Linköping University
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TAOP24, Optimization
Examination
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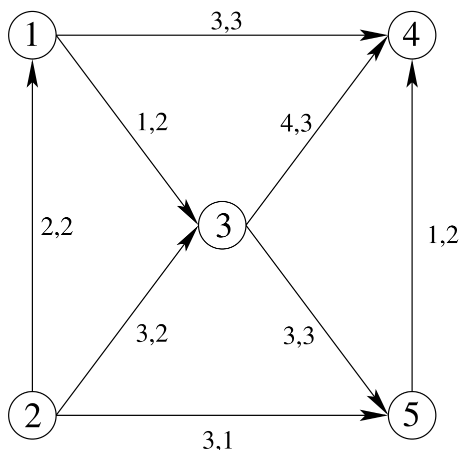
Assignment 1

In the graph below, the arcs are labeled with cost and upper bound as follows $\xrightarrow{c,u}$. The lower bounds are all zero. Nodes 1 is sources of strengths 1 and node 2 is sources of strengths 2, while node 4 is a sink of strength 3.



(1p) a) Given the flow: $x_{13} = x_{54} = 1$, $x_{23} = x_{34} = 2$ and all the other $x_{ij} = 0$. Is it a basic feasible solution?

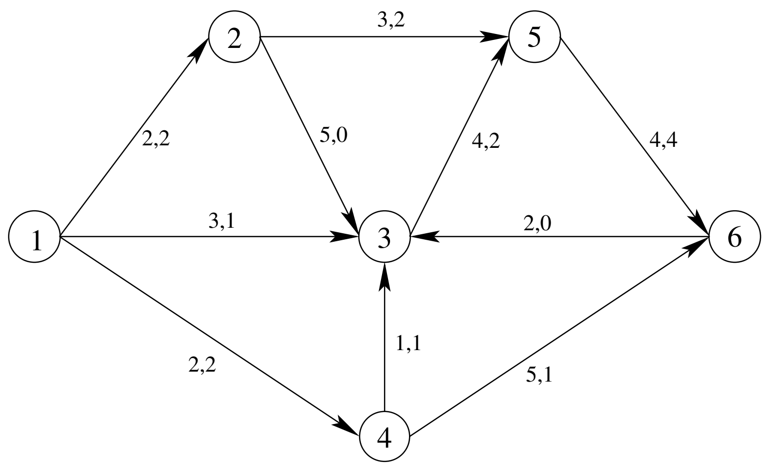
(3p) b) Make two iterations of the network simplex method starting from the basic feasible solution in which $x_{14} = x_{23} = x_{25} = x_{34} = x_{54} = 1$ and all the other $x_{ij} = 0$.

(1p) c) Motivate why the obtained flow is optimal or not.

(2p) d) Find ALL values of c_{25} for which it is guaranteed that the flow cost obtained in b) cannot be decreased.

Assignment 2

Consider the maximum flow problem defined by the graph in the figure below, in which the arcs are labeled with capacity and flow as follows $\xrightarrow{u,x}$.



The lower bounds are all zero. The source is node 1, and the sink is node 6.

(2p) a) Find capacity of the cut $N_s = \{1, 4, 5\}$. Find flow across this cut.

(3p) b) Use the max-flow algorithm for finding the maximum flow from 1 to 6. Start from the given flow. Present the results of using the modified Dijkstra's algorithm for finding a flow increasing path from 1 to 6. Show how arc's feasible directions are updated.

(2p) c) Find a minimum cut in this graph.

Assignment 3

Consider the following integer knapsack problem.

$$\begin{array}{rl} \max & x_1 + 2x_2 + 3x_3 \\ \text{s.t.} & 3x_1 + 4x_2 + 5x_3 \leq a \\ & x_1, x_2, x_3 \geq 0, \quad \text{integer}, \end{array}$$

where a is an integer parameter.

(1p) a) Use dynamic programming to find *ALL* optimal solutions x^* for a=7.

(3p) b) Use dynamic programming to find all optimal solutions x^* and z^* for each value of $a = 3, 4, 5, 6, 7, 8$.

Assignment 4

Consider the convex function

$$f(x) = \max\{-x,\ x/2,\ x-1\}.$$

Use the graph of this function to answer to the following questions.

(1p) a) What is the subdifferential for $x = 0$?

(3p) b) What is the subdifferential for $x = 2$?

(2p) c) Use the subdifferential in $x = 0$ to motivate that it is a global minimizer.

Assignment 5

(3p) a) Consider the problem of minimizing a strictly convex quadratic function. Show that the first two directions generated by the conjugate gradient method are conjugate.

