Integer (liner) programming problem

$$(IP) \quad \min \; c^T x \qquad\qquad (\max).$$
$$\text{s.t.} \quad Ax \le b \qquad\qquad (=, \ge)$$
$$x \ge 0, \; x \text{ is integer}$$

$x, c \in R^n$, $b \in R^m$, $A \in R^{m \times n}$

Special cases

- $x_i \in \{0, 1\}$, $i = 1, \dots, n$
- mixed IP: 0/1 and integer
  (continuous and integer)

Feasible region

$$F = \{x : \; Ax \le b, \; x \ge 0, \; x \text{ is integer}\}$$

Feasible region in LP-relaxation

$$F^{LP} = \{x : \; Ax \le b, \; x \ge 0\}$$

Convex hull of F

$$F^C = \{x : \; x = \sum_{i=1}^{|F|} \lambda_i x^i, \; x^i \in F, \; \sum_{i=1}^{|F|} \lambda_i = 1, \; \lambda_i \ge 0 \; \forall i\}$$

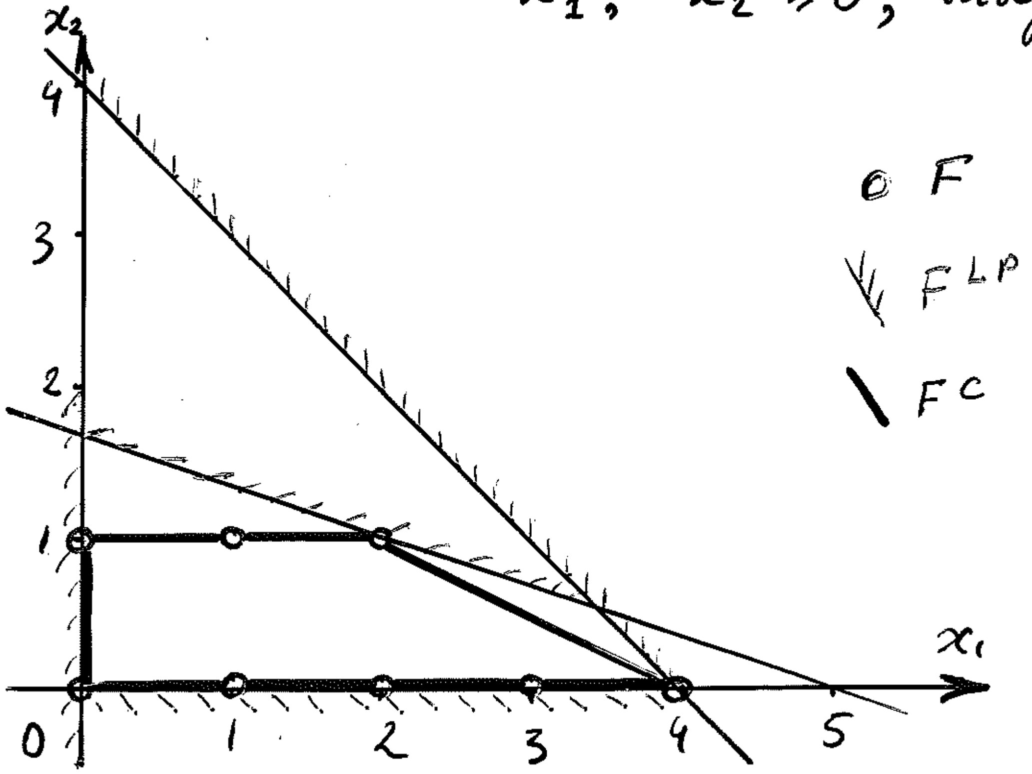
$$F \subset F^C \subset F^{LP}.$$

Ex. $\max\ z = x_1 + 5x_2$

s.t. $x_1 + x_2 \le 4$

$x_1 + 3x_2 \le 5$

$x_1,\ x_2 \geq 0$, integer



$F^C = \{x : x_1 + 2x_2 \le 4,\ x_2 \le 1,\ x_1 \geq 0,\ x_2 \geq 0\}$

$\max\ z = x_1 + 5x_2$ s.t. $x \in F \Rightarrow x^*_{IP} = \binom{2}{1}$, $z^*_{IP} = 7$

$\max\ z = x_1 + 5x_2$ s.t. $x \in F^{LP} \Rightarrow x^*_{LP} = \binom{0}{5/3}$, $z^*_{LP} = 8\,1/3$

