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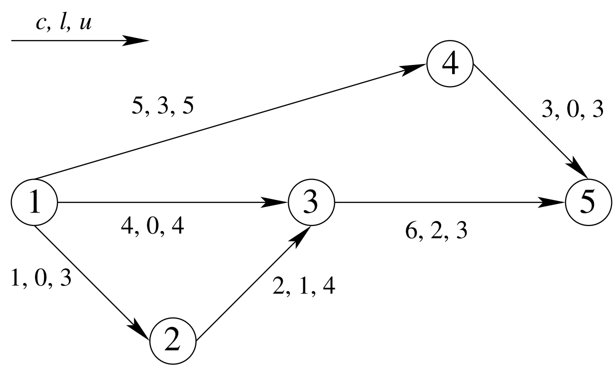
Optimization TAOP24
Examination
of 4 July 2019

Assignment 1

Consider the following balanced network in which node 1 is a source and node 5 is a sink of a proper strength.



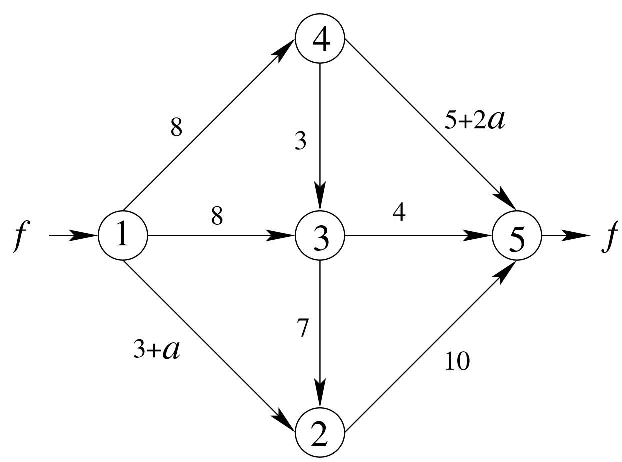
(3p) a) Make one iteration of the network simplex method starting from the basic feasible solution defined by the following values of the basic variables $x_{12} = 1$, $x_{13} = 1$, $x_{14} = 3$, $x_{45} = 3$.

(1p) b) Motivate why the obtained flow is optimal.

(3p) c) Does the minimal cost decrease or not when arc (3,4) with $c = 1$, $l = 0$ and $u = 3$ is introduced. Use the obtained optimal flow to answer to this question.

Assignment 2

Consider the following graph in which nodes 1 and 5 are source and sink, respectively. The arcs are labeled with capacities, where a is a non-negative parameter.



(3p) a) Use the max-flow algorithm for finding the maximum flow f for $a = 0$. Start with zero flow. Show the results of using the modified Dijkstra's algorithm for finding a flow increasing path from node 1 to node 5. Show how arc's feasible directions are updated.

(3p) b) Find the dependence of the maximum flow f on a for all values of $a \geq 0$.

Assignment 3 (3p)

Use the dynamic programming to solve the problem:

$$\begin{array}{ll} \max & 4x_1 + 6x_2 + 3x_3 \\ \text{s.t.} & 5x_1 + 7x_2 + 4x_3 \le 9 \\ & x_1, x_2, x_3 \ge 0 \text{ and integer} \end{array}$$

Assignment 4

Let the functions $f_1(x)$ and $f_2(x)$ be convex in R^n.

(3p) a) Consider the function $F(x) = a_1 f_1(x) + a_2 f_2(x)$, where a_1 and a_2 are positive constants. Prove that if $g_1 \in \partial f_1(\bar{x})$ and $g_2 \in \partial f_2(\bar{x})$, then $a_1 g_1 + a_2 g_2 \in \partial F(\bar{x})$.

(5p) b) Consider the function $F(x) = \max\{f_1(x), f_2(x)\}$. Let the point $\bar{x}$ be such that $f_1(\bar{x}) = f_2(\bar{x})$. Prove that if $g_1 \in \partial f_1(\bar{x})$ and $g_2 \in \partial f_2(\bar{x})$, then $\lambda g_1 + (1-\lambda) g_2 \in \partial F(\bar{x})$ for any $\lambda \in [0, 1]$.

Assignment 5

(3p) a) Suppose that the conjugate gradient method is used for minimizing a quadratic function with a positively definite Hessian matrix A. Let d_0 and d_1 be the first two search directions generated by this method. Show that these directions are conjugate.

(3p) b) Consider the problem of minimizing the function $f(x) = x_1^3/3 + \ln(x_2)$. Use the Levenberg-Marquardt method to generate a search direction at the point $x^{(0)} = (-1,\ 1)^T$.

