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TAOP24, Optimization
Examination
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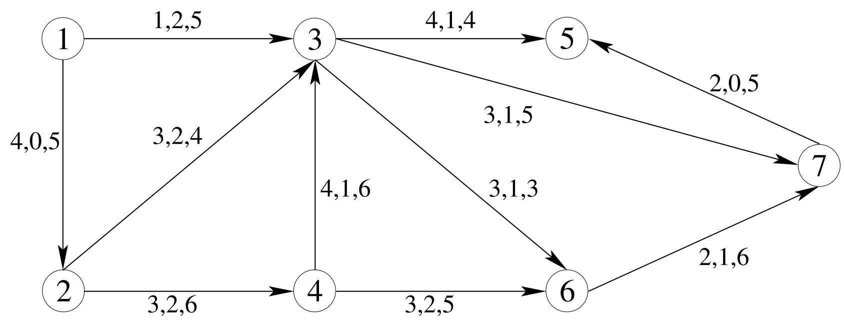
## Assignment 1

Consider the balanced network in the graph below in which the arcs are labeled with cost, lower and upper bounds as follows $\xrightarrow{c,l,u}$. Nodes 1 and 2 are sources of capacity 4 and 4. Nodes 5 and 7 are sinks of capacity 5 and 3, respectively.



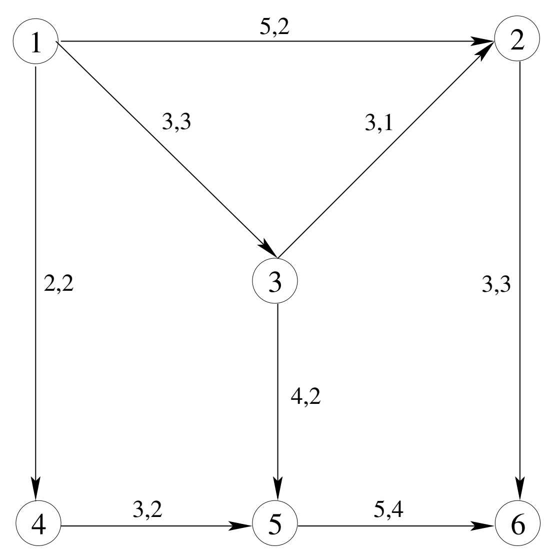

**(2p)** a) Consider the flow $x_{12} = 1$, $x_{13} = 3$, $x_{23} = 2$, $x_{24} = 3$, $x_{35} = 1$, $x_{36} = 3$, $x_{37} = 2$, $x_{43} = 1$, $x_{46} = 2$, $x_{67} = 5$, $x_{75} = 4$. Is it a basic feasible solution?

**(3p)** b) Use the network simplex method to solve the min-cost flow problem starting from the flow given in a). Make at most three iterations.

**(1p)** c) Is the obtained optimal flow unique?

## Assignment 2

Consider the maximum flow problem defined by the graph in the figure below, in which the arcs are labeled with capacity and flow as follows $\xrightarrow{u,x}$. The lower bounds are all zero. The source is node 1, and the sink is node 6.



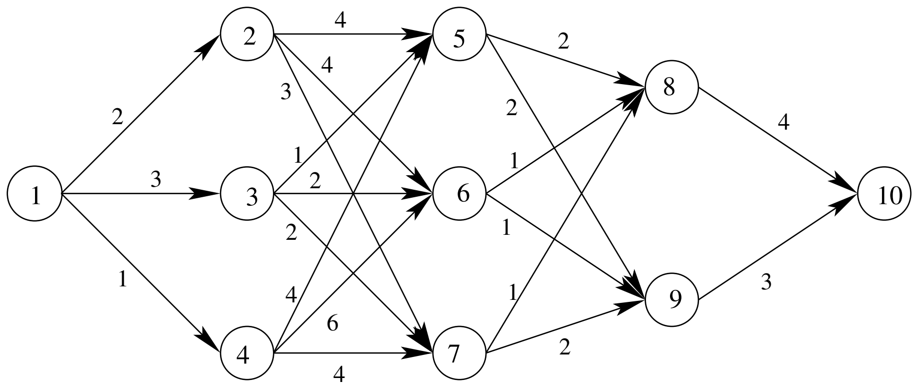

**(2p)** a) Find capacity of the cut $N_s = \{1, 3, 5\}$. Find flow across this cut.

**(3p)** b) Use the max-flow algorithm for finding the maximum flow from node 1 to node 6. Start from the given flow. Present the results of using the modified Dijkstra's algorithm for finding a flow increasing path from node 1 to node 6. Show how arc's feasible directions are updated.

**(3p)** c) Find a minimum cut.

**Assignment 3**

Consider the following network




**(2p)** a) Use dynamic programming to find at least *one* shortest path from node 1 to node 10.

**(1p)** b) Find *all* the shortest paths from node 1 to node 10.

**(1p)** c) Use the results of a) for finding a shortest path from node 2 to node 10. Explain how you used these results.