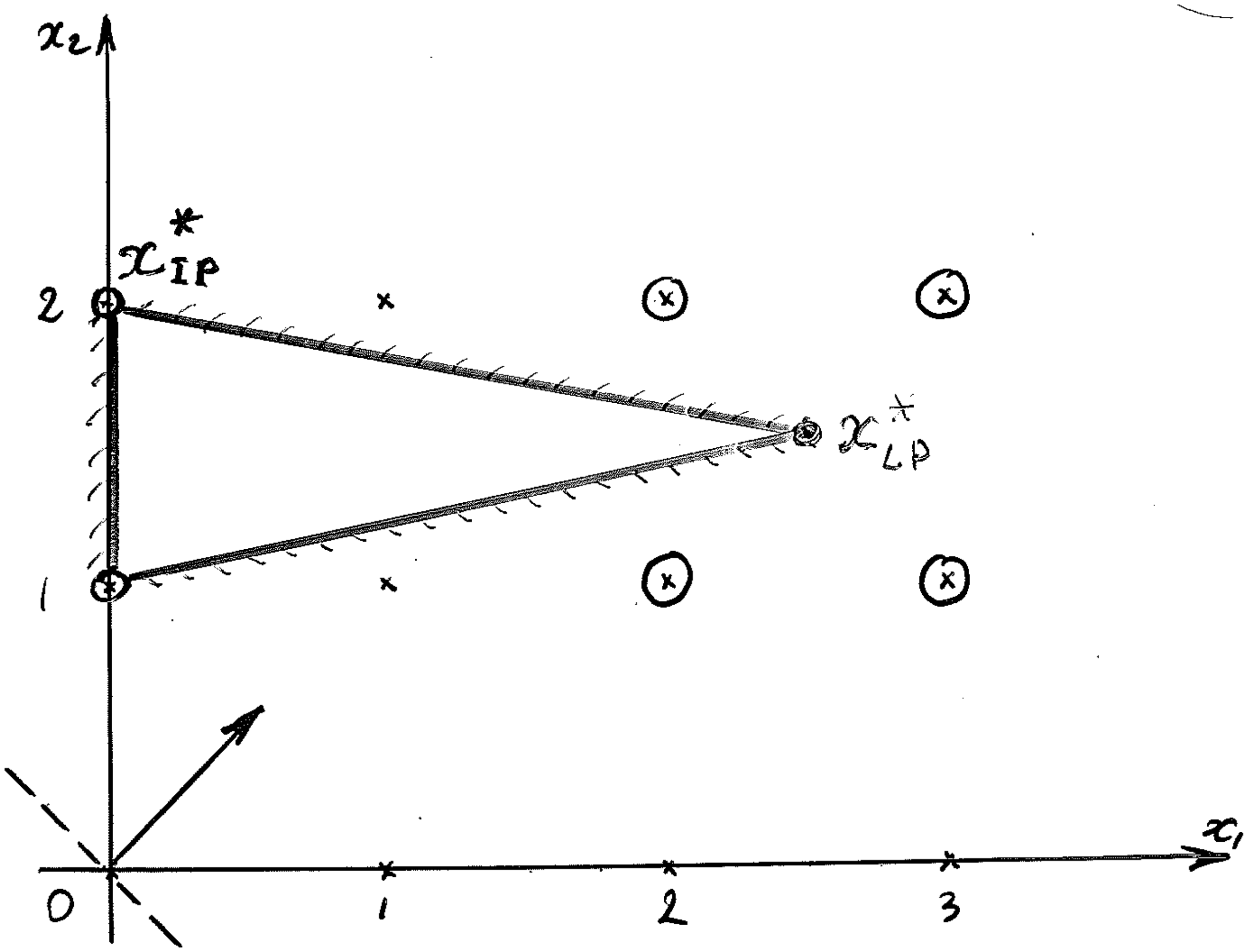
$\max\ z = x_1 + 5x_2$ s.t. $x \in F^C \Rightarrow x^*_C = \binom{2}{1}$, $z^*_C = 7$

(this is an LP-problem in the λ-variables)

$$F \subset F^C \subset F^{LP} \overset{\max}{\Longrightarrow} z^*_{IP} = z^*_C \le z^*_{LP}$$

$$\overset{\min}{\Longrightarrow} z^*_{IP} = z^*_C \geq z^*_{LP}$$

Rounding of x^*_{LP}



$\Rightarrow$ Rounding of x^*_{LP} may give infeasible points

Formulating IP problems

1) Given m constraints $a_i^T x \le b_i$, $i=1,\dots,m$, at least k of which should be satisfied

Introduce:

$$\begin{cases} y_i = \begin{cases} 1, & \text{implies that } a_i^T x \le b_i \text{ holds} \\ 0, & \text{may not imply this} \end{cases} \\ \sum y_i \ge k \\ a_i^T x \le b_i + \underset{\text{large enough}}{M}(1-y_i), \quad i=1,\dots,m \end{cases}$$

Then:

$$y_i = 1 \implies a_i^T x \le b_i$$

$$y_i = 0 \implies a_i^T x \le b_i + M$$

2) Variable $x \in \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$.

