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TAOP24, Optimization
Examination
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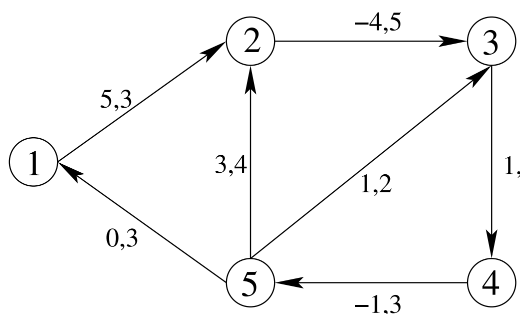
Assignment 1

In the graph below, the arcs are labeled with cost and upper bound as follows $\xrightarrow{c,u}$. The lower bounds are all zero. All the nodes are transshipment nodes, which means that there are no sources or sinks.

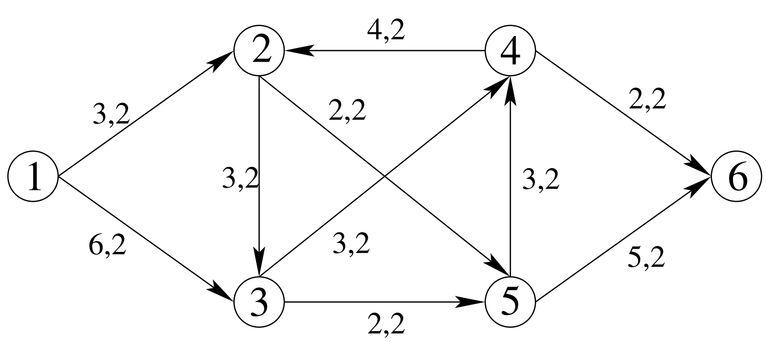


(3p) a) Make two iterations of the network simplex method starting from the zero flow ($x_{ij} = 0$ for all (i, j)). Choose as the starting basic feasible solution the one in which x_{51}, x_{53} and x_{45} are non-basic variables.

(1p) b) Is the obtained flow optimal or not? Motivate why.

Assignment 2

Consider the maximum flow problem defined by the graph in the figure below, in which the arcs are labeled with capacity and flow as follows $\xrightarrow{u,x}$. The source is node 1, and the sink is node 6.



(2p) a) Find capacity of the cut $N_s = \{1, 2, 5\}$. Find flow across this cut.

(3p) b) Use the max-flow algorithm for finding the maximum flow from node 1 to node 6. Start from the given flow. Present the results of using the modified Dijkstra's algorithm for finding a flow increasing path from node 1 to node 6. Show how arc's feasible directions are updated.

(2p) c) Find a minimum cut in this graph.

Assignment 3

Consider the following integer knapsack problem.

$$\begin{array}{ll} \max & 2x_1 + x_2 + 3x_3 \\ \text{s.t.} & 5x_1 + 4x_2 + 6x_3 \leq 10 \\ & x_1, x_2, x_3 \geq 0, \quad \text{integer} \end{array}$$

(3p) a) Use dynamic programming to find at least *one* optimal solution.

(1p) b) Use dynamic programming to find all optimal solutions. Explain how the results of a) are used here.

Assignment 4

Consider the convex function

$$f(x) = \begin{cases} x_1^2 - 2x_1 + x_2^2, & x_1 < 0, \\ x_1 + x_2^2, & x_1 \geq 0. \end{cases}$$

(5p) a) Show that $g_1 = (1,\ 0)^T$ and $g_2 = (-2,\ 0)^T$ are subgradients of $f(x)$ in the point $\bar{x} = (0,\ 0)^T$.

(2p) b) Use the results of a) to show that $\bar{x}$ is a global minimizer of $f(x)$.

Assignment 5

(5p) a) The conjugate gradient method is applied to the minimization of a function of three variables. Initially, $\nabla f(x_0) = (-1,\ 1,\ 1)^T$. After one iteration, the first two components of $\nabla f(x_1)$ are 1 and -1, respectively. Find the search direction d_1.

(3p) b) Consider the problem of minimizing the function $f(x) = 2x_1 + x_1x_2 + 2x_2$. Use the Levenberg-Marquardt method to generate a search direction at the point $x^{(0)} = (1,\ 1)^T$.

