Exempel på användning av

MATRISER

i olika kurser

Flervariabelanalys

$$\frac{\partial f}{\partial s}+\frac{\partial f}{\partial t}y-\left(\frac{\partial f}{\partial s}+\frac{\partial f}{\partial t}x\right)=y-x \quad\Rightarrow\quad (y-x)\frac{\partial f}{\partial t}=y-x$$

ska gälla alla x,y (inte bara $y=x$) $\Rightarrow \dfrac{\partial f}{\partial t}=1 \Rightarrow f=t+\underbrace{g(s)}_{\text{godt.}}$

$\Rightarrow f(x,y)=xy+g(x+y)$

Bivillkor $f(x,0)=x\cdot 0+g(x+0)=x^2 \Rightarrow g(x)=x^2 \Rightarrow f(x,y)=xy+(x+y)^2$

Ex Variabelbyte till polära koordinater

$u(x,y)=u(\underbrace{\rho\cos\varphi}_{x},\underbrace{\rho\sin\varphi}_{y})$

$$\frac{\partial u}{\partial \rho}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial \rho}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial \rho}=\frac{\partial u}{\partial x}\cos\varphi+\frac{\partial u}{\partial y}\sin\varphi$$

$$\frac{\partial u}{\partial \varphi}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial \varphi}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial \varphi}=\frac{\partial u}{\partial x}(-\rho\sin\varphi)+\frac{\partial u}{\partial y}\rho\cos\varphi$$

Matrisform
$$\left(\frac{\partial u}{\partial \rho},\ \frac{\partial u}{\partial \varphi}\right)=\left(\frac{\partial u}{\partial x}\ \ \frac{\partial u}{\partial y}\right)\begin{pmatrix}\frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \varphi}\\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \varphi}\end{pmatrix}=\left(\frac{\partial u}{\partial x}\ \ \frac{\partial u}{\partial y}\right)\underbrace{\begin{pmatrix}\cos\varphi & -\rho\sin\varphi\\ \sin\varphi & \rho\cos\varphi\end{pmatrix}}_{\text{variabelbytes-matris}}$$

Ex Linjärt koordinatbyte $\begin{cases} s=ax+by\\ t=cx+dy\end{cases}$ a,b,c,d konstanter ger

konstant variabelbytesmatris $\begin{pmatrix}\frac{\partial s}{\partial x} & \frac{\partial s}{\partial y}\\ \frac{\partial t}{\partial x} & \frac{\partial t}{\partial y}\end{pmatrix}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ linjär algebra!

Ex Bestäm alla lösningar till $x\dfrac{\partial^2 f}{\partial x\partial y}-y\dfrac{\partial^2 f}{\partial y^2}-\dfrac{\partial f}{\partial y}=0$

(Partiell diff ekv av ordn 2)

Ledning: inför $\begin{cases} u=x\\ v=xy\end{cases} \Rightarrow \begin{cases} x=u\\ y=v/x=v/u\end{cases}$

Kedjeregeln $\dfrac{\partial f}{\partial y}=\dfrac{\partial f}{\partial u}\underbrace{\dfrac{\partial u}{\partial y}}_{0}+\dfrac{\partial f}{\partial v}\underbrace{\dfrac{\partial v}{\partial y}}_{x}=x\dfrac{\partial f}{\partial v}\quad (\forall f)\quad$ "$\dfrac{\partial}{\partial y}=x\dfrac{\partial}{\partial v}$"

$$\frac{\partial^2 f}{\partial y^2}=\frac{\partial}{\partial y}\left(x\frac{\partial f}{\partial v}\right)=x\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial v}\right)=x\cdot x\frac{\partial}{\partial v}\left(\frac{\partial f}{\partial v}\right)=x^2\frac{\partial^2 f}{\partial v^2}$$

$$\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial}{\partial x}\left(x\frac{\partial f}{\partial v}\right)\underset{\text{produkt}}{=}1\cdot\frac{\partial f}{\partial v}+x\frac{\partial}{\partial x}\Big(\underbrace{\frac{\partial f}{\partial v}}_{g}\Big)=\frac{\partial f}{\partial v}+x\left(\overset{1}{\frac{\partial u}{\partial x}}\frac{\partial}{\partial u}+\overset{y}{\frac{\partial v}{\partial x}}\frac{\partial}{\partial v}\right)\frac{\partial f}{\partial v}=$$

$$=\frac{\partial f}{\partial v}+x\frac{\partial^2 f}{\partial u\partial v}+xy\frac{\partial^2 f}{\partial v^2}$$

$/\ \dfrac{\partial g}{\partial x}=\dfrac{\partial g}{\partial u}\underbrace{\dfrac{\partial u}{\partial x}}_{1}+\dfrac{\partial g}{\partial v}\underbrace{\dfrac{\partial v}{\partial x}}_{y}=\dfrac{\partial^2 f}{\partial u\partial v}+y\dfrac{\partial^2 f}{\partial v^2}\ /$

Föreläsningsanteckningar

(5) För ett företag finns totalt 25 veckors sammanhörande data över veckoomsättningen (variabel Y) och den veckovisa reklamkostnaden (variabel x_1), då reklamen har skett på två olika sätt, A alternativt B. Den binära variabeln x_2 har värdet 0 för A och värdet 1 för B.

Man anpassar en modell till data: $Y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \varepsilon$, där ε är normalfördelad med väntevärde 0 och varians σ^2. Från datorutskriften får man

Predictor	Coef	SE Coef
Constant	44.751	7.062
x1	0.8174	0.1307
x2	13.667	3.464

$$S = 8.486$$

$$(\mathbf{X}\mathbf{X}^T)^{-1} = \begin{pmatrix} 0.692616 & -0.012025 & -0.083333 \\ -0.012025 & 0.000237 & 0.000000 \\ -0.083333 & 0.000000 & 0.166667 \end{pmatrix}.$$

(a) Påverkas den genomsnittliga veckoomsättningen signifikant (5%-nivån) av om reklamen sker enligt A eller B? (1.5p)

(b) Hur stor kan veckoomsättningen bli om reklamkostnaden väljs till 50 och reklamen sker enligt A? Beräkna ett 95%, tvåsidigt prediktionsintervall. (1.5p)

Ledning: Påminna dig om att för den okända variabeln Y_0 gäller det att

$$\frac{Y_0 - u^T\hat{\beta}}{s \cdot \sqrt{u^T(\mathbf{X}^T\mathbf{X})^{-1}u + 1}} \sim t(n-k-1)$$

där $\hat{\beta}$ är skattningen av $\beta = (\beta_0, \beta_1, \beta_2)^T$ och $u^T = (1, u_1, u_2) = (1, 50, 0)$.