Introduce:

$$y_1, y_2, y_3, y_4 \in \{0,1\}$$

$$x = \alpha_1 y_1 + \alpha_2 y_2 + \alpha_3 y_3 + \alpha_4 y_4$$

$$y_1 + y_2 + y_3 + y_4 = 1$$

3) Either-or constraints. At least one of the constraints $f(x) \le 0$ and $g(x) \le 0$ shold be satisfied. Introduce:

large enough

$$y = \begin{cases} 0, & f \le 0 \\ 1, & g \le 0 \end{cases}, \qquad \begin{aligned} f(x) &\le My \\ g(x) &\le M(1-y) \end{aligned}$$

4) If-then constraints. If $f(x) > 0$, then $g(x) \geq 0$.

Introduce:

$$y = \begin{cases} 0, & \text{if } f(x) > 0 \\ 1, & \text{if } f(x) \leq 0 \end{cases}$$

$$-g(x) \leq My$$

$$f(x) \leq M(1-y)$$

Ex. $\max f(x) = 3x_1 + 5x_2 + 7x_3$

$$\text{s.t. } g(x) = x_1 + 2x_2 + 3x_3 - 4 \le 0 \quad \leftarrow u \ge 0$$

$$x \in X = \{0,1\}^3$$

<u>Integer</u> knapsack problem $\Rightarrow$ not convex

$x^* = (1,0,1)$, $f(x^*) = 10$

$$L(x,u) = 3x_1 + 5x_2 + 7x_3 \underset{\text{max!}}{\underset{\uparrow}{-}} u(x_1 + 2x_2 + 3x_3 - 4) =$$

$$= 4u + (3-u)x_1 + (5-2u)x_2 + (7-3u)x_3$$

$$\varphi(u) = 4u + \max_{x_1=0,1}(3-u)x_1 + \max_{x_2=0,1}(5-2u)x_2 + \max_{x_3=0,1}(7-3u)x_3$$

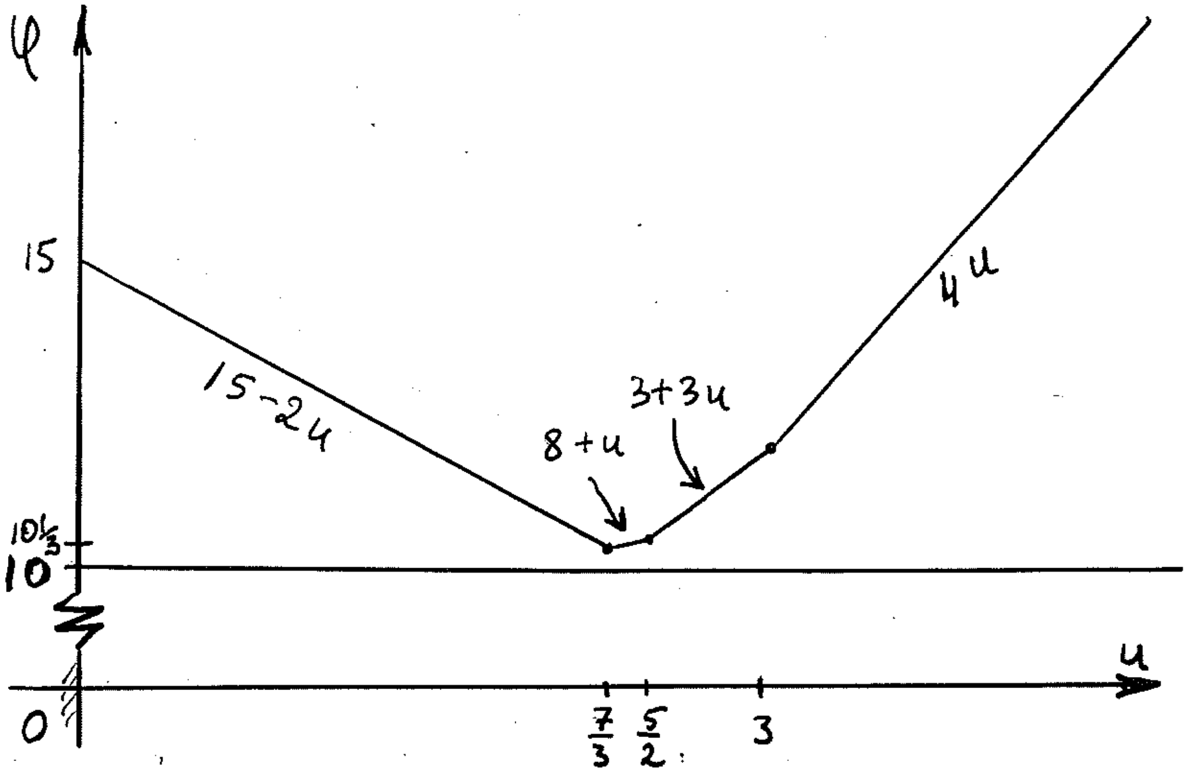
$$\varphi(u) = \begin{cases} 15-2u, & \text{if } 0 \le u \le \frac{7}{3} & \leftarrow x(u) = (1,1,1) \\ 8+u, & \text{if } \frac{7}{3} \le u \le \frac{5}{2} & \leftarrow x(u) = (1,1,0) \\ 3+3u, & \text{if } \frac{5}{2} \le u \le 3 & \leftarrow x(u) = (1,0,0) \\ 4u, & \text{if } 3 \le u & \leftarrow x(u) = (0,0,0) \end{cases}$$