Assignment 6 (3p)

Consider the quadratic programming problem

$$\begin{array}{ll} \min & f(x) = (x_1^2 + x_2^2)/2 \\ \text{s.t.} & x_1 - x_2 \le 0 \\ & x_2 \ge 1 \\ & x_1 \ge -1 \end{array}$$

Solve it using the active set method. Start from the point $x^{(0)} = (2,\ 2)^T$ and the set of active constraints.

Assignment 7

Consider the problem

$$\begin{array}{ll} \min & x \\ \text{s.t.} & -1 \le x \le 1 \end{array}$$

Let x_* denote the solution to this problem. The quadratic penalty method and the logarithmic barrier method reduce this constrained optimization problem to a sequence of unconstrained minimization of a function $F(x, c)$ for a series of values of the parameter c. Let $x(c)$ denote a minimizer of $F(x, c)$.

(3p) a) Find $x(c)$ for the quadratic penalty method and show that $x(c) \to x_*$.

(3p) b) Find $x(c)$ for the logarithmic barrier method and show that $x(c) \to x_*$.

(1p) c) Consider the problem in Assignment 6. Find $F(x, 1)$ for the quadratic penalty method in the point $x = (2,\ 0)^T$.

ANSWERS

Assignment 1

a) By the conservation of flow, the starting flow is the following: $x_{12} = 1$, $x_{13} = 1$, $x_{14} = 3$, $x_{23} = 1$, $x_{35} = 2$, $x_{45} = 3$. The node prices $y = (0,\ 1,\ 4,\ 5,\ 8)^T$. Reduced costs:
$\bar{c}_{23} = -1$ & $x_{23} = l_{23} \ \Rightarrow$ NOT OK
$\bar{c}_{35} = 2$ & $x_{35} = u_{35} \ \Rightarrow$ OK
Then x_{23} enters the basis. This results in the cycle 1–2–3–1. It is possible to send 1 unit in the cycle with x_{13} exiting the basis. This gives the new flow: $x_{12} = 2$, $x_{13} = 0$, $x_{14} = 3$, $x_{23} = 2$, $x_{35} = 2$, $x_{45} = 3$.

b) The new node prices $y = (0,\ 1,\ 3,\ 5,\ 8)^T$. Reduced costs:
$\bar{c}_{13} = 1$ & $x_{13} = l_{13} \ \Rightarrow$ OK
$\bar{c}_{35} = 1$ & $x_{35} = l_{35} \ \Rightarrow$ OK
Thus, the optimality conditions are satisfied.
The optimal value of the total flow cost = 42.

c) The reduced cost $\bar{c}_{34} = -1$ indicates that the total flow cost would decrease, if it could be possible to send a flow along this arc. Consider the possibility for x_{34} to enter the basis. This results in the cycle 3–4–1–2–3. It is possible to send at most 0 units in the cycle with x_{14} exiting the basis. This does not change the flow: $x_{12} = 2$, $x_{13} = 0$, $x_{14} = 3$, $x_{23} = 2$, $x_{34} = 0$, $x_{35} = 2$, $x_{45} = 3$. The list of basic and nonbasic variables changed, which results in the new node prices $y = (0,\ 1,\ 3,\ 4,\ 7)^T$. Reduced costs:
$\bar{c}_{13} = 1$ & $x_{13} = l_{13} \ \Rightarrow$ OK
$\bar{c}_{14} = 1$ & $x_{14} = l_{14} \ \Rightarrow$ OK
$\bar{c}_{35} = 3$ & $x_{35} = l_{35} \ \Rightarrow$ OK
Thus, the optimality conditions are satisfied.
The optimal value of the total flow cost does not change and remains the same, 42.

Assignment 2

a) We start with zero flow ($f = 0$).
The modified Dijkstra's algorithm gives the flow increasing path 1—3—2—5 whose capacity is 7. Then $f = 7$.
The 2nd flow increasing path is 1—4—5 with the capacity 5. This makes $f = 12$.
Then the modified Dijkstra's algorithm admits alternative sequence of paths. Each of the alternative gives the same optimal flow. We present here one of the alternative continuations.
The 3rd flow increasing path is 1—2—5 with the capacity 3. This makes $f = 15$.
The 4th flow increasing path is 1—4—3—5 with the capacity 3. This makes $f = 18$.
The 5th flow increasing path is 1—3—5 with the capacity 1. This makes $f = 19$.
Then the modified Dijkstra's algorithm shows that there is no flow increasing path from node 1 to node 5, which means that the obtained flow is maximal. On each arc, the optimal arc flow is equal to the arc capacity.

b) If $0 < a \leq 3$, we start with the flow which is maximal for $a = 0$. Then the modified Dijkstra's algorithm gives the flow increasing path 1—2—3—4—5 with the capacity

a. This makes the maximal flow $f = 19 + a$. The corresponding optimal arc flows are presented in Fig. 1.
If $a > 3$, we start with the optimal flow which corresponds in Fig. 1 to $a = 3$. The modified Dijkstra's algorithm shows that the same flow $f = 22$ remains maximal for any $a > 3$.