(6) För att få en uppfattning om hur många fiskar det finns i en sjö märker man alla 98 fiskar i en fångst och släpper tillbaks dessa i sjön. Senare gör man en ny fångst och finner att av de 60 fiskarna i denna var 28 märkta.

(a) Skatta totalantalet fiskar i sjön med momentmetoden. (1p)

(b) Skatta totalantalet fiskar i sjön med maximum-likelihood-metoden. (2p)

Ledning: Väntevärdet av en *hypergeometriskt fördelning* med parametrarna n, N, och m, dvs. en fördelning med sannolikhetsfunktion

$$p(x) = \frac{\binom{m}{x}\binom{N-m}{n-x}}{\binom{N}{n}}, \quad x = \max\{0, n+m-N\}, \ldots, \min\{m, n\},$$

är $n\frac{m}{N}$.

Statistisk teori, grundkurs

Tentamensuppgifter

Reglerteknik

Sammanfattn. Fö 11: Återkoppling från skattade tillstånd

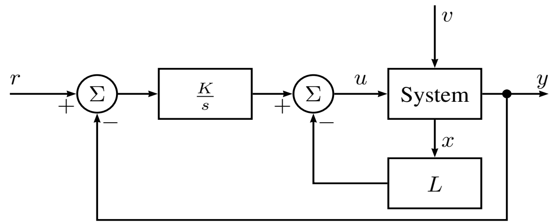
System: $\dot{x} = Ax + Bu$
Observatör: $\dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x})$
Återkoppling: $u = -L\hat{x} + \ell_0 r = -Lx + L\tilde{x} + \ell_0 r$
Slutna systemet:

$$\frac{d}{dt}\begin{bmatrix} x \\ \tilde{x} \end{bmatrix} = \begin{bmatrix} A - BL & BL \\ 0 & A - KC \end{bmatrix}\begin{bmatrix} x \\ \tilde{x} \end{bmatrix} + \begin{bmatrix} \ell_0 B \\ 0 \end{bmatrix} r, \quad y = \begin{bmatrix} C & 0 \end{bmatrix}\begin{bmatrix} x \\ \tilde{x} \end{bmatrix}$$

Egenvärden: ges av $A - BL$, $A - KC$
Överföringsfunktion $r \to y$: $G_c(s) = C(sI - (A - BL))^{-1}B\ell_0$

Integralverkan

För att ta bort stationärt fel, gör en yttre återkoppling med integrator:



Det slutna systemet får då överföringsfunktionen ($K = -\ell_{n+1}$):

$$\begin{pmatrix} \dot{x} \\ \dot{x}_{n+1} \end{pmatrix} = \begin{pmatrix} A - BL & -B\ell_{n+1} \\ -C & 0 \end{pmatrix}\begin{pmatrix} x \\ x_{n+1} \end{pmatrix} + \begin{pmatrix} F \\ 0 \end{pmatrix} v + \begin{pmatrix} 0 \\ 1 \end{pmatrix} r$$

Utblick – DARPA Urban Challenge

En kul reglerteknisk utmaning som ligger i tiden.

- Förarlösa bilar ska köra en komplicerad bana på 10 km i stadsmiljö på max 6 timmar.
- Bilarna ska i princip klara av att ta körkort

Vinnaren fick \$2M $\Rightarrow$

Utblick – Viktigt att kunna sin reglerteknik

Om man inte har rätt poler i observatören så att den är för långsam... eller har fel tecken i regulatorn... kan det gå så här!

`http://www.youtube.com/watch?v=bnv5JP8gL_k`

Föreläsningsmaterial

TSDT16 Felrättande koder

Hints for the exercises in session 5

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1. (5.1)

(a) The parity polynomial $\mathbf{h}(X)$ is calculated as follows

$$\mathbf{h}(X) = \frac{X^{15}+1}{\mathbf{g}(X)} = X^{11}+X^8+X^7+X^5+X^3+X^2+X+1.$$

(b) According to Theorem 5.7 the dual code is generatied by the polynomial

$$\begin{aligned} X^k\mathbf{h}(X^{-1}) &= X^{11}\mathbf{h}(X^{-1}) \\ &= X^{11}+X^{10}+X^9+X^8+X^6+X^4+X^3+1. \end{aligned}$$

(c) According to (5.15) the systematic for of the generator matrix is obtained from the polynomials $\mathbf{b}_i(X)$, $i = 0, 1, \ldots, k-1$. These polynomials are obtained as the remainders from the polynomial division of X^{n-k+i} by $\mathbf{g}(X)$. Alternatively, $\mathbf{b}_{i+1}(X)$, $i = 0, 1, \ldots, k-2$ is obtained as the remainder from the polynomial division of $X\mathbf{b}_i(X)$ by $\mathbf{g}(X)$. The systematic parity-check matrix is either obtained in the same way for the the generator polynomial of the dual code given in (b) or by the standard correspondence given by (3.4) and (3.7). The matrices $\mathbf{G}$ and $\mathbf{H}$ are given below.

$$\mathbf{G} = \begin{pmatrix}
1&1&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&1&1&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&1&1&0&0&1&0&0&0&0&0&0&0&0\\
1&1&0&1&0&0&0&1&0&0&0&0&0&0&0\\
1&0&1&0&0&0&0&0&1&0&0&0&0&0&0\\
0&1&0&1&0&0&0&0&0&1&0&0&0&0&0\\
1&1&1&0&0&0&0&0&0&0&1&0&0&0&0\\
0&1&1&1&0&0&0&0&0&0&0&1&0&0&0\\
1&1&1&1&0&0&0&0&0&0&0&0&1&0&0\\
1&0&1&1&0&0&0&0&0&0&0&0&0&1&0\\
1&0&0&1&0&0&0&0&0&0&0&0&0&0&1
\end{pmatrix}$$

$$\mathbf{H} = \begin{pmatrix}
1&0&0&0&1&0&0&1&1&0&1&0&1&1&1\\
0&1&0&0&1&1&0&1&0&1&1&1&1&0&0\\
0&0&1&0&0&1&1&0&1&0&1&1&1&1&0\\
0&0&0&1&0&0&1&1&0&1&0&1&1&1&1
\end{pmatrix}$$

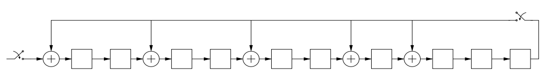
2. (5.3)

First we have to show that the polynomial $\mathbf{g}(X)$ divides $X^{21}+1$. This can be seen from the fact that the parity polynomial $\mathbf{h}(X)$ is

$$\mathbf{h}(X) = \frac{X^{21}+1}{\mathbf{g}(X)} = X^{11}+X^8+X^7+X^2+1.$$

The dimension of the cyclic code is $k = n - \deg \mathbf{g}(X) = 11$.

