Penalty method

$$\boxed{\begin{array}{l} \min f(x) \\ \text{s.t. } g(x) \le 0 \\ \quad\;\; h(x) = 0 \end{array}} \implies \boxed{\begin{array}{l} \min F(x,c) \\ x \in R^n \end{array}}$$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

x^* is approximated by $x(c)$

Denote: $a^+ = \begin{cases} a, & a \geq 0 \\ 0, & a < 0 \end{cases}$

Penalty function:

$$P(x) = \sum_{i=1}^{l} (h_i(x))^2 + \sum_{i=1}^{m} (g_i^+(x))^2$$

Properties:

- $P(x) = 0 \quad \forall x \in X$
- $P(x) > 0 \quad \forall x \,\bar{\in}\, X$

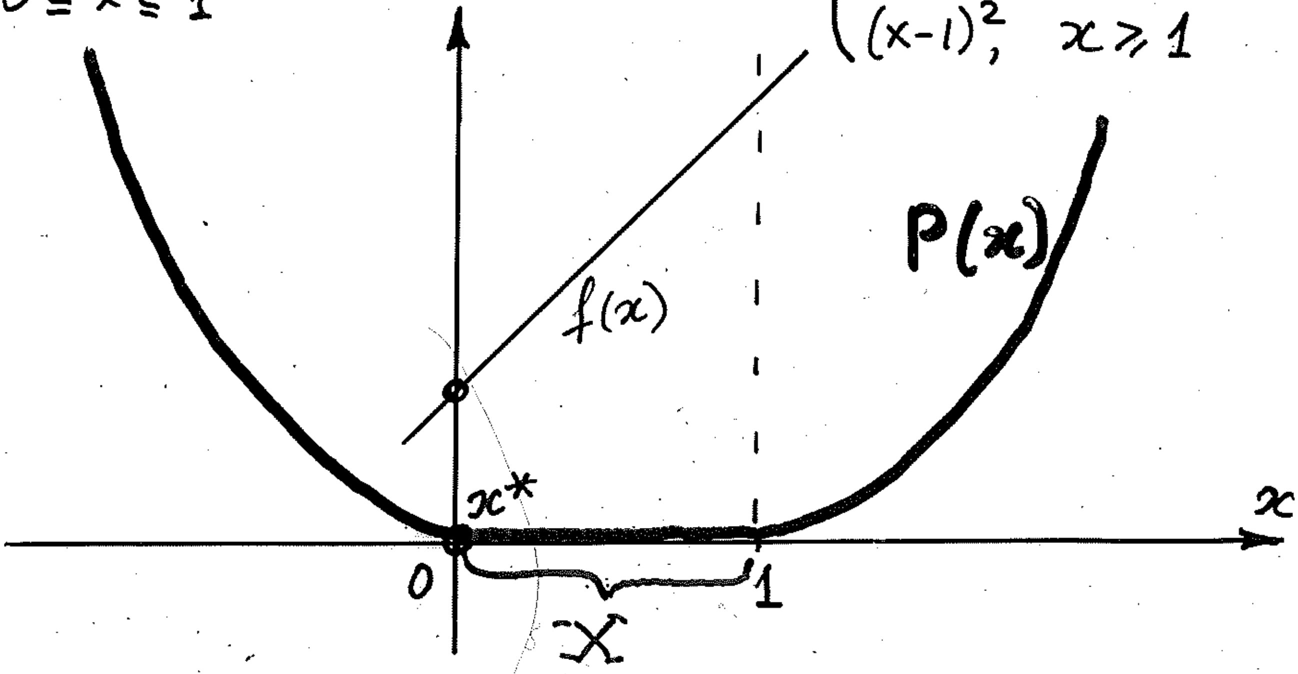
New objective function:

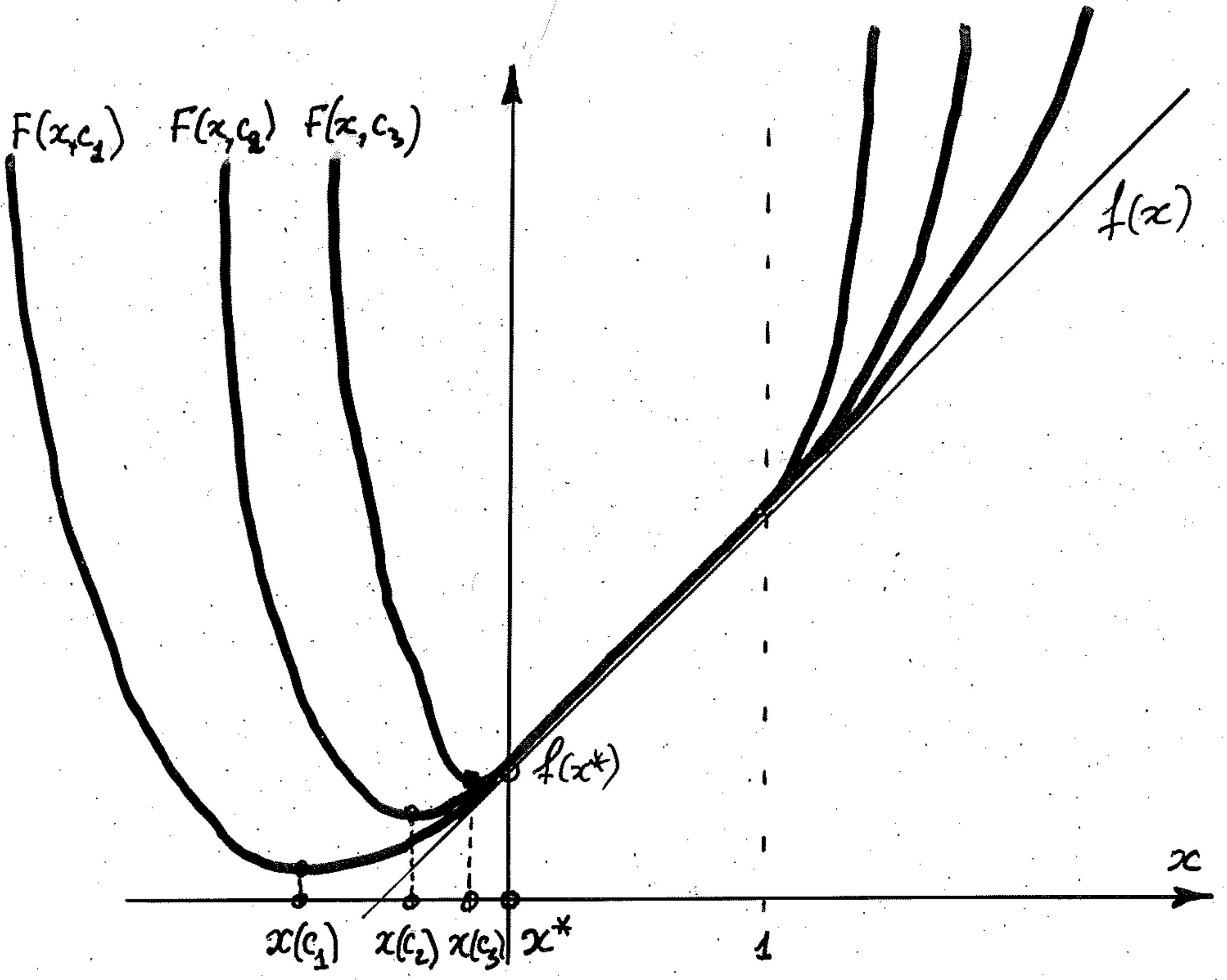
$$F(x,c) = f(x) + c\,P(x), \quad c > 0$$

f, g, h are smooth $\implies F(x,c)$ is smoot

$$\min\ x + 1/2$$
$$0 \le x \le 1$$

$$P(x) = \begin{cases} x^2, & x \le 0 \\ 0, & 0 \le x \le 1 \\ (x-1)^2, & x \ge 1 \end{cases}$$





$$c_1 < c_2 < c_3$$

$$\boxed{\begin{array}{c} x(c) \to x^* \\ c \to +\infty \end{array}}$$

Advantages:

- NLP $\Rightarrow$ UCM (smooth)
- $x(c) \approx x^*$

Disadvantages:

- c should be large enough to give a good approximation $x(c)$ to x^*
- $F''(x,c)$ changes too sharply at the boundary of $\underline{X}$
- UCM methods slow down with $c \to \infty$

Recommendations:

- Solve sequentially $\min F(x,c)$ for a few increasing values of $c = c_1, c_2, c_3, \dots$
- Use $x(c_i)$ as an initial point in $\min F(x, c_{i+1})$
- Don't solve the UCM problems with too high accuracy
- Refine the obtained (final) approximation to x^* with a faster (more accurate) NLP method.

Barrier methods

$$\boxed{\begin{array}{l} \min f(x) \\ \text{s.t. } g(x) \le 0 \end{array}} \implies \boxed{\min_{x \in \mathbb{R}^n} F(x, c)}$$

$$x^* \approx x(c)$$

Barrier functions:

$$P(x) = \sum_{i=1}^{m} -\ln(-g_i(x))$$

$$P(x) = \sum_{i=1}^{m} -\frac{1}{g_i(x)}$$

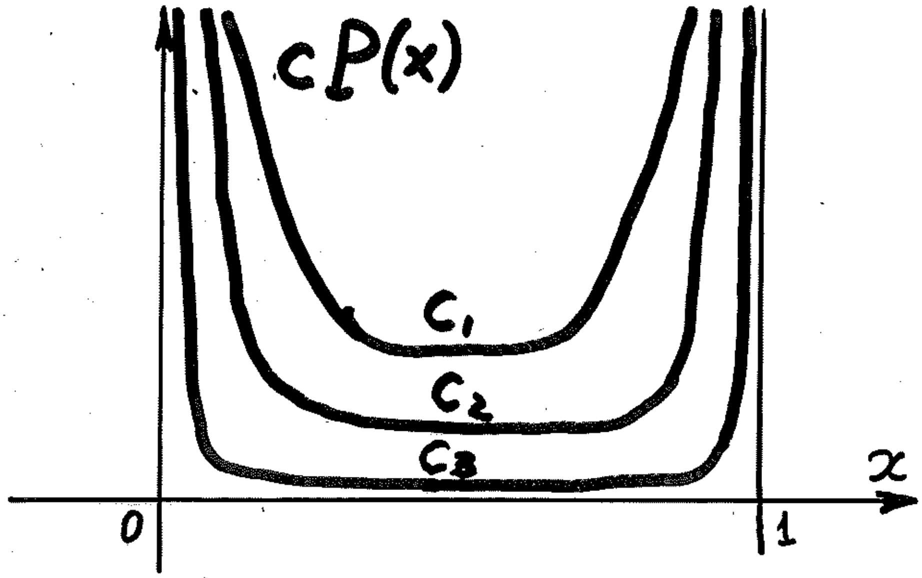
Properties:

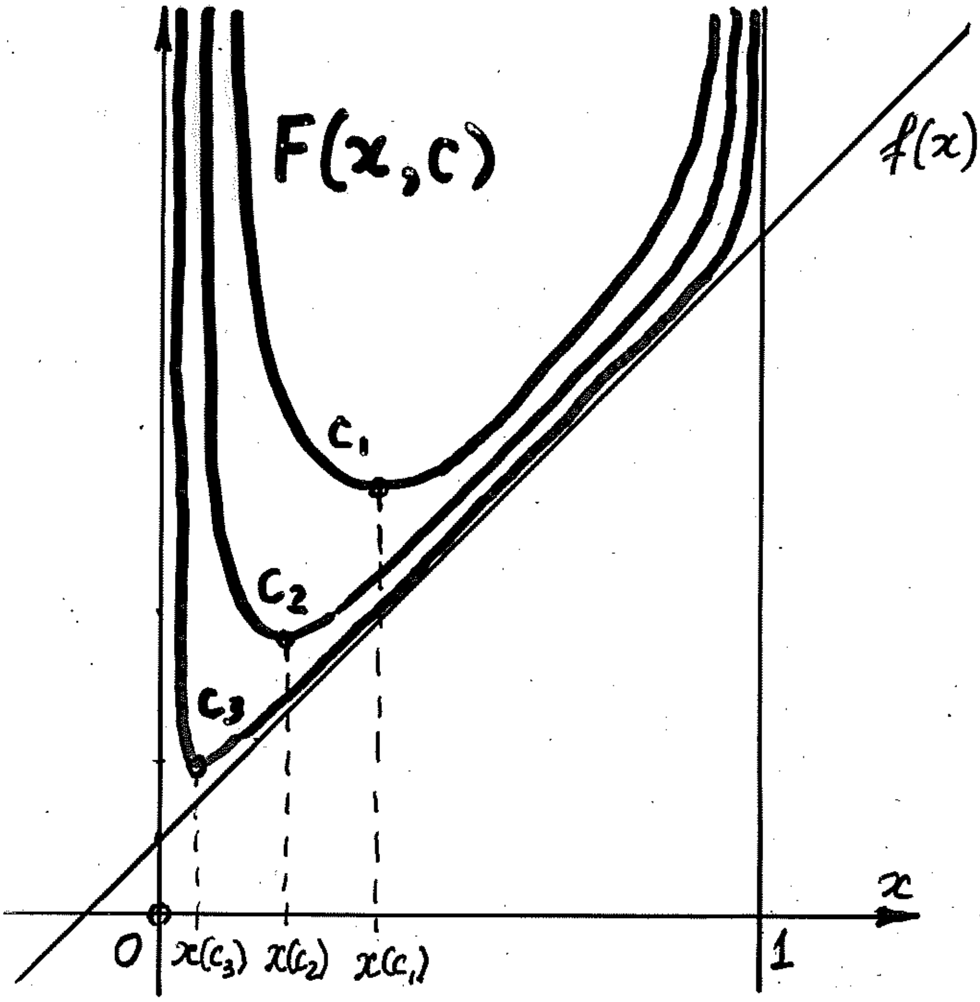
- $P(x) \to +\infty, \quad g_i \to -0$
- $P(x)$ is smooth for all $x: \; g(x) < 0$

$$\min_{x \in \mathbb{R}^n} F(x, c) = f(x) + c\,P(x)$$

$$\boxed{\begin{array}{c} f \text{ and } g \\ \text{are smooth} \end{array}} \overset{\forall c \geq 0}{\implies} \boxed{\begin{array}{l} F(x, c) \text{ is smooth} \\ \forall x: \; g(x) < 0 \end{array}}$$

$$\begin{array}{ll} \min & x + 1/8 \\ \text{s.t.} & 0 \leq x \leq 1 \end{array} \quad \Longrightarrow \quad \begin{array}{l} P(x) = -\ln x - \ln(1-x) \\ F(x,c) = x + 1/8 + c[-\ln x - \ln(1-x)] \end{array}$$





$$\begin{array}{l} x(c) \to x^* \\ F(x(c),c) \to f(x^*) \end{array} , \quad c \to 0$$