Linköping University
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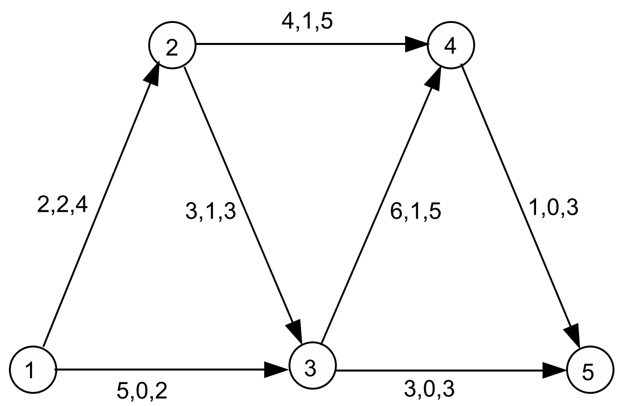
TAOP24, Optimization
Examination
of 1 November 2019

Assignment 1

Consider the network



where the arcs are labeled with cost, lower and upper bounds as follows $\xrightarrow{c,l,u}$. Node 1 is a source with a strength 5, while nodes 4 and 5 are sinks with the strength 3 and 2, respectively.

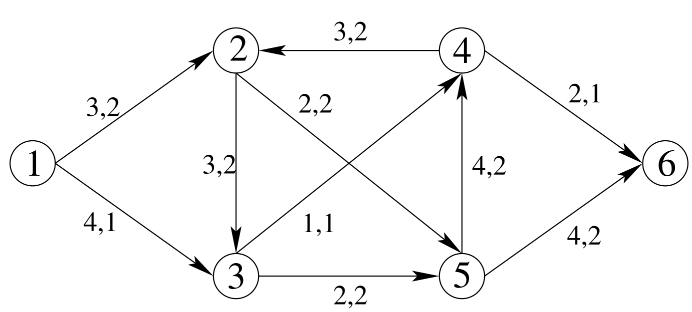
(1p) a) Given the flow $x_{12} = 3$, $x_{13} = 2$, $x_{23} = 1$, $x_{24} = 2$, $x_{34} = 2$, $x_{35} = 1$ and $x_{45} = 0$. Is it a basic feasible solution?

(3p) b) Given the starting basic feasible solution $x_{12} = 4$, $x_{13} = 1$, $x_{23} = 2$, $x_{24} = 2$, $x_{34} = 1$, $x_{35} = 2$ and $x_{45} = 0$. Make one iteration of the network simplex method. Is the obtained solution optimal? If it is optimal, is it unique?

(2p) c) Find ALL values of c_{23} in the network above for which the optimal flow is the same as for $c_{23} = 3$.

Assignment 2

Consider the maximum flow problem defined by the graph in the figure below, in which the arcs are labeled with capacity and flow as follows $\xrightarrow{u,x}$. The source is node 1, and the sink is node 6.



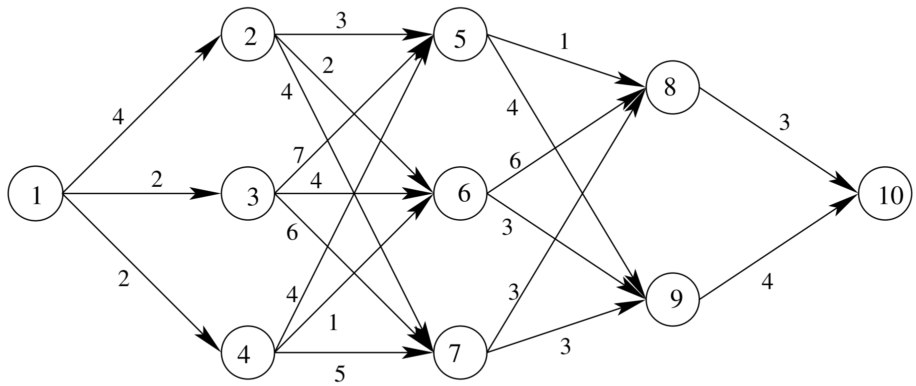
(2p) a) Find capacity of the cut $N_s = \{1, 3, 4\}$. Find flow across this cut.

(3p) b) Use the max-flow algorithm for finding the maximum flow from node 1 to node 6. Start from the given flow. For each iteration, present the results of using the modified Dijkstra's algorithm for finding a flow increasing path from node 1 to node 6. Show how arc's feasible directions are updated.

(2p) c) Find a minimum cut in this graph.

Assignment 3

Consider the following network



(2p) a) Use dynamic programming to find at least *one* shortest path from node 1 to node 10.

(1p) b) Find *all* the shortest paths from node 1 to node 10.

(2p) c) Use the results of a) for finding the shortest path from node 2 to node 10. Explain how you used these results.