(3p) b) Prove that the conjugate directions are linearly independent.

(2p) c) Consider the problem of minimizing the function $f(x) = x_1^2 - x_2^2/2 - 4x_1 - x_2$. Use the Levenberg-Marquardt method to generate a search direction at the point $\bar{x} = (0,\ 0)^T$.

Assignment 6 (4p)

Consider the quadratic programming problem

$$\begin{array}{ll} \min & f(x) = x_1^2/2 + 2x_2^2 \\ \text{s.t.} & x_1 \geq 2x_2 + 1 \\ & x_2 \geq 0 \end{array}$$

Solve it using the active set method starting from the point $x^{(0)} = (4,\ 1)^T$.

Assignment 7

Consider the problem

$$\begin{array}{ll} \min & x \\ \text{s.t.} & (x-1)(x-2) \leq 0 \end{array}$$

Let x_* denote the solution to this problem. The quadratic penalty method and the logarithmic barrier method reduce this constrained optimization problem to a sequence of unconstrained minimization of a function $F(x, c)$ for a series of values of the parameter c. Let $x(c)$ denote a minimizer of $F(x, c)$.

(3p) **a)** Find $x(c)$ for the logarithmic barrier method and show that $x(c) \to x_*$.

(1p) b) Consider the problem in Assignment 6. For the quadratic penalty method, find the value of $F(x, 1)$ in the point $x = (2,\ 1)^T$.

ANSWERS

Assignment 1

a) It is not a basic feasible solution, because the flow is infeasible (the conservation flow is violated in nodes 3 and 5).

b) The basic tree is composed of the edges (1,4), (2,3), (3.4), (5,4). The node prices $y = (0,\ -4,\ -1,\ 3,\ 2)^T$. Reduced costs and optimality:
$\bar{c}_{13} = 2$ & $x_{13} = l_{13} \ \Rightarrow$ OK
$\bar{c}_{21} = -2$ & $x_{21} = l_{21} \ \Rightarrow$ NOT OK
$\bar{c}_{25} = -3$ & $x_{25} = u_{25} \ \Rightarrow$ OK
$\bar{c}_{35} = 0$ & $x_{35} = l_{35} \ \Rightarrow$ OK
Then x_{21} enters the basis. This results in the cycle 2–1–4–3–2. It is possible to send 1 unit in the cycle with x_{23} exiting the basis. (The alternative is that x_{34} is exiting the basis.)

c) Consider the case when x_{23} is exiting the basis. The new node prices $y = (0,\ -2,\ -1,\ 3,\ 2)^T$. Reduced costs and optimality:
$\bar{c}_{13} = 2$ & $x_{13} = l_{13} \ \Rightarrow$ OK
$\bar{c}_{23} = 2$ & $x_{23} = l_{23} \ \Rightarrow$ OK
$\bar{c}_{25} = -1$ & $x_{25} = u_{25} \ \Rightarrow$ OK
$\bar{c}_{35} = 0$ & $x_{35} = l_{35} \ \Rightarrow$ OK
Thus, the optimality conditions are satisfied.
The optimal value of the total flow cost = 12.

d) The optimality conditions give $\bar{c}_{25} = c_{25} + y_2 - y_5 \leq 0 \Rightarrow c_{25} \leq y_5 - y_2 \Rightarrow c_{25} \leq 4$.

Assignment 2

a) The capacity of the given cut is equal to $2 + 3 + 1 + 5 + 4 = 15$. The flow across this cut is equal to $2 + 1 + 1 + 1 + 4 - 2 - 2 = 5$.

b) The modified Dijkstra's algorithm gives the flow increasing path 1—3—4—6 whose capacity is 1. Then $f = 6$. The modified Dijkstra's algorithm shows that there is no flow increasing path. This means that the obtained flow is maximal.

c) A minimum cut is defined by $N_s = \{1, 2, 3, 5\}$. Its capacity is equal to 6 (= max flow).

Assignment 3

a) The dynamic programming algorithm gives the two optimal solutions:
$x' = (0,\ 0,\ 1)^T$ and $x'' = (1,\ 1,\ 0)^T$.

b) To find optimal solutions x^* and z^* for each value of $a = 3, 4, 5, 6, 7, 8$, the table for Stage 1 should include $s_1 = 3, 4, 5, 6, 7, 8$:

$x_1 \backslash s_1$	3	4	5	6	7	8
0	0	2	3	3	3	4
1	1	1	1	1	3	4
2	-	-	-	2	2	2
$f_1(s_1)$	1	2	3	3	3	4
$x_1^*(s_1)$	1	0	0	0	0/1	0/1