Assignment 6 (3p)

Consider the quadratic programming problem

$$\begin{array}{ll} \min & f(x) = (x_1^2 + x_2^2)/2 \\ \text{s.t.} & x_2 \leq x_1 \\ & x_1 \geq 1 \\ & x_1 + x_2 \geq 0 \end{array}$$

Solve it using the active set method. Start from the point $x^{(0)} = (2,\ 2)^T$ and the active set in this point.

Assignment 7

Consider the problem

$$\begin{array}{ll} \min & 1 - x \\ \text{s.t.} & -1 \leq x \leq 1 \end{array}$$

Let x_* denote the solution to this problem. The quadratic penalty method and the logarithmic barrier method reduce this constrained optimization problem to a sequence of unconstrained minimization of a function $F(x, c)$ for a series of values of the parameter c. Let $x(c)$ denote a minimizer of $F(x, c)$.

(3p) a) Find $x(c)$ for the quadratic penalty method and show that $x(c) \to x_*$.

(3p) b) Find $x(c)$ for the logarithmic barrier method and show that $x(c) \to x_*$.

(1p) c) Consider the problem in Assignment 6. For the quadratic penalty method, find the value of $F(x, 1)$ in the point $x = (1,\ 3)^T$.

ANSWERS

Assignment 1

a) The basic tree is composed of the edges (1,2), (2,3), (3.4), (5,2). The node prices $y = (0,\ 5,\ 1,\ 2,\ 2)^T$. Reduced costs and optimality:
$\bar{c}_{51} = 2$ & $x_{51} = l_{51} \Rightarrow$ OK
$\bar{c}_{45} = -1$ & $x_{45} = l_{45} \Rightarrow$ NOT OK
$\bar{c}_{53} = 2$ & $x_{53} = l_{53} \Rightarrow$ OK
Then x_{45} enters the basis. This results in the cycle 2—3—4—5—2. It is possible to send 3 unit in the cycle with x_{45} exiting the basis. (This variable moves from the lower bound to the upper bound.) The basic tree does not change.

At the next iteration, the node prices $y = (0,\ 5,\ 1,\ 2,\ 2)^T$ do not change. Reduced costs and optimality:
$\bar{c}_{51} = 2$ & $x_{51} = l_{51} \Rightarrow$ OK
$\bar{c}_{45} = -1$ & $x_{45} = u_{45} \Rightarrow$ OK
$\bar{c}_{53} = 2$ & $x_{53} = l_{53} \Rightarrow$ OK

b) The reduced costs show that the optimality conditions are satisfied.

Assignment 2

a) The capacity of the given cut is equal to $6 + 3 + 3 + 5 = 17$. The flow across this cut is equal to $2 + 2 + 2 + 2 - 2 - 2 = 4$.

b) The modified Dijkstra's algorithm gives the flow increasing path 1—3—2—4—5—6 whose capacity is 2. Then $f = 6$. The modified Dijkstra's algorithm shows that there is no flow increasing path. This means that the obtained flow is maximal.

c) A minimum cut is defined by $N_s = \{1, 2, 3, 4\}$. Its capacity is is equal to 6 (= max flow).

Assignment 3

The dynamic programming algorithm gives the two optimal solutions:
$x' = (2,\ 0,\ 0)^T$ and $x'' = (0,\ 1,\ 1)^T$.

Assignment 4

a) By the definition of $g \in \partial f(\bar{x})$,

$$f(x) \geq f(\bar{x}) + g^T(x - \bar{x}), \quad \forall x \in R^n. \tag{1}$$

First, consider $g = g_1$, for which (1) takes the form:

$$f(x) \geq (1,\ 0)x, \quad \forall x \in R^2.$$

If $x \in R^2$ is any vector such that $x_1 < 0$, then $x_1^2 - 2x_1 + x_2^2 \geq x_1$ holds. If $x \in R^2$ is any vector such that $x_1 \geq 0$, then $x_1 + x_2^2 \geq x_1$ holds. Thus, $g_1 \in \partial f(\bar{x})$.
Second, consider $g = g_2$, for which (1) takes the form:

$$f(x) \geq (-2,\ 0)x, \quad \forall x \in R^2.$$

If $x \in R^2$ is any vector such that $x_1 < 0$, then $x_1^2 - 2x_1 + x_2^2 \geq -2x_1$ holds. If $x \in R^2$ is any vector such that $x_1 \geq 0$, then $x_1 + x_2^2 \geq -2x_1$ holds. Thus, $g_2 \in \partial f(\bar{x})$.

b) We know that $\partial f(\bar{x})$ is a convex set, i.e. if $g_1, g_2 \in \partial f(\bar{x})$, then $\lambda g_1 + (1-\lambda) g_2 \in \partial f(\bar{x})$ for any $\lambda \in [0, 1]$. Obviously,

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 0 \end{pmatrix} + (1-\lambda) \begin{pmatrix} -2 \\ 0 \end{pmatrix}$$

holds for $\lambda = 2/3$, i.e. $0 \in \partial f(\bar{x})$. This means that $\bar{x}$ is a global minimizer of $f(x)$.