An alternative:

$$\varphi(u) = \max_{x \in \{0,1\}^3} \{4u + (3-u)x_1 + (5-2u)x_2 + (7-3u)x_3\}$$

$$= \max\{\underset{(0,0,0)}{4u}, \underset{(0,0,1)}{7+u}, \underset{(0,1,0)}{5+2u}, \underset{(0,1,1)}{12-u}, \underset{(1,0,0)}{3+3u}, \underset{(1,0,1)}{10}, \underset{(1,1,0)}{8+u}, \underset{(1,1,1)}{15-2u}\}$$

$2^3 = 8$



$$(\mathcal{D}) \min_{u \geq 0} \varphi(u) \Rightarrow u^* = \frac{7}{3}, \quad \varphi(u^*) = 10\frac{1}{3}$$

$\varphi(u^*) > f(x^*)$, because (P) is not convex

the dual gap $= \varphi(u^*) - f(x^*) = \frac{1}{3} > 0$

Nondifferentiable Optimization

$$\min_{x \in R^n} f(x)$$

A. f is convex

If $f \in C^1$ then

$$f(x) \geq f(\bar{x}) + \nabla f(\bar{x})^T (x - \bar{x}), \quad \forall x, \bar{x}$$

Def. A vector $\bar{\gamma} \in R^n$ such that

$$f(x) \geq f(\bar{x}) + \bar{\gamma}^T (x - \bar{x}), \quad \forall x$$

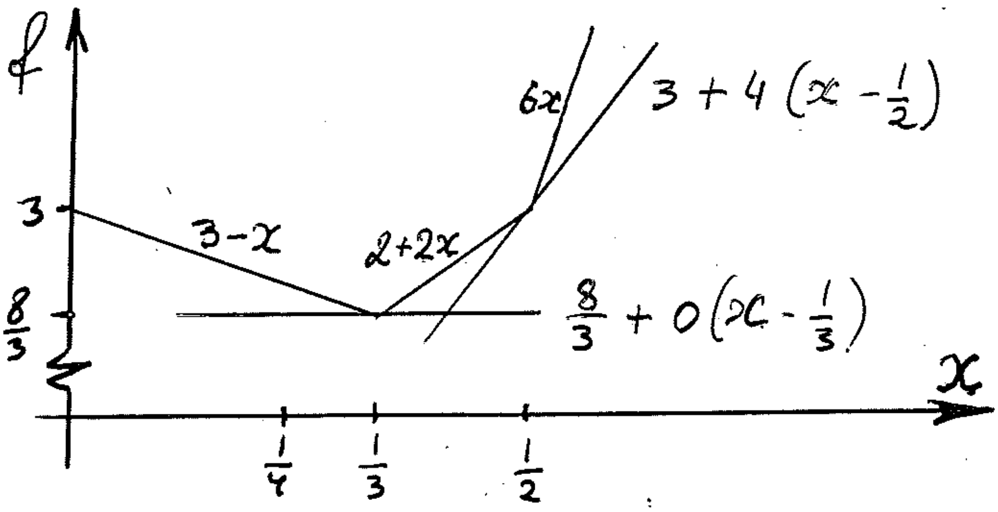
is called subgradient of f at $\bar{x}$.

Def. The set of subgradients of f at $\bar{x}$ is called subdifferential and is denoted by $\partial f(\bar{x})$.