Assignment 4

Consider the convex function

$$f(x) = \max\{-2x,\ x,\ 3x - 2\}.$$

Use the graph of this function for finding subdifferentials.

(2p) a) Find the subdifferential in $x = -1$ and also in $x = 1$.

(3p) b) Show algebraically that any element of the subdifferential in $x = 1$ satisfies the definition of subgradient?

(2p) c) Use the subdifferential in $x = 0$ to motivate that it is a global minimizer.

Assignment 5

(4p) a) Let $d_1, \dots, d_n$ be conjugate directions for a symmetric positively definite matrix $A \in R^{n\times n}$. Suppose that these vectors are orthonormal, i.e.

$$d_i^T d_j = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

Prove that $d_1, \dots, d_n$ are eigenvectors of A. Find the corresponding eigenvalues.

(2p) b) Consider the problem of minimizing the function $f(x) = x_1^2 x_2^2 - 4x_1x_2$. Use the Levenberg-Marquardt method to generate a search direction at the point $\bar{x} = (1,\ 1)^T$.

Assignment 6 (4p)

Consider the quadratic programming problem

$$\begin{array}{ll} \min & f(x) = (x_1 - 1)^2/2 + x_2^2/2 \\ \text{s.t.} & x_1 + x_2 \geq 0 \\ & x_1 - x_2 \leq 2 \\ & x_2 \geq 0 \end{array}$$

Solve it using the active set method starting from the point $x^{(0)} = (3,\ 1)^T$ and the active set in this point.

Assignment 7

(3p) **a)** Consider the problem

$$\begin{array}{ll} \min & x \\ \text{s.t.} & x^2 \leq x \end{array}$$

Let x_* denote the solution to this problem. The logarithmic barrier method reduces this constrained optimization problem to a sequence of unconstrained minimization of a function $F(x, c)$ for a series of values of the parameter c. Let $x(c)$ denote a minimizer of $F(x, c)$. Find $x(c)$ for the logarithmic barrier method and show that $x(c) \to x_*$.

(2p) b) Consider the problem in Assignment 6. For the quadratic penalty method, find the value of $F(x, 1)$ in the point $x = (0,\ -1)^T$.

ANSWERS

Assignment 1

a) It is not a basic feasible solution, because the flow is infeasible (the conservation flow is violated in nodes 4 and 5).

b) The total flow cost = 39, and the node prices $y = (0,\ 2,\ 5,\ 6,\ 8)^T$. Reduced costs:
$\bar{c}_{12} = 0$ & $x_{12} = u_{12} \Rightarrow$ OK
$\bar{c}_{34} = 5$ & $x_{34} = l_{34} \Rightarrow$ OK
$\bar{c}_{45} = -1$ & $x_{45} = l_{45} \Rightarrow$ NOT OK
Then x_{45} enters the basis. This results in the cycle 4–5–3–2–4. It is possible to send 1 unit in the cycle with x_{23} exiting the basis. The new total flow cost = 38, and the new node prices $y = (0,\ 3,\ 5,\ 7,\ 8)^T$. Reduced costs:
$\bar{c}_{12} = -1$ & $x_{12} = u_{12} \Rightarrow$ OK
$\bar{c}_{23} = 1$ & $x_{23} = l_{23} \Rightarrow$ OK
$\bar{c}_{34} = 4$ & $x_{34} = l_{34} \Rightarrow$ OK
Thus, the optimality conditions are satisfied. The obtained optimal solution is unique, because there is no zero reduced cost.

c) The optimality conditions give $\bar{c}_{23} = c_{23} + y_2 - y_3 \geq 0 \Rightarrow c_{23} \geq y_3 - y_2 \Rightarrow c_{23} \geq 2$.

Assignment 2

a) The capacity of the given cut is equal to $3 + 2 + 3 + 2 = 10$. The flow across this cut is equal to $2 + 2 + 2 + 1 - 2 - 2 = 3$.

b) The modified Dijkstra's algorithm gives the flow increasing path 1—3—2—4—5—6 whose capacity is 2. Then $f = 5$. The modified Dijkstra's algorithm shows that there is no flow increasing path. This means that the obtained flow is maximal.

c) A minimum cut is defined by $N_s = \{1, 2, 3\}$. Its capacity is equal to 5 (= max flow).

Assignment 3

a) The length of shortest path from node 1 to node 10 is equal to 10. All shortest paths are listed in b).

b) All the alternative shortest paths:
1–4–5–8–10
1–4–6–9–10

c) For stage 2, we have $f_2(2) = 7$, which is the length of the shortest path from node 3 to node 10. The values $x_2^*(2) = 5$, $x_3^*(5) = 8$ and $x_4^*(8) = 10$ define the shortest path: 2–5–8–10.