The syndrom computation circuit as given in Figure 5.5 is in this case as follows.



The received vector $\mathbf{r}$ corresponding to the received polynomial $\mathbf{r}(X) = 1 + X^5 + X^{17}$ is

$$\mathbf{r} = (1,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0).$$

The contents of the syndrom calculation shift register af-

Felrättande koder

Övningsuppgifter

Mekanik

Formelblad i stelkroppsmekanik Y

Beteckningar:

$\mathcal{A}, \mathcal{B}$: godtyckliga punkter

$\mathcal{C}, \mathcal{D}$: kroppsfixa punkter

$\mathcal{P}$: fix punkt i en inertialram

$\mathcal{O}$: fix punkt i kroppen och i en inertialram

$\mathcal{M}$: momentancentrum

$\mathcal{G}$: masscentrum

$\boldsymbol{r}_{\mathcal{AB}}$: $\boldsymbol{r}_{\mathcal{AB}} = \overrightarrow{\mathcal{AB}}$

$\boldsymbol{V}$: godtycklig vektor

$d_{\perp a}$ $(d_{\perp v})$: vinkelräta avståndet mellan $\mathcal{A}$ och $\boldsymbol{a}_{\mathcal{G}}$ $(\boldsymbol{v}_{\mathcal{G}})$

$a_{\mathcal{G}\perp}$ $(v_{\mathcal{G}\perp})$: komposant av $\boldsymbol{a}_{\mathcal{G}}$ $(\boldsymbol{v}_{\mathcal{G}})$ vinkelrät mot $\boldsymbol{r}_{\mathcal{AG}}$

Kinematik

- Polära koordinater:

$$\boldsymbol{v} = \dot{r}\hat{\boldsymbol{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}, \qquad \boldsymbol{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{\boldsymbol{r}} + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{\boldsymbol{\theta}}$$

- O'Briens ekvation:

$$\left(\frac{d\boldsymbol{V}}{dt}\right)_i = \left(\frac{d\boldsymbol{V}}{dt}\right)_j + \boldsymbol{\omega}_{j/i} \times \boldsymbol{V}$$

- Hastighets- och accelerationssamband:

$$\boldsymbol{v}_{\mathcal{A}} = \boldsymbol{v}_{\mathcal{C}} + \boldsymbol{\omega}_k \times \boldsymbol{r}_{\mathcal{CA}} + \boldsymbol{v}_{\mathcal{A}/k}$$
$$\boldsymbol{a}_{\mathcal{A}} = \boldsymbol{a}_{\mathcal{C}} + \dot{\boldsymbol{\omega}}_k \times \boldsymbol{r}_{\mathcal{CA}} + \boldsymbol{\omega}_k \times (\boldsymbol{\omega}_k \times \boldsymbol{r}_{\mathcal{CA}}) + 2\,\boldsymbol{\omega}_k \times \boldsymbol{v}_{\mathcal{A}/k} + \boldsymbol{a}_{\mathcal{A}/k}$$

Då $\mathcal{A}$ är (liksom $\mathcal{C}$) en fix punkt i kropp k, gäller $\boldsymbol{v}_{\mathcal{A}/k} = \boldsymbol{0}$ och $\boldsymbol{a}_{\mathcal{A}/k} = \boldsymbol{0}$.

Kinetik

- Kraft- och momentlagar

$$\boldsymbol{F} = \dot{\boldsymbol{p}} = m\boldsymbol{a}_{\mathcal{G}}$$
$$\boldsymbol{M}_{\mathcal{P}} = \dot{\boldsymbol{h}}_{\mathcal{P}}, \qquad \boldsymbol{M}_{\mathcal{G}} = \dot{\boldsymbol{h}}_{\mathcal{G}}, \qquad \boldsymbol{M}_{\mathcal{G}} = \boldsymbol{I}_{\mathcal{G}}\boldsymbol{\alpha} + \boldsymbol{\omega} \times \boldsymbol{h}_{\mathcal{G}}, \qquad \boldsymbol{M}_{\mathcal{O}} = \boldsymbol{I}_{\mathcal{O}}\boldsymbol{\alpha} + \boldsymbol{\omega} \times \boldsymbol{h}_{\mathcal{O}}$$

- Förflyttningssatser

$$\boldsymbol{h}_{\mathcal{A}} = \boldsymbol{h}_{\mathcal{B}} + \boldsymbol{r}_{\mathcal{AB}} \times m\boldsymbol{v}_{\mathcal{G}}$$
$$\boldsymbol{M}_{\mathcal{A}} = \boldsymbol{M}_{\mathcal{B}} + \boldsymbol{r}_{\mathcal{AB}} \times \boldsymbol{F}$$

- Omskriven momentlag

$$\boldsymbol{M}_{\mathcal{A}} = \dot{\boldsymbol{h}}_{\mathcal{G}} + \boldsymbol{r}_{\mathcal{AG}} \times m\boldsymbol{a}_{\mathcal{G}}$$
$$M_{\mathcal{A}} = I_{\mathcal{G}}\alpha + ma_{\mathcal{G}}d_{\perp a}, \quad M_{\mathcal{A}} = I_{\mathcal{G}}\alpha + mr_{\mathcal{AG}}a_{\mathcal{G}\perp} \qquad \text{(2D)}$$

- Rörelsemängdsmoment

$$\boldsymbol{h}_{\mathcal{O}} = \boldsymbol{I}_{\mathcal{O}}\boldsymbol{\omega}, \qquad \boldsymbol{h}_{\mathcal{G}} = \boldsymbol{I}_{\mathcal{G}}\boldsymbol{\omega}, \qquad \boldsymbol{h}_{\mathcal{M}} = \boldsymbol{I}_{\mathcal{M}}\boldsymbol{\omega},$$
$$h_{\mathcal{A}} = I_{\mathcal{G}}\omega + mv_{\mathcal{G}}d_{\perp v}, \quad h_{\mathcal{A}} = I_{\mathcal{G}}\omega + mr_{\mathcal{AG}}v_{\mathcal{G}\perp} \qquad \text{(2D)}$$

