $s_4 = s_3 + x_3 - 2$

$x_3 \backslash s_3$	0	1	2	3	4
0	–	–	7	6.5	6
1	–	11	10.5	10	9.5
2	12	11.5	11	10.5	7
3	12.5	12	11.5	8	–
4	13	12.5	9	–	–
5	13.5	10	–	–	–
$f_3(s_3)$	12	10	7	6.5	6
$x_3^*(s_3)$	2	5	0	0	0

Stage 2: $s_3 = s_2 + x_2 - 3$

$x_2 \backslash s_2$	0	1	2	3	4
0	–	–	–	12	10.5
1	–	–	16	14.5	12
2	–	17	15.5	13	13
3	18	16.5	14	14	14
4	17.5	15	15	15	–
5	16	16	16	–	–
$f_2(s_2)$	16	15	14	12	10.5
$x_2^*(s_2)$	5	4	3	0	0

Stage 1:
$s_2 = s_1 + x_1 - 1 = x_1 - 1$

$x_1 \backslash s_1$	0
0	–
1	20
2	20.5
3	21
4	20.5
5	20.5
$f_1(s_1)$	20
$x_1^*(s_1)$	1

Backtracking: $s_1 = 0 \Rightarrow x_1 = 1 \Rightarrow s_2 = 0 \Rightarrow x_2 = 5 \Rightarrow s_3 = 2 \Rightarrow x_3 = 0 \Rightarrow s_4 = 0 \Rightarrow x_4 = 4$. Total cost = 20. The optimal plan is to produce one unit in month 1, five units in month 2 and four units in month 4.