Rem. If f is concave

$$f(x) \leq f(\bar{x}) + \bar{\gamma}^T (x - \bar{x}) \qquad \forall x.$$

Ex. $f(x) = \begin{cases} 3 - x & \text{if } 0 \le x \le 1/3 \\ 2+2x, & \text{if } 1/3 \le x \le 1/2 \\ 6x, & \text{if } 1/2 \le x \end{cases}$



$\bar{x} = \frac{1}{2}$, $\bar{\gamma} = 4$ is a subgradient, because

$f(x) \geq f(\frac{1}{2}) + \bar{\gamma}^T (x - \frac{1}{2}) = 3 + 4(x - \frac{1}{2})$, $\forall x$

$\partial f(\frac{1}{2}) = [2, 6]$

$\partial f(\frac{1}{3}) = [-1, 2]$

$\partial f(\frac{1}{4}) = \{-1\}$

$0 \in \partial f(\frac{1}{3}) = [-1, 2]$

$\Rightarrow f(x) \geq \frac{8}{3} + 0(x - \frac{1}{3})$, $\forall x$

$\Rightarrow f(x) \geq f(\frac{1}{3}) = \frac{8}{3}$, $\forall x$

$\Rightarrow x^* = \frac{1}{3}$

Rem.

- f is differentiable in $\bar{x}$ $\Rightarrow$ $\partial f(\bar{x}) = \{\nabla f(\bar{x})\}$
- $\partial f(\bar{x}) = \{\bar{\gamma}\}$ (one element) $\Rightarrow$ f is differentiable in $\bar{x}$ and $\nabla f(\bar{x}) = \bar{\gamma}$

Corollary. f is differentiable in $\bar{x}$ $\Longleftrightarrow$ subgradient $\bar{\gamma}$ is unique in $\bar{x}$ $(\bar{\gamma} = \nabla f(\bar{x}))$

Th. x^* is optimal $\Longleftrightarrow$ $0 \in \partial f(x^*)$

Proof. Suppose $0 \in \partial f(x^*)$ $\Rightarrow$

$f(x) \geq f(x^*) + 0^T(x - x^*), \ \forall x$ $\Rightarrow$

$f(x) \geq f(x^*), \ \forall x$ $\Rightarrow$ x^* is optimal

Suppose x^* is optimal $\Rightarrow$

$f(x) \geq f(x^*), \ \forall x$ $\Rightarrow$

$f(x) \geq f(x^*) + 0^T(x - x^*), \ \forall x \Rightarrow 0 \in \partial f(x^*)$

Th. If $x(\bar{\lambda})$ solves $\min\limits_{x \in X} f(x) + \lambda^T g(x)$

$\Rightarrow g(x(\bar{\lambda})) \in \partial \varphi(\bar{\lambda})$

(φ is concave)

Proof. $\varphi(\lambda) = \min_{x\in X} f(x) + \lambda^T g(x)$

$\leq f(x(\bar\lambda)) + \lambda^T g(x(\bar\lambda)) =$

$= \underbrace{f(x(\bar\lambda)) \underline{+ \bar\lambda^T g(x(\bar\lambda))}}_{\varphi(\bar\lambda)} + \lambda^T g(x(\bar\lambda)) \underline{- \bar\lambda^T g(x(\bar\lambda))}$

$= \varphi(\bar\lambda) + g^T(x(\bar\lambda))\cdot(\lambda - \bar\lambda), \quad \forall \lambda$

$\Rightarrow g(x(\bar\lambda)) \in \partial\varphi(\bar\lambda)$

Rem. $x(\bar\lambda)$ is unique $\Rightarrow$
φ is differentiable in $\bar\lambda$ with $\nabla\varphi(\bar\lambda) = g(x(\bar\lambda))$

$$\left.\begin{array}{l} X \text{ is convex} \\ f \text{ is } \underline{\text{strictly}} \text{ convex} \\ g_i \text{ is convex } \forall i \end{array}\right\} \Rightarrow \varphi(\lambda) \text{ is differentiable}$$

Subgradientoptimering på Lagrange-dual

Primalt problem:

$$\begin{array}{ll} \min & f(x) \\ \text{då} & g(x) \leq 0 \\ & x \in X. \end{array}$$

Lagrange-dualt problem:

$$\max_{u \geq 0} h(u)$$

där

$$h(u) = \min_{x \in X} f(x) + u^T g(x).$$

Subgradientoptimering:

0. Välj $u^0 \geq 0$, sätt $k = 0$ och $LBD_{-1} = -\infty$.

1. Lös Lagrange-relaxationen $\min_{x \in X} f(x) + (u^k)^T g(x)$.
Optimum: $x(u^k) \Rightarrow h(u^k) = f(x(u^k)) + (u^k)^T g(x(u^k))$.
Sätt $LBD_k = \max\{LBD_{k-1}, h(u^k)\}$.

