Group A

1–3 Figures 18–20 show the networks for Problems 1–3. Find the maximum flow from source to sink in each network. Find a cut in the network whose capacity equals the maximum flow in the network. Also, set up an LP that could be used to determine the maximum flow in the network.

FIGURE 18
Network for Problem 1

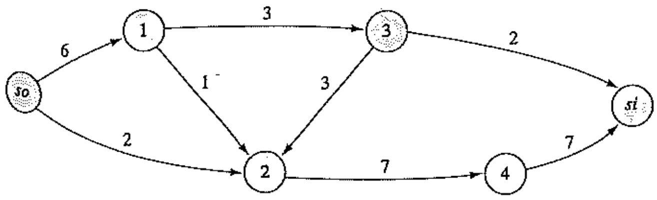


FIGURE 19
Network for Problem 2

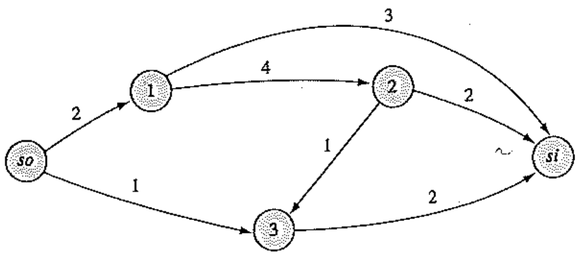


FIGURE 20
Nework for Problem 3

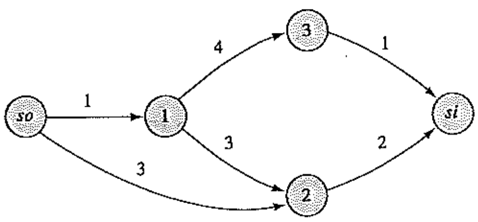


FIGURE 21
Network for Problem 4

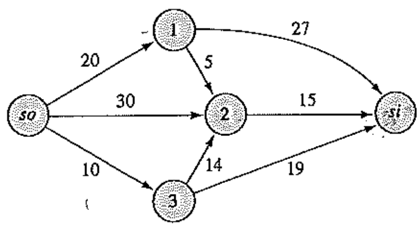
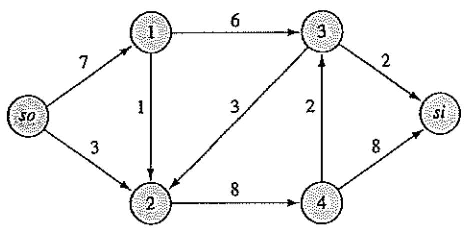


FIGURE 22
Network for Problem 5



4–5 For the networks in Figures 21 and 22, find the maximum flow from source to sink. Also find a cut whose capacity equals the maximum flow in the network.

6 Seven types of packages are to be delivered by five trucks. There are three packages of each type, and the capacities of the five trucks are 6, 4, 5, 4, and 3 packages, respectively. Set up a maximum-flow problem that can be used to determine whether the packages can be loaded so that no truck carries two packages of the same type.

7 Four workers are available to perform jobs 1–4. Unfortunately, three workers can do only certain jobs: worker 1, only job 1; worker 2, only jobs 1 and 2; worker 3, only job 2; worker 4, any job. Draw the network for the maximum-flow problem that can be used to determine whether all jobs can be assigned to a suitable worker.

8 The Hatfields, Montagues, McCoys, and Capulets are going on their annual family picnic. Four cars are available to transport the families to the picnic. The cars can carry the following number of people: car 1, four; car 2, three; car 3, three; and car 4, four. There are four people in each family, and no car can carry more than two people from any one family. Formulate the problem of transporting the maximum possible number of people to the picnic as a maximum-flow problem.

9–10 For the networks in Figures 23 and 24, find the maximum flow from source to sink. Also find a cut whose capacity equals the maximum flow in the network.

Group B

11 Suppose a network contains a finite number of arcs and the capacity of each arc is an integer. Explain why the Ford–Fulkerson method will find the maximum flow in the finite number of steps. Also show that the maximum flow from source to sink will be an integer.

12 Consider a network flow problem with several sources and several sinks in which the goal is to maximize the total flow into the sinks. Show how such a problem can be converted into a maximum flow problem having only a single source and a single sink.