- Arbete och energi

$$U = \Delta T + \Delta V_g + \Delta V_e$$

där en kraft $\boldsymbol{F}$ resp. ett kraftparsmoment $\boldsymbol{C}$ utför arbetet

$$U = \int \boldsymbol{F}\cdot d\boldsymbol{r} \quad \text{resp.} \quad U = \int \boldsymbol{C}\cdot\boldsymbol{\omega}\, dt$$
$$U = \int \boldsymbol{F}\cdot d\boldsymbol{r} \quad \text{resp.} \quad U = \int C\, d\theta \qquad \text{(2D)}$$

Plan rörelse

$$T = \frac{1}{2}mv_{\mathcal{G}}^2 + \frac{1}{2}I_{\mathcal{G}}\omega^2$$
$$T = \frac{1}{2}I_{\mathcal{O}}\omega^2, \qquad T = \frac{1}{2}I_{\mathcal{M}}\omega^2$$

Tredimensionell rörelse

$$T = \frac{1}{2}m\boldsymbol{v}_{\mathcal{G}}\cdot\boldsymbol{v}_{\mathcal{G}} + \frac{1}{2}\boldsymbol{\omega}\cdot\boldsymbol{h}_{\mathcal{G}}$$
$$T = \frac{1}{2}\boldsymbol{\omega}\cdot\boldsymbol{h}_{\mathcal{O}}$$

- Impuls och impulsmoment

$$\int_{t_1}^{t_2} \boldsymbol{F}\, dt = \boldsymbol{p}_2 - \boldsymbol{p}_1 = m\boldsymbol{v}_{\mathcal{G}_2} - m\boldsymbol{v}_{\mathcal{G}_1}$$
$$\int_{t_1}^{t_2} \boldsymbol{M}_{\mathcal{P}}\, dt = \boldsymbol{h}_{\mathcal{P}_2} - \boldsymbol{h}_{\mathcal{P}_1}, \qquad \int_{t_1}^{t_2} \boldsymbol{M}_{\mathcal{G}}\, dt = \boldsymbol{h}_{\mathcal{G}_2} - \boldsymbol{h}_{\mathcal{G}_1}$$

- Tröghetssamband

$$\boldsymbol{I}_{\mathcal{A}} = \int \left(\rho_{\mathcal{A}}^2\boldsymbol{E} - \boldsymbol{\rho}_{\mathcal{A}} \otimes \boldsymbol{\rho}_{\mathcal{A}}\right) dm, \qquad [\boldsymbol{I}_{\mathcal{A}}] = \int \left(\rho_{\mathcal{A}}^2[\boldsymbol{E}] - [\boldsymbol{\rho}_{\mathcal{A}}][\boldsymbol{\rho}_{\mathcal{A}}]^T\right) dm$$

$$[\boldsymbol{I}_{\mathcal{A}}] = \begin{bmatrix} I_{\mathcal{A}xx} & I_{\mathcal{A}xy} & I_{\mathcal{A}xz} \\ I_{\mathcal{A}xy} & I_{\mathcal{A}yy} & I_{\mathcal{A}yz} \\ I_{\mathcal{A}xz} & I_{\mathcal{A}yz} & I_{\mathcal{A}zz} \end{bmatrix}$$

$$I_{\mathcal{A}xx} = \int (y^2 + z^2)\, dm, \qquad I_{\mathcal{A}xy} = -\int xy\, dm$$

$$I_{\mathcal{A}xx} = I_{\mathcal{G}xx} + m(d_y^2 + d_z^2)$$
$$I_{\mathcal{A}xy} = I_{\mathcal{G}xy} - md_xd_y, \qquad \text{där} \quad (d_x,\, d_y,\, d_z) = [\boldsymbol{r}_{\mathcal{AG}}]$$

Algebra

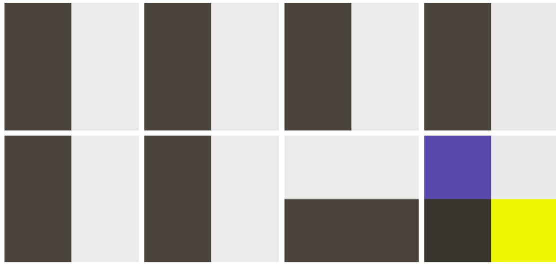
$$\boldsymbol{a}\cdot(\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}\cdot(\boldsymbol{c} \times \boldsymbol{a})$$
$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}\,(\boldsymbol{a}\cdot\boldsymbol{c}) - \boldsymbol{c}\,(\boldsymbol{a}\cdot\boldsymbol{b})$$
$$(\boldsymbol{a} \otimes \boldsymbol{b})\boldsymbol{c} = \boldsymbol{a}(\boldsymbol{b}\cdot\boldsymbol{c}), \qquad [\boldsymbol{a} \otimes \boldsymbol{b}] = [\boldsymbol{a}][\boldsymbol{b}]^T$$

Formelsamling

Digital bildbehandling

Color edge detection

There is not optimal to just sum the magnitude of the gradient for the 3 components.



R+G+B=RGB

R+G+B=RGB

$$g_{xx} = \left|\frac{\partial R}{\partial x}\right|^2 + \left|\frac{\partial G}{\partial x}\right|^2 + \left|\frac{\partial B}{\partial x}\right|^2 \quad (6.7\text{-}5)$$

$$g_{yy} = \left|\frac{\partial R}{\partial y}\right|^2 + \left|\frac{\partial G}{\partial y}\right|^2 + \left|\frac{\partial B}{\partial y}\right|^2 \quad (6.7\text{-}6)$$

$$g_{xy} = \frac{\partial R}{\partial x}\frac{\partial R}{\partial y} + \frac{\partial G}{\partial x}\frac{\partial G}{\partial y} + \frac{\partial B}{\partial x}\frac{\partial B}{\partial y} \quad (6.7\text{-}7)$$

A better method…

Fig. 6.45

Color edge detection

somewhat better

Fig. 6.46

…a better method.