Thus, the maximal flow $f = \begin{cases} 19 + a, & \text{if } 0 \leq a \leq 3, \\ 22, & \text{if } a > 3. \end{cases}$

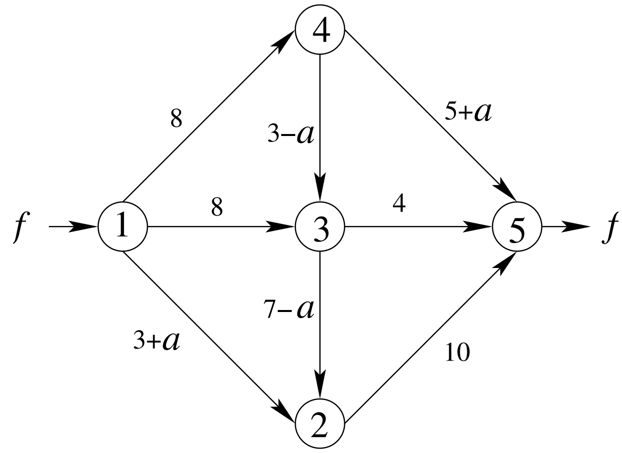


Figure 1: Optimal flow for $0 \leq a \leq 3$.

Assignment 3

The optimal solution is $x^* = (1,\ 0,\ 1)$, $z^* = 7$.

Assignment 4

a) The fact that $g_1 \in \partial f_1(\bar{x})$ and $g_2 \in \partial f_2(\bar{x})$ implies

$$f_1(x) \geq f_1(\bar{x}) + g_1^T(x - \bar{x}), \qquad \forall x \in R^n, \tag{1}$$

$$f_2(x) \geq f_2(\bar{x}) + g_2^T(x - \bar{x}), \qquad \forall x \in R^n. \tag{2}$$

Let a_1 and a_2 be any positive scalars. Multiplying (1) by a_1 and (2) by a_2, and then summing up the resulting inequalities, we obtain

$$F(x) \geq F(\bar{x}) + g^T(x - \bar{x}), \qquad \forall x \in R^n, \tag{3}$$

where $F(x) = a_1 f_1(x) + a_2 f_2(x)$, $F(\bar{x}) = a_1 f_1(\bar{x}) + a_2 f_2(\bar{x})$ and $g = a_1 g_1 + a_2 g_2$. From inequality (3) we finally conclude that $g \in \partial F(\bar{x})$.

b) Let λ be any scalar from the interval $[0, 1]$. Multiplying (1) by λ and (2) by $(1 - \lambda)$, and then summing up the resulting inequalities, we obtain

$$\lambda f_1(x) + (1-\lambda) f_2(x) \geq [\lambda f_1(\bar{x}) + (1-\lambda) f_2(\bar{x})] + [\lambda g_1 + (1-\lambda) g_2]^T (x - \bar{x}), \qquad \forall x \in R^n, \tag{4}$$

where $\lambda f_1(\bar{x}) + (1 - \lambda) f_2(\bar{x}) = \max\{f_1(\bar{x}), f_2(\bar{x})\}$, because $f_1(\bar{x}) = f_2(\bar{x})$. Moreover, $\max\{f_1(x), f_2(x)\} \geq \lambda f_1(x) + (1-\lambda) f_2(x)$, because $\max\{f_1(x), f_2(x)\} \geq f_i(x)$ obviously holds for $i = 1, 2$. Thus, we obtain inequality (3) in which $F(x) = \max\{f_1(x), f_2(x)\}$, $F(\bar{x}) = \max\{f_1(\bar{x}), f_2(\bar{x})\}$ and $g = \lambda g_1 + (1-\lambda) g_2$. This finally proves that $g \in \partial F(\bar{x})$.