2. Beräkna subgradienten $\gamma^k = g(x(u^k))$ och en steglängd $t_k > 0$.

3. Beräkna nytt iterat som $u^{k+1} = \max\{0, u^k + t_k \gamma^k\}$ (där maximum tas komponentvis).

4. Sätt $k = k + 1$ och gå till steg 1.

Polyak-steglängder:

$$t_k = \lambda_k \frac{UBD_k - h(u^k)}{\|\gamma^k\|_2^2}, \quad k = 0, 1, 2, \ldots,$$

där

— värdena på *relaxationsparametern* λ_k väljs så att $0 < \varepsilon_1 \leq \lambda_k \leq 2 - \varepsilon_2 < 2$ gäller för alla k

— $UBD_k \geq h^*$, dvs UBD_k utgör en *optimistisk uppskattning* av h^*

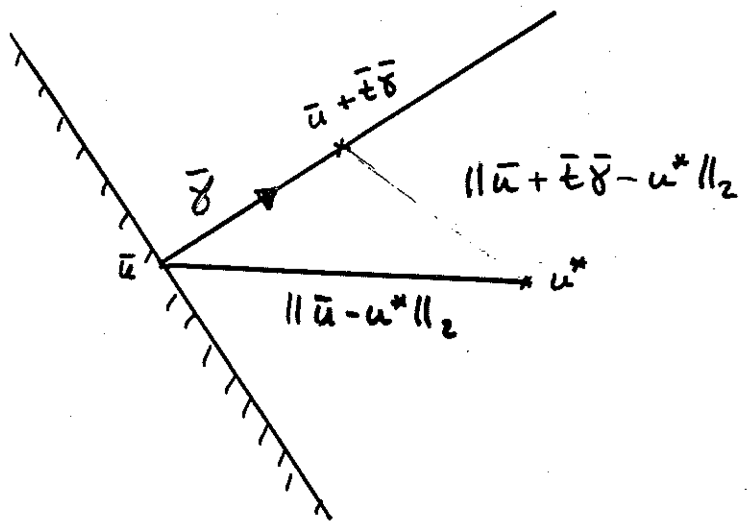
— UBD_k kan vara *iterationsoberoende*, dvs för alla k väljs $UBD_k = UBD$

— vanligen ges UBD_k av den bästa kända primala tillåtna lösningen (som typiskt är heuristiskt genererad)

— relaxationsparametern kan till exempel väljas enligt $\lambda_0 = 2 - \varepsilon_3$ och $\lambda_k = (1 - \varepsilon_4)\lambda_{k-1}$, $k = 1, 2, \ldots$, där $\varepsilon_3, \varepsilon_4 \geq 0$ är små.

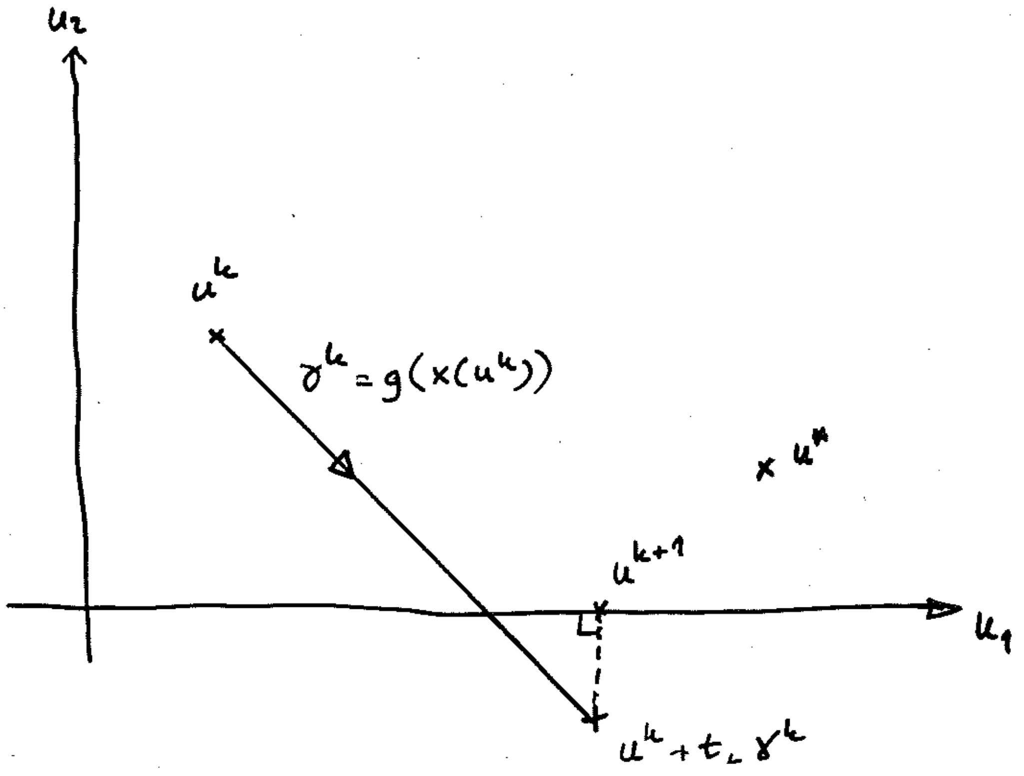
$$\boxed{\textbf{SATS}\quad \left.\begin{array}{l} u^* \text{ optimal i } (D) \\ \bar{u} \geq 0 \\ \bar{\gamma} \in \partial h(\bar{u}) \end{array}\right\} \; u^* \in \{ u \mid \bar{\gamma}^T (u - \bar{u}) \geq 0 \}}$$