(6.7-8)

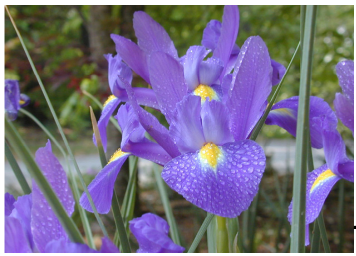
$$\theta(x,y) = \frac{1}{2}\tan^{-1}\left[\frac{2g_{xy}}{g_{xx}-g_{yy}}\right]$$

(6.7-9)

$$F_\theta(x,y) = \left\{\frac{1}{2}\left[(g_{xx}+g_{yy}) + (g_{xx}-g_{yy})\cos 2\theta(x,y) + 2g_{xy}\sin 2\theta(x,y)\right]\right\}^{0.5}$$

Conversion from color to gray scale in MATLAB

True color image



Gray scale image

$$\text{GrayV} = 0.2989\cdot R + 0.5870\cdot G + 0.1140\cdot B \;\; (\text{MatLab: rgb2gray})$$

The YCbCr Color model

Not in G&W!

- Used for image compression, JPEG.
- The relation between R, G, B for the luminance component Y is similar to MATLAB:s rgb2gray.
- Decouples luminance and chromaticity.
- Uses color differences Cb, Cr instead of hue and saturation for chromaticity.
- The fact that chromaticity can be more compressed is utilized. There are higher frequencies in the luminance component.
- Note that luminance Y and intensity I are different. Probably Y describes the subjective brightness better. However, equal amount of R,G, and B in an RGB-image gives a gray-scale image.

$$\begin{pmatrix} Y \\ Cb \\ Cr \end{pmatrix} = \left(\begin{pmatrix} 16 \\ 128 \\ 128 \end{pmatrix} + \begin{pmatrix} 65.481 & 128.553 & 24.966 \\ -37.797 & -74.203 & 112.000 \\ 112.000 & -93.786 & -18.214 \end{pmatrix}\begin{pmatrix} R \\ G \\ B \end{pmatrix}\right) / 255$$

Föreläsningsmaterial

Optimeringslära, grundkurs

Föreläsning om linjärprogrammering

Exempel

$$\begin{array}{rrcrcl}
\max & -x_1 & + & x_2 & & \\
\text{då} & x_1 & + & x_2 & \geq & 2 \\
 & -x_1 & + & 2x_2 & \leq & 6 \\
 & x_1 & & & \leq & 0 \\
 & & & x_2 & \geq & 1
\end{array}$$

$$x_1^{ny} = -x_1 \quad \text{och} \quad x_2^{ny} = x_2 - 1 \Rightarrow$$

$$\begin{array}{rrcrcl}
\max & x_1^{ny} & + & x_2^{ny} + 1 & & \\
\text{då} & -x_1^{ny} & + & x_2^{ny} & \geq & 2-1 \\
 & x_1^{ny} & + & 2x_2^{ny} & \leq & 6-2 \\
 & x_1^{ny} & , & x_2^{ny} & \geq & 0
\end{array}$$

$$\Rightarrow \quad \begin{array}{rl}
\min & -x_1^{ny} - x_2^{ny}(-1) \\
\text{då} & -x_1^{ny} + x_2^{ny} - s_1 = 1 \\
 & x_1^{ny} + 2x_2^{ny} + s_2 = 4 \\
 & x_1^{ny},\ x_2^{ny},\ s_1,\ s_2 \geq 0
\end{array}$$

$$\Rightarrow \quad \begin{array}{rl}
\min & (-1,-1,0,0)\,x \\
\text{då} & \begin{pmatrix} -1 & 1 & -1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \\
 & x \geq 0
\end{array}$$

där $x = (x_1^{ny}, x_2^{ny}, s_1, s_2)^T$.

Arnold's cat map (comp. 6.6.1)

Two-dimensional, area-preserving map ⇒ can model Poincaré map of two-degree of freedom Hamiltonian system.

$$\begin{cases} x(j+1) = x(j) + y(j) \\ y(j+1) = x(j) + 2y(j) \end{cases} \qquad x, y \text{ defined modulo } 1$$

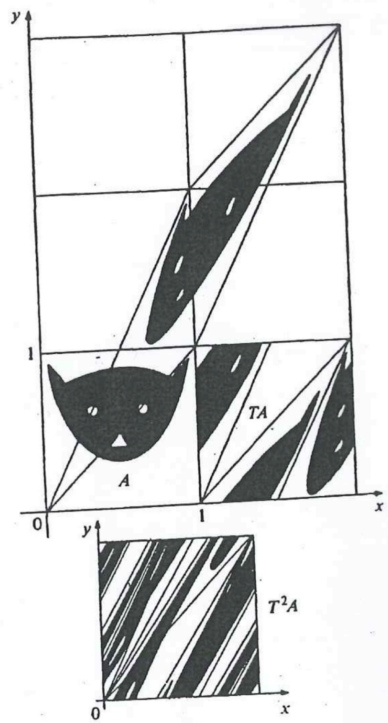
(special case $\alpha = 1$, $\gamma = 1$ of Exercise 6.10)

Jacobian: $\bar{Df} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \Rightarrow \det(\bar{Df}) = 1 \Rightarrow$ Area-preserving.

Cat is stretched according to the equations, and then "cut in pieces" and re-assembled according to the modulo 1 operation.

Unrecognizable after two iterations!

Mixing!



The cat map is a chaotic K-system (even a C-system!), since the Lyapunov exponents are

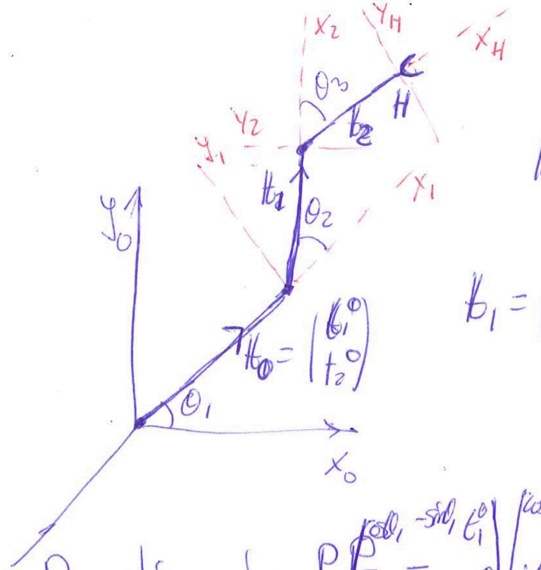
$$\lambda_{1,2} = \ln\left(\frac{3 \pm \sqrt{5}}{2}\right) \approx \pm 0.962$$

(Obtained from $\lambda_{1,2} = \lim_{N\to\infty} \frac{1}{N} \ln|\hat{h}_{1,2}(N)|$, using that $\hat{h}_{1,2}(N) = \tilde{h}_{1,2}^N$ since map is piecewise linear! See compendium and Exercise 6.10!)