Assignment 5

a) The point $x_1 = x_0 + t_0 d_0$ is a minimizer of $f(x)$ along the direction $d_0 = -\nabla f(x_0)$, which means that $d_0^T \nabla f(x_1) = 0$. This equation defines the last component in $\nabla f(x_1) = (1,\ -1,\ 2)^T$. We can now calculate

$$d_1 = -\nabla f(x_1) + \frac{\|\nabla f(x_1)\|^2}{\|\nabla f(x_0)\|^2} d_0 = (1,\ -1,\ -4)^T.$$

b) $\nabla f(1,1) = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$. The Hessian matrix $H(1,1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ has the eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 1$. In the Levenberg-Marquardt method, the search direction $p = -G^{-1}\nabla f$, where $G = H + \gamma I$. Since, in our case, the minimal eigenvalue λ_1 is not positive, the scalar parameter γ should be chosen so that $\gamma > -\lambda_1$. If to choose $\gamma = 2$, then $p = (-1,\ -1)^T$.

Assignment 6

$x^* = (1,\ 0)^T$.

Assignment 7

a) In this problem,

$$F(x,c) = 1 - x + c \cdot \begin{cases} (x+1)^2, & \text{if } x < -1, \\ 0, & \text{if } -1 \leq x \leq 1, \\ (x-1)^2, & \text{if } 1 \leq x. \end{cases}$$

It is a continuously differentiable function with

$$F_x'(x,c) = \begin{cases} -1 + 2c(x+1), & \text{if } x < -1, \\ -1, & \text{if } -1 \leq x \leq 1, \\ -1 + 2c(x-1), & \text{if } 1 \leq x. \end{cases}$$

Therefore, the minimizer $x(c)$ should be a stationary point of $F(x,c)$, i.e.

$$F_x'(x(c), c) = 0.$$

- If to assume that $x(c) < -1$, the corresponding stationary point $x(c) = -1 + 1/(2c)$ breaks this assumption.

• The interval $-1 \leq x \leq 1$ has no stationary points.
• If to assume that $x(c) > 1$, the corresponding stationary point $x(c) = 1 + 1/(2c)$ belongs to this area.
Thus, there exists a unique stationary point $x(c) = 1 + 1/(2c)$. Note that $F(x, c)$ is monotonically decreasing ($F'_x(x, c) < 0$) when $x < x(c)$, and $F(x, c)$ is monotonically increasing ($F'_x(x, c) > 0$) when $x > x(c)$. This means that $x(c)$ is a unique global minimizer of $F(x, c)$.
Obviously, $x(c) \to x_* = 1$ with $c \to +\infty$.

b) $F(x, c) = 1 - x - c\ln(x + 1) - c\ln(1 - x)$ is a convex function of $x \in (-1, 1)$ for any $c > 0$. The first derivative of this function has two roots.
The root $-c - \sqrt{1 + c^2} < -1$, i.e. it is infeasible.
The root $-c + \sqrt{1 + c^2} \in (-1, 1)$, i.e. it is feasible.
Thus, $x(c) = -c + \sqrt{1 + c^2}$. Obviously, $x(c) \to x_* = -1$ with $c \to +0$.

c) We rewrite the inequality-type constraints in the form $g_i(x) \leq 0$, $i = 1, 2, 3$, where $g_1(x) = x_2 - x_1$, $g_2(x) = 1 - x_1$ and $g_3(x) = -x_1 - x_2$. The quadratic penalty function $P(x) = [g_1^+(x)]^2 + [g_2^+(x)]^2 + [g_3^+(x)]^2$. For $x = (1\ 3)^T$:

$$g_1(1,3) = 2 \implies g_1^+(1,3) = 2,$$
$$g_2(1,3) = 0 \implies g_2^+(1,3) = 0,$$
$$g_3(1,3) = -4 \implies g_3^+(1,3) = 0,$$

Thus, $P(1, 3) = 4$. For the given $x = (1\ 3)^T$, $F(x, 1) = f(x) + 1 \cdot P(x) = 9$.