Tolkning: $\bar{\gamma}$ pekar in i ett halvrum där u^* ligger.



$\bar{t}$ = steglängd

Konsekvens: Avståndet till u^* minskar om steget $\bar{t} > 0$ längs $\bar{\gamma}$ är tillräckligt litet.



Ex 1 $\quad \min\ f(x) = (x_1-7)^2 + (x_2-1)^2$

då $\quad g(x) = x_1 + 2x_2 - 4 \leq 0$

$\quad x \in R^2$

$x^* = \begin{pmatrix} 6 \\ -1 \end{pmatrix} \quad f^* = 5$

$$\boxed{h(u) = -4u + \min_{x_1 \in R} \{ (x_1-7)^2 + u x_1 \} + \min_{x_2 \in R} \{ (x_2 - 1) + 2u x_2 \}}$$

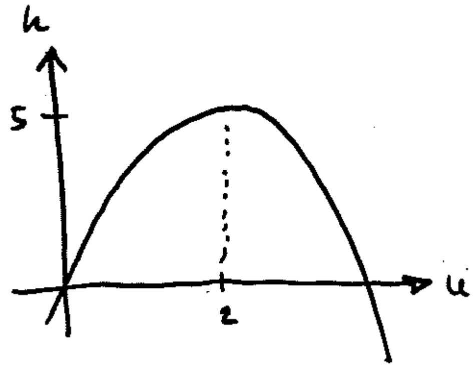
h kan tecknas explicit.

$$\frac{\partial}{\partial x_1} \{ (x_1-7)^2 + u x_1 \} = 2(x_1 - 7) + u = 0 \quad \Rightarrow \quad x_1(u) = 7 - \frac{u}{2}$$

$\ldots \qquad x_2(u) = 1 - u$

$$\boxed{x(u) = \begin{pmatrix} 7 - \frac{u}{2} \\ 1 - u \end{pmatrix}}$$

Insättning ger $\quad h(u) = 5u(1 - \frac{1}{4}u)$



$\max\limits_{u \geq 0} h(u) \quad \Rightarrow \quad u^* = 2 \quad h^* = 5$

$$\boxed{x(u^*) = \begin{pmatrix} 7 - \frac{u^*}{2} \\ 1 - u^* \end{pmatrix} = \begin{pmatrix} 6 \\ -1 \end{pmatrix}}$$

$x^* = x(u^*) = \begin{pmatrix} 6 \\ -1 \end{pmatrix}$ löser (P).