Kaos och icke-linjära fenomen

Föreläsning om kaos

Geometri med tillämpningar



$$t_2 = \begin{pmatrix} t_2^2 \\ t_1^2 \end{pmatrix}$$

$$t_1 = \begin{pmatrix} t_1^1 \\ t_2^1 \end{pmatrix}$$

$$t_0 = \begin{pmatrix} t_1^0 \\ t_2^0 \end{pmatrix}$$

Position for P $= \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 & t_1^0 \\ \sin\theta_1 & \cos\theta_1 & t_2^0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta_2 & -\sin\theta_2 & t_1^1 \\ \sin\theta_2 & \cos\theta_2 & t_2^1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta_3 & -\sin\theta_3 & t_1^2 \\ \sin\theta_3 & \cos\theta_3 & t_2^2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_1^H \\ p_2^H \\ 1 \end{pmatrix}$

A camera C (with frame C) observes a ball moving on a table. The ball has a frame B rigidly attached to it with the origin at the centre of the ball. The table T has a frame with the origin, and the X and Y axes in the plane of the table.

The camera reports the following position (and speed) for the ball

$$ {}^{C}_{B}P = \begin{pmatrix} 0.43 & -0.75 & 0.50 & 14.83 \\ 0.78 & 0.05 & -0.61 & 1.1 \\ 0.43 & 0.66 & 0.61 & -1.1 \\ 0 & 0 & 0 & 1.00 \end{pmatrix}$$

and

$$ {}^{C}_{T}P = \begin{pmatrix} 0.86 & 0 & 0.5 & 10.00 \\ 0.35 & 0.7 & -0.61 & 0 \\ -0.35 & 0.7 & 0.61 & 0 \\ 0 & 0 & 0 & 1.00 \end{pmatrix}$$

How does the table "sees" ~~the~~ the approximation of the ball ??

Clearly $\boxed{{}^{T}_{B}P = \left({}^{C}_{T}P\right)^{-1} {}^{C}_{B}P}$ position of B with respect to T

Föreläsning om robotik

Kvantmekanik

TFFY54 **Quantum Mechanics, 6 ECTS credits.**
/Kvantmekanik/

For: **FyN** **MFYS** **MPN** **Y**

Prel. scheduled hours: 54
Rec. self-study hours: 106

Area of Education: Science

Main field of studies: Physics, Applied Physics

Advancement level (G1, G2, A): A

Aim:
The purpose of the course is to give the student a deeper understanding of quantum mechanics and to further develop the students ability to solve quantum mechanical problems. After completed course, the student should be able to:

- derive results based on the postulates of quantum mechanics.
- use various representations of quantum mechanics.
- solve quantum mechanical problems that involve topics listed in the course content.

Prerequisites: (valid for students admitted to programmes within which the course is offered)
Modern physics, linear algebra and fourier transform.

Note: Admission requirements for non-programme students usually also include admission requirements for the programme and threshhold requirements for progression within the programme, or corresponding.

Supplementary courses:
Quantum dynamics; Relativistic quantum mechanics; Elementary particle physics.

Organisation:
The course is divided into lectures and lessons (problem solving sessions).

Course contents:
Historical background. Wave-particle dualism. Wave packets. The time-dependent Schrödinger equation. Probability current density. Expectation values. Hermitian operators. Time-independent Schrödinger equation. Boundary conditions. Properties of eigenfunctions. General solution to the Schrödinger equation. Time evolution operator. The Dirac notation. State space. Adjoint operators. Unitary operators. Commutator. Rigorous proof of the uncertainty principle. Heisenberg's matrix representation. Ehrenfest's theorem. The postulates of quantum mechanics. Harmonic oscillator with operator method. Operators as generators of translation and rotation. Symmetries and conservation laws. Generalized angular momentum. Spherical harmonics. Pauli spin matrices. Spin dynamics. Spherical symmetric potential. The hydrogen atom in magnetic fields. Spin-orbit term. Conceptual problems. Approximative methods: non-degenerate and degenerate perturbation theory; the variational method.

Kursplan

Medicinsk bildanalys

Föreläsningsmaterial om magnetresonanstomografi

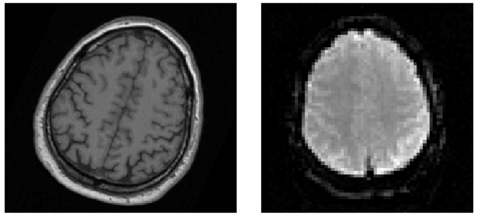


Figure 39: *Image registration is necessary to register the activity map from a low resolution fMRI dataset to a high resolution T_1-volume. The image to the right is though an fMRI slice and not the activity map.*

The methods that will be covered in this chapter are intensity based registration and phase based registration. First the optical flow algorithm will be explained and then the mutual information algorithm will be explained.

4.2 Transformation models

Image registration is normally divided into different areas, depending on the transformation model that is used. One area of transformation models are linear transformations, that include translation, rotation, scaling and other affine transforms. If the transformation is limited to rotation and translation, the registration is *rigid* since the size and the shape of the object can not change. The motion fields for linear transformations can be described by a number of parameters. Another area is *non-rigid* registration where the shape and the size of the object can change, not only globally but locally. These motion fields can however not be described by a number of parameters. In order to get a smooth transition between the local motion fields, it is common to *reguralize* the motion field.

4.3 Intensity based registration using gradient filters

The most common way to register two images is to look at the intensity itself. One way to do this is to calculate the optical flow between the images.

The optical flow registration algorithm is based on 2 assumptions

I. The motion can locally be described as a movement $\Delta \boldsymbol{x}$.

$$I(\boldsymbol{x}, t) = I(\boldsymbol{x} + \Delta \boldsymbol{x}, t + 1) \tag{34}$$

II. The image can locally be described as a leaning plane

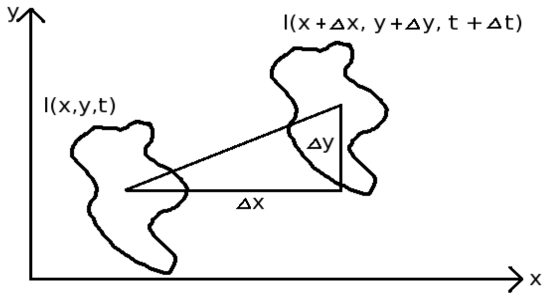


Figure 40: *An object in the image has moved between two timepoints. The movement in the x-direction is Δx, the movement in the y-direction is Δy and the time difference between the images is Δt. The two images and the time difference are given, the goal is to calculate the motion vector that describes the movement between the images.*

$$I(\boldsymbol{x} + \boldsymbol{v}(\boldsymbol{x}), t + 1) = I(\boldsymbol{x}, t + 1) + \nabla \boldsymbol{I}^T \boldsymbol{v} \tag{35}$$

where $\nabla \boldsymbol{I} = [\nabla_x I, \nabla_y I]^T$.