Assignment 5

a) Consider the quadratic optimization problem

$$\min_{x \in R^n} q(x) = \frac{1}{2} x^T A x + b^T x + c,$$

where $A > 0$. Denote $g_k = \nabla q(x_k)$. In the conjugate gradient method,

$$d_0 = -g_0, \tag{5}$$

$$d_1 = -g_1 + \frac{\|g_1\|^2}{\|g_0\|^2}(-g_0). \tag{6}$$

We need to prove that $d_1^T A d_0 = 0$. Since $t = t_0$ minimizes $q(x_0 + t d_0)$, we have $g_1 \perp d_0$, and hence, $g_1 \perp g_0$ due to (5). We assume that $g_0 \neq 0$, which implies $t_0 \neq 0$. Note that $A(u - v) = \nabla q(u) - \nabla q(v)$, $\forall u, v \in R^n$. Then $A d_0 = (g_1 - g_0)/t_0$. Using this relation and (6), we get

$$d_1^T A d_0 = \left[-g_1 + \frac{\|g_1\|^2}{\|g_0\|^2}(-g_0) \right]^T (g_1 - g_0)/t_0.$$

Here the orthogonality $g_1 \perp g_0$ finally gives $d_1^T A d_0 = 0$.

b) $\nabla f(-1, 1) = (1\ 1)^T$. The Hessian matrix $H(1, 0) = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}$ has the eigenvalues $\lambda_1 = -2$ and $\lambda_2 = -1$. In the Levenberg-Marquardt method, the search direction $p = -G^{-1} \nabla f$, where

$$G = H + \gamma I = \begin{pmatrix} -2 + \gamma & 0 \\ 0 & -1 + \gamma \end{pmatrix}.$$

Thus,

$$p = -\begin{pmatrix} \frac{1}{-2+\gamma} & 0 \\ 0 & \frac{1}{-1+\gamma} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2-\gamma} \\ \frac{1}{1-\gamma} \end{pmatrix}.$$

Since, in our case, the minimal eigenvalue λ_1 is not positive, the scalar parameter γ should be chosen so that $\gamma > -\lambda_1$. If to choose $\gamma = 3$, then $p = \begin{pmatrix} -1 \\ -\frac{1}{2} \end{pmatrix}$.

Assignment 6

$x^* = (0,\ 1)^T$.

Assignment 7

a) In this problem,

$$F(x, c) = x + c \cdot \begin{cases} (x+1)^2, & \text{if } x < -1, \\ 0, & \text{if } -1 \le x \le 1, \\ (x-1)^2, & \text{if } 1 \le x. \end{cases}$$

It is a continuously differentiable function with

$$F'_x(x,c) = \begin{cases} 1+2c(x+1), & \text{if } x < -1, \\ 1, & \text{if } -1 \le x \le 1, \\ 1+2c(x-1), & \text{if } 1 \ge x. \end{cases}$$

Therefore, the minimizer $x(c)$ should be a stationary point of $F(x,c)$, i.e.

$$F'_x(x(c),c) = 0.$$

• If to assume that $x(c) < -1$, the corresponding stationary point $x(c) = -1 - 1/(2c)$ belongs to this area.
• The interval $-1 \le x \le 1$ contains no stationary point.
• If to assume that $x(c) > 1$, the corresponding stationary point $x(c) = 1 - 1/(2c)$ breaks this assumption.
Thus, there exists a unique stationary point $x(c) = -1 - 1/(2c)$. Note that $F(x,c)$ is monotonically decreasing ($F'_x(x,c) < 0$) when $x < x(c)$, and $F(x,c)$ is monotonically increasing ($F'_x(x,c) > 0$) when $x > x(c)$. This means that $x(c)$ is a unique global minimizer of $F(x,c)$.
Obviously, $x(c) \to x_* = -1$ with $c \to +\infty$.

b) $F(x,c) = x - c\ln(x+1) - c\ln(1-x)$ is a convex function of $x \in (-1,1)$ for any $c > 0$. The first derivative of this function has two roots.
The root $c + \sqrt{1+c^2} > 1$, i.e. it is infeasible.
The root $c - \sqrt{1+c^2} \in (-1,1)$, i.e. it is feasible.
Thus, $x(c) = c - \sqrt{1+c^2}$. Obviously, $x(c) \to x_* = -1$ with $c \to +0$.

c) To write the inequality-type constraints in the form $g_i(x) \le 0$, $i = 1,2,3$, we denote $g_1(x) = x_1 - x_2$, $g_2(x) = 1 - x_2$ and $g_3(x) = -1 - x_1$. The quadratic penalty function $P(x) = [g_1^+(x)]^2 + [g_2^+(x)]^2 + [g_3^+(x)]^2$. For $x = (2\ 0)^T$:
$g_1(2,0) = 2 \implies g_1^+(2,0) = 2,$
$g_2(2,0) = 1 \implies g_2^+(2,0) = 1,$
$g_3(2,0) = -3 \implies g_3^+(2,0) = 0,$
Thus, $P(2,0) = 5$. For the given $x = (2\ 0)^T$, $F(x,1) = f(x) + 1 \cdot P(x) = 7$.