$g(x^*) = x_1^* + 2x_2^* - 4 = 0 \leq 0 \quad$ ok

$f(x^*) = (x_1^* - 7)^2 + (x_2^* - 1)^2 = 5 = h^* \quad$ ok

Ex2

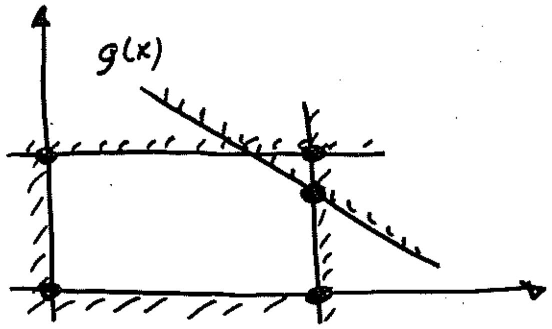
$$\max \; f(x) = x_1 + x_2$$

då $2x_1 + 3x_2 \le 6 \;:\; g(x) \le 0 \mid u \ge 0$

$x_1 \le 2$

$x_2 \le 1$

$x_1, x_2 \ge 0$



$x^* = \begin{pmatrix} 2 \\ 2/3 \end{pmatrix}$

$f^* = 8/3$

$$L(x,u) = x_1 + x_2 - u(2x_1 + 3x_2 - 6)$$

maxproblem

$$h(u) = \max_{x \in X} L(x,u) = \max \; 6u + (1-2u)x_1 + (1-3u)x_2$$

då $x_1 \le 2$

$x_2 \le 1$

$x_1, x_2 \ge 0$

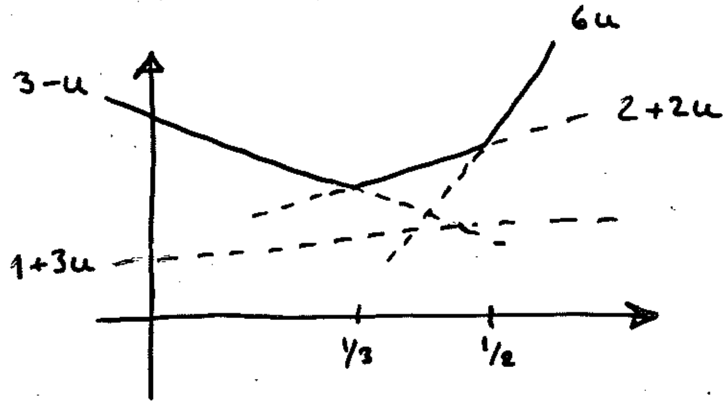
$(0,0)$: $6u$

$(2,0)$: $2+2u$

$(0,1)$: $1+3u$

$(2,1)$: $3-u$

$$h(u) = \max\{6u, 2+2u, 1+3u, 3-u\}$$



$u^* = \frac{1}{3} \Rightarrow (2,1)$ och $(2,0)$ optimala $\Rightarrow x_1^* = 2 \quad 0 \le x_2^* \le 1$

$h^* = \frac{8}{3}$

- $u^{*T} g(x^*) = 0 \quad \Rightarrow \quad \underbrace{\frac{1}{3}}_{>0} \left(2x_1^* + 3x_2^* - 6\right) = 0 \quad \Rightarrow$

$\Rightarrow \; 2x_1^* + 3x_2^* = 6 \quad \underset{x_1^* = 2}{\Rightarrow} \quad x_2^* = \frac{2}{3} \in [0,1]$ ok!

- $g(x^*) = 0 \quad \leq 0$ ok!

$\therefore \quad x^* = \begin{pmatrix} 2 \\ 2/3 \end{pmatrix} \quad f(x^*) = \frac{8}{3} \quad = h^*$

Ex 3 $\quad \max \; f(x) = 3x_1 + 5x_2 + 7x_3$

$g(x) = x_1 + 2x_2 + 3x_3 - 4 \leq 0 \quad | \; u \geq 0$

$x \in X = \{0,1\}^3$

$x^* = (1,0,1) \quad f^* = 10$

$$\boxed{h(u) = 4u + \max_{x_1 = 0/1} (3-u)x_1 + \max_{x_2 = 0/1} (5-2u)x_2 + \max_{x_3 = 0/1} (7-3u)x_3}$$

$$h(u) = \begin{cases} 15 - 2u & 0 \leq u \leq 7/3 & x(u) = (1,1,1) \\ 8 + u & 7/3 \leq u \leq 5/2 & x(u) = (1,1,0) \\ 3 + 3u & 5/2 \leq u \leq 3 & x(u) = (1,0,0) \\ 4u & u \geq 3 & x(u) = (0,0,0) \end{cases}$$

$\min_{u \geq 0} h(u) \quad \Rightarrow \quad u^* = \frac{7}{3} \qquad h^* = 10\,1/3$

Dualgap $\quad h^* - f^* = 10\,1/3 - 10 = 1/3$

$X(u^*) = \{(1,1,1), (1,1,0)\} \qquad g(1,1,1) = 2 \nleq 0 \Rightarrow (1,1,1)$ otillåten

$g(1,1,0) = -1 \leq 0 \Rightarrow (1,1,0)$ tillåten

$u^{*T} g(1,1,0) = \frac{7}{3} \cdot (-1) \neq 0 \quad x^*$ kan ej bestämmas så här, pga att $h^* > f$.