The first assumption says that the intensity does not change between the two images. The second assumption says that the image locally can be described with a first order Taylor expansion.

By using the first assumption we can write that the pixel value at the position (x, y) at timepoint t is equal to the pixel value at $(x + \Delta x, y + \Delta y)$ at timepoint $t + \Delta t$, where Δx is the movement in the x-direction, Δy is the movement in the y-direction and Δt is the time difference between the images. If the intensity changed between the two images, this would not be true.

$$I(x + \Delta x, y + \Delta y, t + \Delta t) = I(x, y, t) \tag{36}$$

Using the other assumption, we apply a first order Taylor expansion to the expression above and get

$$\Delta x \nabla_x I + \Delta y \nabla_y I - \Delta t (I_1 - I_2) = 0 \tag{37}$$

where I_2 is the second image and I_1 is the first image and $I_1 - I_2$ is an estimate of the time derivative.

If we divide each term by Δt we get

$$\frac{\Delta x}{\Delta t} \nabla_x I + \frac{\Delta y}{\Delta t} \nabla_y I - (I_1 - I_2) = 0 \tag{38}$$

and since $\frac{\Delta x}{\Delta t}$ is the velocity in the x-direction v_x and $\frac{\Delta y}{\Delta t}$ is the velocity in the y-direction v_y we finally get the classical flow equation

$$v_x \nabla_x I + v_y \nabla_y I - (I_1 - I_2) = 0 \tag{39}$$

$\nabla_x I$ is here the derivative of the image in the x-direction and $\nabla_y I$ is the derivative of the image in the y-direction. A more compact way to write this is

$$\nabla I^T \boldsymbol{v} - \Delta I = 0 \tag{40}$$

where $\Delta I = I(\boldsymbol{x}, t) - I(\boldsymbol{x}, t + 1) = I_1 - I_2$.

The derivative of the image tells us the possible directions of the movement, since we only can detect movement if there is some structure in the image. If an area of the image is flat, it is impossible to know if there has been any movement or not. The velocity tells us how big the movement is, and the scalar product projects the velocity on the gradient of the image.

To get the derivative of the image, we can apply a number of gradient filters. The Sobel-operator is a common way of calculating the gradient of an image, it will give the gradient and a lowpass effect in one direction, and a lowpass effect only in the perpendicular direction.

The Sobel-operator consists of two 3 x 3 filters, one that is used to calculate the x-gradient and one that is used to calculate the y-gradient.

$$G_x = \begin{bmatrix} +1 & 0 & -1 \\ +2 & 0 & -2 \\ +1 & 0 & -1 \end{bmatrix} /8 \tag{41}$$

$$G_y = \begin{bmatrix} +1 & +2 & +1 \\ 0 & 0 & 0 \\ -1 & -2 & -1 \end{bmatrix} /8 \tag{42}$$

The gradient magnitude can then be calculated as

$$G = \sqrt{\nabla_x I^2 + \nabla_y I^2} \tag{43}$$

and the direction of the gradient can be calculated as

$$\phi = \arctan \left(\frac{\nabla_y I}{\nabla_x I} \right) \tag{44}$$

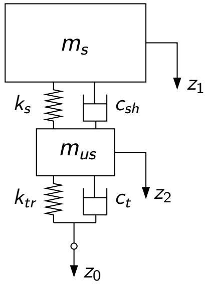
where $\nabla_x I$ is the filter response from G_x and $\nabla_y I$ is the filter response from G_y.

Fordonsdynamik med reglering

Föreläsningsmaterial

Kvartsbilsmodell

En modell med den fjädrade massan m_s (karossmassa) och den ofjädrade massan m_{us} (hjul- och axelmassa).



Dynamiska ekvationer

$$m_s\ddot{z}_1 + c_{sh}(\dot{z}_1 - \dot{z}_2) + k_s(z_1 - z_2) = 0$$
$$m_{us}\ddot{z}_2 + c_{sh}(\dot{z}_2 - \dot{z}_1) + k_s(z_2 - z_1) + c_t\dot{z}_2 + k_{tr}z_2 = c_t\dot{z}_0 + k_{tr}z_0$$

Odämpat system

Systemet har lösningar på formen:

$$z_1 = Z_1\cos(\omega_n t - \varphi)$$
$$z_2 = Z_2\cos(\omega_n t - \varphi)$$

Vinkelfrekvenserna ω_n ges av den karakteristiska ekvationen

$$\det(-\omega_n^2 M + A) = 0$$

och egenvektorerna

$$Z = \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}$$

ges av det homogena ekvationssystemet

$$(-\omega_n^2 M + A)Z = 0$$

Odämpat system

Utan dämpning och med $z_0 = 0$ får vi

$$m_s\ddot{z}_1 + k_s z_1 - k_s z_2 = 0$$
$$m_{us}\ddot{z}_2 - k_s z_1 + (k_s + k_{tr})z_2 = 0$$

Kan skrivas

$$M\ddot{\mathbf{z}} + A\mathbf{z} = 0$$

där matriserna

$$M = \begin{bmatrix} m_s & 0 \\ 0 & m_{us} \end{bmatrix}, \quad A = \begin{bmatrix} k_s & -k_s \\ -k_s & k_s + k_{tr} \end{bmatrix}$$

är positivt definita och symmetriska.

Dämpat system

Tar vi med dämpning får vi systemet

$$M\ddot{\mathbf{z}} + C\dot{\mathbf{z}} + A\mathbf{z} = \mathbf{f}(t)$$

där

$$C = \begin{bmatrix} c_{sh} & -c_{sh} \\ -c_{sh} & c_{sh} + c_t \end{bmatrix}, \quad \mathbf{f}(t) = \begin{pmatrix} 0 \\ c_t\dot{z}_0 + k_{tr}z_0 \end{pmatrix}$$

Genom att transformera systemet får vi

$$G(s)\tilde{z}(s) = \tilde{f}(s)$$

där

$$G(s) = s^2 M + sC + A$$

Solutions TATA71 2011-12-15

1. (a) The substitution $y(x) = 1/z(x)$, $z(x) \neq 0$ gives $y'(x) = -z'(x)/z(x)^2$ and $z' = \frac{1}{1+x^2}$.
So $z(x) = \arctan x + C$ and $y(x) = 1/(\arctan x + C)$.
(b) By separating variables we get $y'/y^2 = -\frac{1}{1+x^2}$ if $y(x) \neq 0$. So $-1/y = -\arctan x + C$ and again $y(x) = 1/(\arctan x + C)$. We also have to check whether the condition $y(x) \neq 0$ does not exclude a possible solution. Substitution into the equation shows that $y(x) = 0$ is also a solution.
(c) From the IC $1 = y(0) = \frac{1}{C}$ the solution is $y(x) = 1/(\arctan x + 1)$. The second IC $y(0) = 0$ is satisfied by $y(x) = 0$.