**Assignment 4**

Consider the function $f(x) = \|x\| = \sqrt{x^T x}$, where $x \in R^n$.

**(2p)** a) Show that $f(x)$ is a convex function in $R^n$.

**(3p)** b) Find the subdifferential of $f(x)$ for $x \neq 0$.

**(5p)** c) Find the subdifferential of $f(x)$ for $x = 0$.

**Assignment 5 (4p.)**

Consider the problem of minimizing $f(x) = x_1^2/2 + x_2^2$. Solve it using the conjugate gradient method starting from the point $x^{(0)} = (2,\ 1)^T$.

**Assignment 6 (3p.)**

Consider the quadratic programming problem

$$\begin{array}{ll} \min & (x_1^2 + x_2^2)/2 \\ \text{s.t.} & x_1 + 1 \leq x_2 \\ & x_1 \geq 0 \end{array}$$

Solve it using the active set method starting from the point $x^{(0)} = (2,\ 4)^T$. Start with the set of active constraints in this point.

**Assignment 7**

Consider the problem

$$\begin{array}{ll} \min & x \\ \text{s.t.} & x^2 + x \leq 0 \end{array}$$

Let $x_*$ denote the solution to this problem. The penalty method and the logarithmic barrier method reduce this constrained optimization problem to a sequence of unconstrained minimization of a function $F(x, c)$ for a series of values of the parameter $c$. Let $x(c)$ denote a minimizer of $F(x, c)$.

**(3p)** a) Find $x(c)$ for the logarithmic barrier method and show that $x(c) \to x_*$.

**(2p)** b) Consider the problem in Assignment 6. For the quadratic penalty method, find the value of $F(x, 1)$ in the point $x = (2,\ 0)^T$.

# ANSWERS

## Assignment 1

**a)** It is a basic feasible solution, because the flow is feasible (the conservation flow holds in each node and each arc flow is within the boundaries) and the set of basic arcs composes a spanning tree.

**b)** The basic tree is composed of the edges (1,2), (1,3), (2,4), (3,7), (6,7), (7,5). The node prices $y = (0,\ 4,\ 1,\ 7,\ 6,\ 2,\ 4)^T$. Reduced costs and optimality:
$\bar{c}_{23} = 6\ \&\ x_{23} = l_{23} \ \Rightarrow\ $ OK
$\bar{c}_{35} = -1\ \&\ x_{35} = l_{35} \ \Rightarrow\ $ NOT OK
$\bar{c}_{36} = 2\ \&\ x_{36} = u_{36} \ \Rightarrow\ $ NOT OK
$\bar{c}_{43} = 10\ \&\ x_{43} = l_{43} \ \Rightarrow\ $ OK
$\bar{c}_{46} = 8\ \&\ x_{46} = l_{46} \ \Rightarrow\ $ OK
Since $|\bar{c}_{36}| > |\bar{c}_{35}|$, $x_{36}$ enters the basis. This results in the cycle 6–3–7–6. It is possible to send 2 units in the cycle with $x_{36}$ exiting the basis. The basic tree does not change, and the node prices remain the same. Reduced costs and optimality:
$\bar{c}_{23} = 6\ \&\ x_{23} = l_{23} \ \Rightarrow\ $ OK
$\bar{c}_{35} = -1\ \&\ x_{35} = l_{35} \ \Rightarrow\ $ NOT OK
$\bar{c}_{36} = 2\ \&\ x_{36} = l_{36} \ \Rightarrow\ $ OK
$\bar{c}_{43} = 10\ \&\ x_{43} = l_{43} \ \Rightarrow\ $ OK
$\bar{c}_{46} = 8\ \&\ x_{46} = l_{46} \ \Rightarrow\ $ OK
The arc $x_{35}$ enters the basis. This results in the cycle 3–5–7–3. It is possible to send 3 units in the cycle with $x_{35}$ exiting the basis (the alternative is that $x_{37}$ exits the basis). The basic tree does not change, and the node prices remain the same. Reduced costs and optimality:
$\bar{c}_{23} = 6\ \&\ x_{23} = l_{23} \ \Rightarrow\ $ OK
$\bar{c}_{35} = -1\ \&\ x_{35} = u_{35} \ \Rightarrow\ $ OK
$\bar{c}_{36} = 2\ \&\ x_{36} = l_{36} \ \Rightarrow\ $ OK
$\bar{c}_{43} = 10\ \&\ x_{43} = l_{43} \ \Rightarrow\ $ OK
$\bar{c}_{46} = 8\ \&\ x_{46} = l_{46} \ \Rightarrow\ $ OK
Thus, the optimality conditions are satisfied.

**c)** The obtained optimal flow is unique, because there is no zero reduced cost.

## Assignment 2

**a)** The capacity of the given cut is equal to 5 + 2 + 3 + 5 = 15. The flow across this cut is equal to 2 + 2 + 1 + 4 - 2 = 7.