FIGURE 23

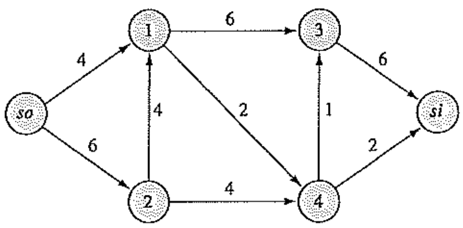
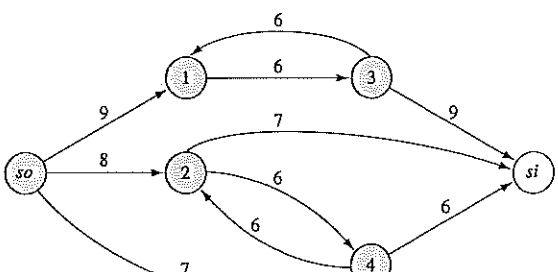


FIGURE 24



SOLUTIONS

1. max $z = x_0$

s.t. $x_{so,1} \leq 6$, $x_{so,2} \leq 2$ $x_{12} \leq 1$, $x_{32} \leq 3$, $x_{13} \leq 3$, $x_{3,si} \leq 2$, $x_{24} \leq 7$, $x_{4,si} \leq 7$

$x_0 = x_{so,1} + x_{so,2}$ (Node so)
$x_{so,1} = x_{13} + x_{12}$ (Node 1)
$x_{12} + x_{32} + x_{s0,2} = x_{24}$ (Node 2)
$x_{13} = x_{32} + x_{3,si}$ (Node 3)
$x_{24} = x_{4,si}$ (Node 4)
$x_{3,si} + x_{4,si} = x_0$ (Node si)
All variables ≥ 0

Initial flow of 0 in each arc. Begin by labeling sink via path of forward arcs (so, 1) - (1, 3) - (3, 2) - (2, 4) - (4, si). Increase flow in each of these feasible arcs by 3, yielding the following feasible flow:

Arc	Flow
so-1	3
so-2	0
1-3	3
1-2	0
2-4	3
3-si	0
3-2	3
4-si	3
Flow to sink 3	

Now label sink by (so-2) - (2-4), (4, si). Each arc is a forward arc and we can increase flow in each arc by 2. This yields the following feasible flow:

Arc	Flow
so-1	3
so-2	2
1-3	3
1-2	0
2-4	5
3-si	0
3-2	3
4-si	5
Flow to sink 5	

Now label sink by (so-1) - (1, 2) - (3, 2) - (3, si). All arcs on this path are forward arcs except for (3, 2), which is a backwards arc. We can increase the flow on each forward arc by 1 and decrease the flow on each backward arc by 1. This yields the following feasible flow:

Arc	Flow
so-1	4
so-2	2
1-3	3

1-2	1
2-4	5
3-si	1
3-2	2
4-si	5

Flow to sink 6

The sink cannot be labeled, so we found the maximum flow of 6 units. The minimum cut is obtained from V' = {3, 2, 4, si}. This cut consists of arcs (1, 3), (1, 2), (so, 2) and has capacity of 3 + 1 + 2 = 6 = maximum flow.

2. max $z = x_0$

s. t. $x_{so,1} \le 2$, $x_{12} \le 4$, $x_{1,si} \le 3$, $x_{2,si} \le 2$, $x_{23} \le 1$, $x_{3,si} \le 2$, $x_{so,3} \le 1$

$x_0 = x_{so,1} + x_{so,3}$ (Node so)

$x_{so,1} = x_{1,si} + x_{12}$ (Node 1)

$x_{12} = x_{23} + x_{2,si}$ (Node 2)

$x_{23} + x_{so,3} = x_{3,si}$ (Node 3)

$x_{1,si} + x_{2,si} + x_{3,si} = x_0$ (Node si)

All variables ≥ 0

We begin with a flow of 0 through each arc. Label the sink by (so, 1)-(1, 2)-(2, 3)-(3, si). We can increase the flow on each of these arcs by one unit, obtaining the following feasible flow:

Arc	Flow
(so,1)	1
(so,3)	0
(1,2)	1
(1,si)	0
(2,3)	1
(2,si)	0
(3,si)	1
Flow to Sink	1

We next label the sink by (so, 1)-(1, 2)-(2, si) and increase the flow in each of these arcs by 1 unit. We obtain the following feasible flow:

Arc	Flow
(so,1)	2
(so,3)	0
(1,2)	2
(1,si)	0
(2,3)	1
(2,si)	1
(3,si)	1
Flow to Sink	2

Now we label the sink by (so, 3)-(3, si), and increase the flow in each of these arcs by 1 unit. We obtain the following feasible flow:

Arc	Flow
(so,1)	2
(so,3)	1
(1,2)	2
(1,si)	0
(2,3)	1
(2,si)	1
(3,si)	2
Flow to sink	3

Now the sink cannot be labeled, so we have obtained a maximal flow. V' = {1, 2, 3, si} yields the minimal cut (arcs (so, 1) and (so, 3)) with a capacity of 2+1=3 = maximal flow.

3. max $z = x_0$

s.t. $x_{so,1} \le 1$, $x_{so,2} \le 3$, $x_{13} \le 4$, $x_{12} \le 3$, $x_{3,si} \le 1$, $x_{2,si} \le 2$

$x_0 = x_{so,1} + x_{so,2}$ (Node so)

$x_{so,1} = x_{13} + x_{12}$ (Node 1)

$x_{12} + x_{so,2} = x_{2,si}$ (Node 2)

$x_{13} = x_{3,si}$ (Node 3)

$x_{3,si} + x_{2,si} = x_0$ (Node si)

All variables ≥ 0

We begin with a flow of 0 through each arc and label the sink by (so-2)-(2-si). We increase the flow in each of these arcs by 2 units. This yields the following feasible flow.

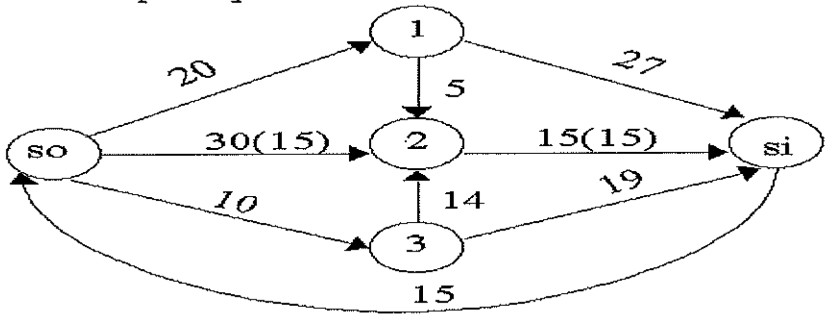
so-1	0
so-2	2
1-2	0
1-3	0
2-si	2
3-si	0
Flow to Sink	2

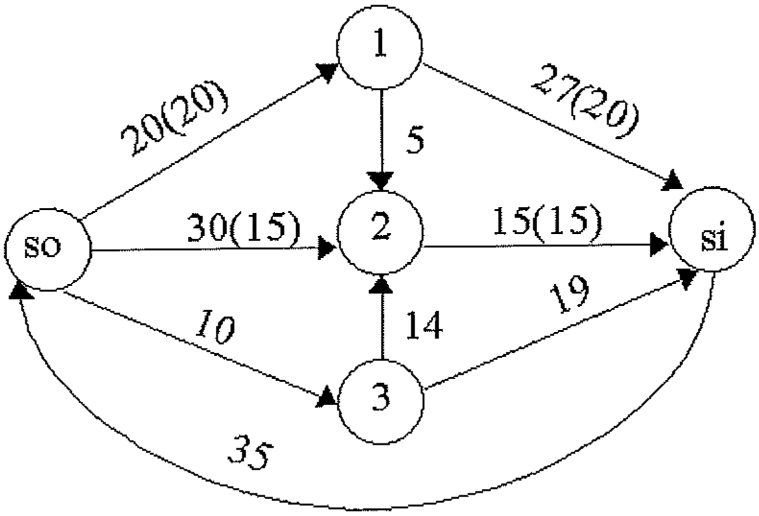
Now label the sink by the chain (so, 1)-(1, 3)-(3, si) and increase the flow in each of these arcs by 1 unit. This yields the following feasible flow:

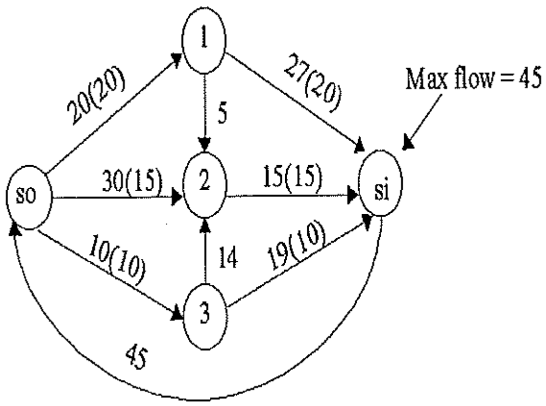
so-1	1
so-2	2
1-2	0
1-3	1
2-si	2
3-si	1
Flow to Sink	3

The sink cannot be labeled, so we have obtained a maximal flow. V' = {si} yields the minimal cut (3, si), (2, si) with a capacity of 1+2 = 3 = maximum flow.

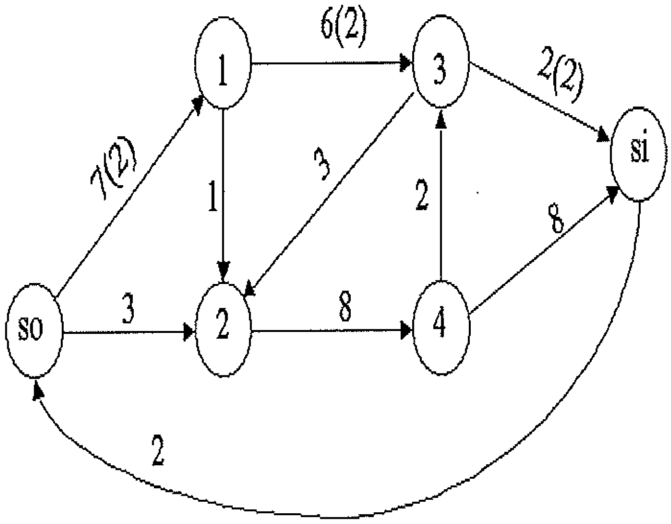
4. Maximum flow is 45. Min Cut Set = {1, 3, and si}. Capacity of Cut Set = 20 + 15 + 10 = 45. See Figure.