The optimal solution for each value of a:

a	x_1^*	x_2^*	x_3^*	z^*
3	1	0	0	1
4	0	1	0	2
5	0	0	1	3
6	0	0	1	3
7	0	0	1	3
7	1	1	0	3
8	0	2	0	4
8	1	0	1	4

Assignment 4

The graph of this function is defined by its equivalent presentation as

$$f(x) = \begin{cases} -x, & \text{if } x \le 0 \\ x/2, & \text{if } 0 \le x \le 2 \\ x - 1, & \text{if } 2 \le x \end{cases}$$

a) $\partial f(0) = [-1,\ 0.5]$.

b) $\partial f(2) = [0.5,\ 1]$.

c) Since $0 \in \partial f(0)$, $x = 0$ is a global minimizer.

Assignment 5

a) Consider the problem of minimizing the quadratic function

$$q(x) = x^T Ax/2 + b^T x + c,$$

where $A > 0$. Then $\nabla q(x) = Ax + b$. Let x_0 be a starting point. The first two search directions generated by the conjugate gradient method have the form

$$d_0 = -g_0, \quad d_1 = -g_1 + \beta_1(-g_0),$$

where $g_0 = \nabla q(x_0)$ and $g_1 = \nabla q(x_1)$. We need to prove that $d_1^T A d_0 = 0$, or equivalently, that $d_1^T A(x_1 - x_0) = 0$. Recalling that $A(x_1 - x_0) = g_1 - g_0$, we have

$$d_1^T A(x_1 - x_0) = (-g_1 - \beta_1 g_0)^T (g_1 - g_0).$$

The exact line search gives $g_1 \perp d_0$, that is $g_1^T g_0 = 0$. Then

$$d_1^T A(x_1 - x_0) = -\|g_1\|^2 + \beta_1 \|g_0\|^2.$$

Since $\beta_1 = \|g_1\|^2 / \|g_0\|^2$, we finally obtain $d_1^T A(x_1 - x_0) = 0$. This implies $d_1^T A d_0 = 0$, which means that d_0 and d_1 are conjugate directions.

b) For the proof, see the lecture notes.

c) In the Levenberg-Marquardt method, the search direction $d = -G^{-1}\nabla f(\bar{x})$, where $G = H + \gamma I$. The Hessian matrix $H(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$ has the eigenvalues $\lambda_1 = 2$ and $\lambda_2 = -1$ which means that $H(0,0)$ is not positively definite. Any value of $\gamma > -\min\{\lambda_1, \lambda_2\} = 1$ makes G positively definite. If to choose $\gamma = 2$, this gives $d = (1,\ 1)^T$.

Assignment 6

$k = 0:\ \ x_0 = (4,\ 1)^T$, $S_0 = \emptyset$, $d_0 = (-4,\ -1)^T$, $t = 1/2$, $x_1 = (2,\ 1/2)^T$.
$k = 1:\ \ S_1 = \{1\}$, $d_1 = (-3/2,\ -3/4)^T$, $v_1 = 1/2$, $t = 2/3$, $x_2 = (1,\ 0)^T$.
$k = 2:\ \ S_2 = \{1, 2\}$, $d_1 = (0,\ 0)^T$, $v_2 = (1,\ 2)^T \geq 0 \ \Rightarrow\ x = (1,\ 0)^T$ is the optimal solution.

Assignment 7

a) Note that the interior of the feasible region here is the interval $(1, 2)$. $F(x, c) = x - c\ln(3x - x^2 - 2)$ is a convex function of $x \in (1, 2)$ for any $c > 0$. The first derivative of this function has two roots.
The root $3/2 + c + \sqrt{c^2 + 1/4} > 2$, i.e. it is infeasible.
The root $3/2 + c - \sqrt{c^2 + 1/4} \in (1, 2)$, i.e. it is feasible.
Thus, $x(c) = 3/2 + c - \sqrt{c^2 + 1/4}$. Obviously, $x(c) \to x_* = 1$ with $c \to 0$.

b) We rewrite the inequality-type constraints in the form $g_i(x) \leq 0$, $i = 1, 2$, where $g_1(x) = 2x_2 - x_1 + 1$ and $g_2(x) = -x_2$. The quadratic penalty function $P(x) = [g_1^+(x)]^2 + [g_2^+(x)]^2$. For $x = (2\ 1)^T$:
$g_1(2, 1) = 1 \implies g_1^+(2, 1) = 1$,
$g_2(2, 1) = -1 \implies g_2^+(2, 1) = 0$.
Thus, $P(2, 1) = 1$. For the given $x = (2\ 1)^T$, $F(x, 1) = f(x) + 1 \cdot P(x) = 5$.