**b)** The modified Dijkstra's algorithm gives the flow increasing path 1—2—3—5—6 whose capacity is 1. Then $f = 8$. The modified Dijkstra's algorithm being applied to the new flow shows that there is no flow increasing path. This means that the obtained flow is maximal.

**c)** The modified Dijkstra's algorithm produces the two alternative minimum cuts $N'_s = \{1, 2\}$ and $N''_s = \{1, 2, 3, 4, 5\}$ (it is sufficient to find at least one of them).

**Assignment 3**

**a)** The length of shortest path from node 1 to node 10 is equal to 9. All shortest paths are listed in b).

**b)** All the alternative shortest paths:
1–3–5–9–10
1–3–6–9–10

**c)** For stage 2, we have $f_2(2) = 7$, which is the length of the shortest path from node 2 to node 10. The values $x_2^*(2) = 6$, $x_3^*(6) = 9$ and $x_4^*(9) = 10$ define a shortest path: 2–6–9–10. The values $x_2^*(2) = 7$, $x_3^*(7) = 9$ and $x_4^*(9) = 10$ define an alternative shortest path: 2–7–9–10. The values $x_2^*(2) = 7$, $x_3^*(7) = 8$ and $x_4^*(8) = 10$ define one more shortest path: 2–7–8–10.

**Assignment 4**

**a)** It is necessary to show that

$$f(\lambda u + (1-\lambda)v) \leq \lambda f(u) + (1-\lambda) f(v), \quad \forall u, v \in R^n, \lambda \in [0, 1].$$

Indeed, from the triangular inequality and the non-negativity of $\lambda$ and $(1-\lambda)$, we have

$$\|\lambda u + (1-\lambda)v\| \leq \|\lambda u\| + \|(1-\lambda)v\|, = \lambda\|u\| + (1-\lambda)\|v\|, \quad \forall u, v \in R^n, \lambda \in [0, 1].$$

**b)** $f(x) = \|x\|$ is differntiable for all $x \neq 0$, and its subdifferential is composed of one vector $x/\|x\|$.

**c)** By the definition of subgradient $g \in R^n$ in $x = 0$, the inequality $\|x\| \geq g^T x$ must hold for all $x \in R^n$. Since $g^T x = \|g\|\|x\| \cos(\varphi)$, the inequality $1 \geq \|g\| \cos(\varphi)$ must hold for any angle $\varphi$ between $g$ and $x$. This is possible if, and only if, $\|g\| \leq 1$. Thus, the subdifferential of $\|x\|$ for $x = 0$ is the set of vectors $\{g \in R^n : \|g\| \leq 1\}$.

**Assignment 5**

$k = 0$: $d_0 = (-2, -2)^T$, $t_0 = 2/3$, $x_1 = (2/3, -1/3)^T$.
$k = 1$: $\beta_1 = 1/9$, $d_1 = 4/9 \cdot (-2, 1)^T$, $t_1 = 3/4$, $x_2 = (0, 0)^T$.
$k = 2$: $\|\nabla q(x_2)\| = 0 \Rightarrow x_2$ is the optimal solution, because $q(x)$ ia a convex function.

**Assignment 6**

$k = 0:$ $x_0 = (2, 4)^T$, $S_0 = \emptyset$, $d_0 = (-2, -4)^T$, $t = 1/2$, $x_1 = (1, 2)^T$.
$k = 1:$ $S_1 = \{1\}$, $d_1 = (-3/2, -3/2)^T$, $v_1 = 1/2$, $t = 2/3$, $x_2 = (0, 1)^T$.
$k = 2:$ $S_2 = \{1, 2\}$, $d_1 = (0, 0)^T$, $v_2 = (1, 1)^T \geq 0 \Rightarrow x = (0, 1)^T$ is the optimal solution.

**Assignment 7**

a) $F(x,c) = x - c\ln(-x^2 - x)$ is a convex function of $x \in (-1,0)$ for any $c > 0$. The first derivative of this function has two roots.
The root $c - 1/2 + \sqrt{c^2 + 1/4} > 0$, i.e. it is infeasible.
The root $c - 1/2 - \sqrt{c^2 + 1/4} \in (-1,0)$, i.e. it is feasible.
Thus, $x(c) = c - 1/2 - \sqrt{c^2 + 1/4}$. Obviously, $x(c) \to x_* = -1$ with $c \to +0$.

b) We rewrite the inequality-type constraints in the form $g_i(x) \leq 0$, $i = 1,2$, where $g_1(x) = x_1 - x_2 + 1$ and $g_2(x) = -x_1$. The quadratic penalty function $P(x) = [g_1^+(x)]^2 + [g_2^+(x)]^2$.
For $x = (2\ 0)^T$:
$g_1(2,0) = 3 \implies g_1^+(2,0) = 3,$
$g_2(2,0) = -2 \implies g_2^+(2,0) = 0,$
Thus, $P(2,0) = 9$. For the given $x = (2\ 0)^T$, $F(x,1) = f(x) + 1 \cdot P(x) = 11$.