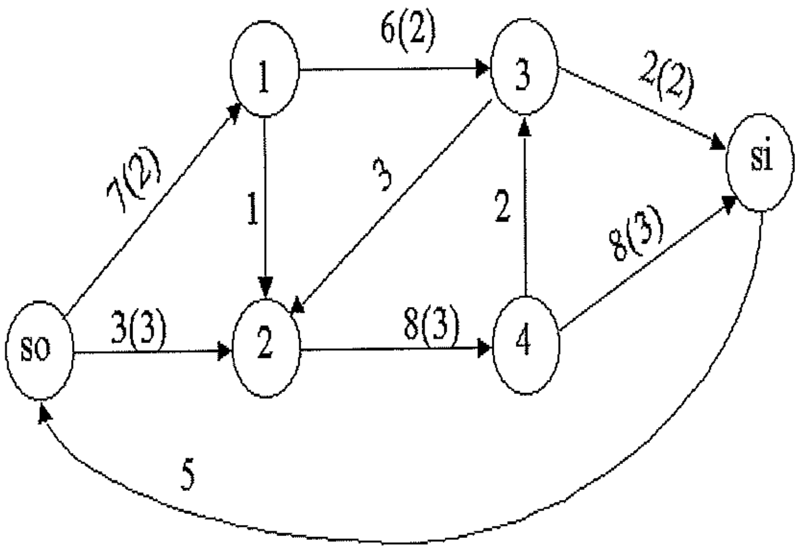


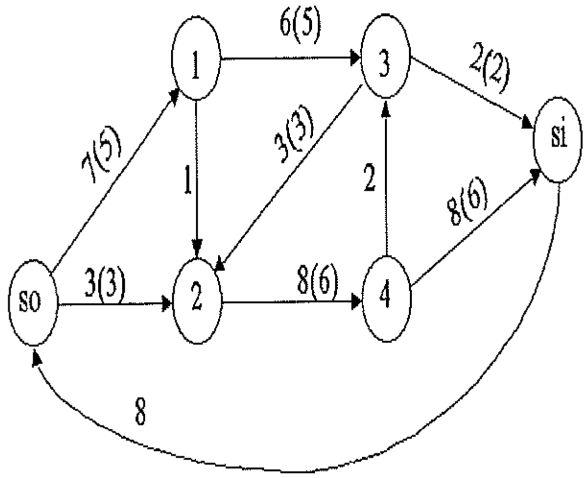


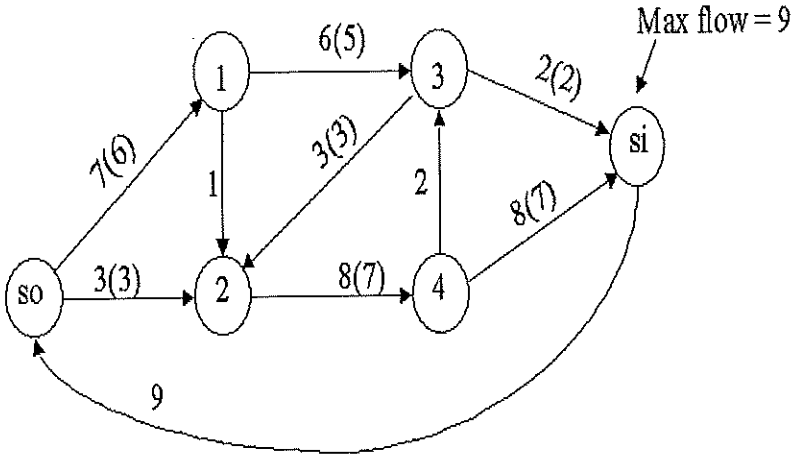


5. Maximum flow = 9. Min Cut Set = {2, 4, si}. Capacity of Cut = 3 + 1 + 3 + 2 = 9. See Figure.

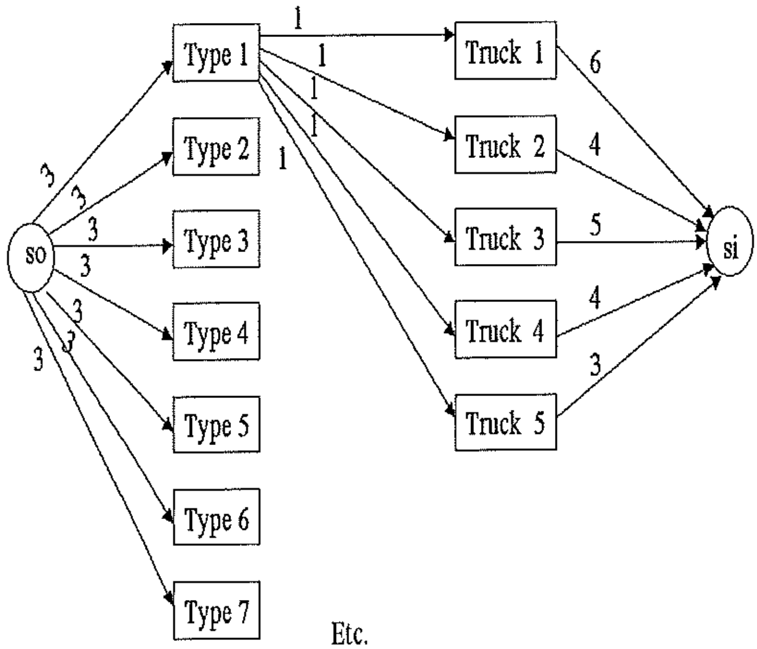

6(2)
2(2)
7(2)
1
3
2
8
si
so
3
8
1
2
3
4
2


6(2)
2(2)
7(2)
1
3
2
8(3)
si
so
3(3)
8(3)
1
2
3
4
5



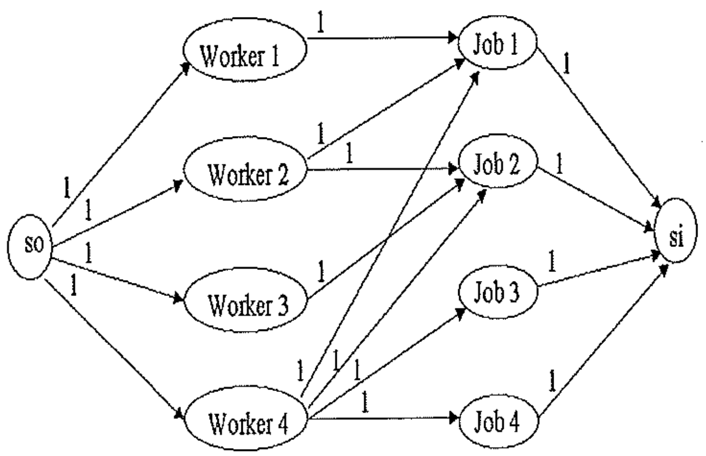


6. See figure. If the maximum flow = 21, then all packages can be loaded.



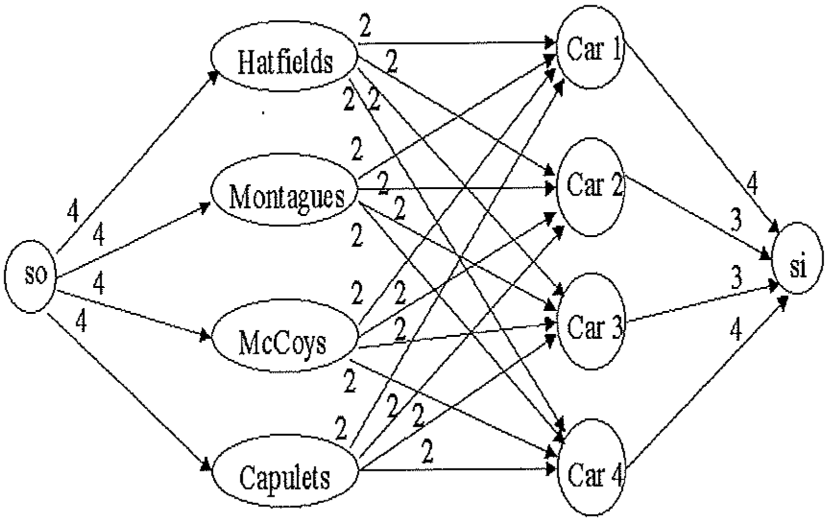
An arc of capacity 1 goes from each 'Type' node to each 'Truck' node.

7. See figure.



If max flow = 4 then all jobs can be completed.

8. See figure.



9. Step 1-Add 4 units of flow along so-1-3-si
   Step 2-Add 2 units of flow along so-2-4-si
   Step 3-Add 1 unit of flow along so-2-4-3-si
   Step 4-Add 1 unit along so-2-1-3-si
   Cannot label si. Maximum flow = 8 Arcs in cut are 3-si and 4-si with capacity of 8. Flow along each arc is as follows:

Arc	Flow
so,1	4
so,2	4
2,1	1
1,3	5
1,4	0
2,4	3
3,si	6
4,3	1
4,si	2

10. Step 1-Add 7 units along so-2-si
    Step 2-Add 6 units along so-1-3-si
    Step 3 Add 6 units along so-4-si

Cannot label si. Maximal flow = 19. Cut set is derived from V' = {si} and consists of arcs (3, si), (2, si), (4, si) with capacity 6+7+6=19

Arc	Flow
so,1	6
so,2	7
so,4	6
1,3	6
2,si	7
3,si	6
4,si	6

11. Suppose maximum flow = k. At each step the flow to the sink increases by an integer amount of at least one unit. Thus at most k iterations of the Ford-Fulkerson method will yield the maximum flow. Also, after each iteration the flow to the sink is an integer, so the optimal flow will be an integer.

12. Construct a "supersource" that has arcs of infinite capacity leading to each real source. Also construct a "supersink' that has infinite capacity arcs leading from each real sink to the supersink.