Assignment 4

The graph of this function is defined by its equivalent presentation as

$$f(x) = \begin{cases} -2x, & \text{if } x \le 0 \\ x, & \text{if } 0 \le x \le 1 \\ 3x - 2, & \text{if } 1 \le x \end{cases}$$

a) $\partial f(-1) = \{-2\}$ and $\partial f(1) = [1,\ 3]$.

b) It is required to show for any $\gamma \in [1,\ 3]$ that

$$f(x) \ge f(1) + \gamma(x-1), \quad \forall x.$$

Indeed, from the mentioned above equivalent presentation of $f(x)$, we obtain:
$-2x \ge 1 + \gamma(x-1), \quad \forall x \le 0, \quad \forall \gamma \in [1,\ 3]$;
$x \ge 1 + \gamma(x-1), \quad \forall x \in [0,1], \quad \forall \gamma \in [1,\ 3]$;
$3x - 2 \ge 1 + \gamma(x-1), \quad \forall x \ge 1, \quad \forall \gamma \in [1,\ 3]$.

c) Since $\partial f(0) = [-2,\ 1]$ and $0 \in \partial f(0)$, $x = 0$ is a global minimizer.

Assignment 5

a) Let $D \in R^{n\times n}$ denote a matrix with the columns $d_1, d_2, \ldots, d_n$. Denote $\lambda_i = d_i^T A d_i$. Let $\Lambda \in R^{n\times n}$ denote a diagonal matrix with the diagonal elements $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then the conjugacy can be written as $D^T A D = \Lambda$. Since D is an orthonormal matrix, we have $AD = D\Lambda$, or equivalently, $Ad_i = \lambda_i d_i$ for $i = 1, 2, \ldots, n$. This means that $d_1, d_2, \ldots, d_n$ are eigenvectors of A, and $\lambda_i = d_i^T A d_i$, $i = 1, 2, \ldots, n$, are the corresponding eigenvalues.

b) In the Levenberg-Marquardt method, the search direction $d = -G^{-1}\nabla f(\bar{x})$, where $G = H + \gamma I$. The Hessian matrix $H(1,1) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ has the eigenvalues $\lambda_1 = \lambda_2 = 2$ which means that $H(1,1)$ is positively definite. In this case, $\gamma = 0$, and then $d = (1,\ 1)^T$.

Assignment 6

$k = 0:\ x_0 = (3,\ 1)^T$, $S_0 = \{2\}$, $d_0 = (-3/2,\ -3/2)^T$, $v_2 = -1/2$, $t = 2/3$, $x_1 = (2,\ 0)^T$.
$k = 1:\ S_1 = \{2, 3\}$, $d_1 = (0,\ 0)^T$, $(v_2,\ v_3) = (-1,\ 1)$, $x_2 = (2,\ 0)^T$.
$k = 2:\ S_2 = \{3\}$, $d_2 = (-1,\ 0)^T$, $v_3 = 0$, $t = 1$, $x_3 = (1,\ 0)^T$.
$k = 3:\ S_3 = \{3\}$, $d_2 = (0,\ 0)^T$, $v_3 = 0 \ge 0 \ \Rightarrow\ x = (1,\ 0)^T$ is the optimal solution.

Assignment 7

a) Note that the interior of the feasible region here is the interval $(0,1)$. $F(x,c) = x - c\ln(x - x^2)$ is a convex function of $x \in (0,1)$ for any $c > 0$. The first derivative of this function has two roots.
The root $1/2 + c + \sqrt{c^2 + 1/4} > 1$, i.e. it is infeasible.
The root $1/2 + c - \sqrt{c^2 + 1/4} \in (0,1)$, i.e. it is feasible.
Thus, $x(c) = 1/2 + c - \sqrt{c^2 + 1/4}$. Obviously, $x(c) \to x_* = 0$ with $c \to 0$.

b) We rewrite the inequality-type constraints in the form $g_i(x) \leq 0$, $i = 1, 2, 3$, where $g_1(x) = -x_1 - x_2$, $g_2(x) = x_1 - x_2 - 2$ and $g_3(x) = -x_2$. The quadratic penalty function $P(x) = [g_1^+(x)]^2 + [g_2^+(x)]^2 + [g_3^+(x)]^2$. For $x = (0 \ -1)^T$:

$g_1(0, -1) = 1 \implies g_1^+(0, -1) = 1,$

$g_2(0, -1) = -1 \implies g_2^+(0, -1) = 0,$

$g_3(0, -1) = 1 \implies g_3^+(0, -1) = 1$

Thus, $P(0, -1) = 2$. For the given $x = (0 \ -1)^T$, $F(x, 1) = f(x) + 1 \cdot P(x) = 3$.