----

2. From $2x - y - 2 = 0$, $3x - 2y - 2 = 0$ the equilibrium point is $(x_0 = 2, y_0 = 2)$. The linear change of variables $x = u + 2$, $y = w + 2$ moves the equilibrium point to $(0,0)$ and (u,w) satisfies the homogeneous system $\dot{u} = 2u - w$, $\dot{w} = 3u - 2w$. $\det\begin{bmatrix} 2-\lambda & -1 \\ 3 & -2-\lambda \end{bmatrix} = 0$ gives eigenvalues $\lambda_1 = 1, \lambda_2 = -1$ and the eigenvectors $w_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $w_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. The general solution is $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} u(t)+2 \\ w(t)+2 \end{bmatrix} = A\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{t} + B\begin{bmatrix} 1 \\ 3 \end{bmatrix}e^{-t} + \begin{bmatrix} 2 \\ 2 \end{bmatrix}$.
For (u,w) - system the origin is an unstable equilibrium of the saddle point type. The line through the origin along the eigenvector w_1 contains two outgoing trajectories and the line along the eigenvector w_2 contains two ingoing trajectories. The remaining trajectories are hyperbolas approaching asymptotically these lines and they are directed consistently with the directions of trajectories along the lines w_1 and w_2.
The phase space diagram for the (x,y) system has the equilibrium point shifted to the point $(2,2)$.

-----

3. The ansatz for solution $y(x) = u(x)z(x) = xu(x)$ gives $(1-x^2)xu'' + 2u' = 0$. The unknown $w = u'$ satisfies a first order separable ODE $(1-x^2)xw' + 2w = 0$. So $w'/w = 2/(x^2-1)x$ and $w'/w = [-2/x] + [1/(x+1)] + [1/(x-1)]$. By integrating $w = \pm e^C(x^2-1)/x^2 = \pm e^C(1-1/x^2)$ and $u(x) = \int w(x)dx = \pm e^C(x + 1/x) + D = \pm e^C(x^2+1)/x + D$. From the second solution $y(x) = ux = e^C(x^2+1) + Dx$ one can take $y(x) = (x^2+1)$ as the simplest linearly independent solution. The determinant of the Wronskian matrix $Det\begin{bmatrix} z & y \\ z' & y' \end{bmatrix} = Det\begin{bmatrix} x & x^2+1 \\ 1 & 2x \end{bmatrix} = x^2 - 1 \neq 0$ for $-1 < x < 1$ proves linear independence.

----

4. Equations $y - 1 = 0$, $x^2 - y = 0$ have two solutions $(x_1 = 1, y_1 = 1)$, $(x_2 = -1, y_2 = 1)$.
The Jacobian matrix is $J = \begin{bmatrix} 0 & 1 \\ 2x & -1 \end{bmatrix}$. At the equilibrium points: $J_1 = J(x_1, y_1) = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}$ has the eigenvalues $\lambda_+ = 1, \lambda_- = -2$ and the eigenvectors $\mathbf{w}_+ = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{w}_- = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$. $J_2 = J(x_2, y_2) = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix}$ has the eigenvalues $\lambda_\pm = -\frac{1}{2}(1 \pm i\sqrt{7})$ and no real eigenvectors. So $(1,1)$ is a saddle point and $(-1,1)$ is a spiral sink. The direction of the vectorfield on the nullclines: $\begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix} = \begin{bmatrix} 0 \\ x^2-1 \end{bmatrix}$ on $y - 1 = 0$, $\begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix} = \begin{bmatrix} y-1 \\ 0 \end{bmatrix}$ on $x^2 - y = 0$ is consistent with this conclusion.

Ordinära differentialekvationer och dynamiska system

Tentamenslösningar

Relativitetsteori

Rumtidens geometri nära ett statiskt, sfäriskt symmetriskt svart hål beskrivs alltså av 4×4-matrisen

$$\begin{bmatrix} 1-\frac{2m}{r} & 0 & 0 & 0 \\ 0 & -\left(1-\frac{2m}{r}\right)^{-1} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{bmatrix}$$

i koordinaterna (t, r, θ, φ)

Konstanten m kallas det svarta hålets <u>geometriska massa</u>.

Föreläsning om svarta hål

Lecture 4
3D graphics part 2

Today's topics:

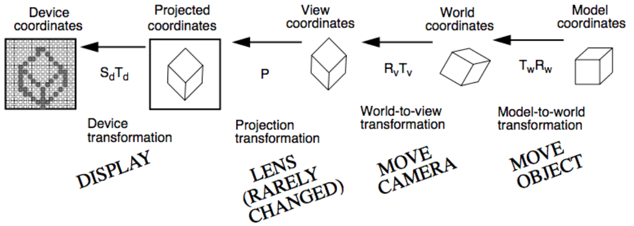
Object representation (intro)

Visible surface detection (intro):
Z-buffer
Back-face culling

Illumination models, lighting

Transformation pipeline

Model coordinates
World coordinates
View coordinates
Projected coordinates
Device coordinates



Transformations in 2D and 3D with homogenous coordinates

$$\begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Projection by using the bottom row with homogenous coordinates

$$\begin{bmatrix} 2n/(r-l) & 0 & A & 0 \\ 0 & 2n/(t-b) & B & 0 \\ 0 & 0 & C & D \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

A, B usually zero

$C = -(f+n)/(f-n)$

$D = -2f*n/(f-n)$

Normalized coordinates and viewing frustum

Föreläsningsmaterial

Elementarpartikelfysik

The group SU(2) (D.J. Sec. 8.4)

(2:1)

"Special unitary group in two dimensions"

Important in description of spin and isospin

Fundamental representation of rotation group: $D^{(1/2)}_{m,m'}$ ($j=1/2$)

$m, m' = \pm 1/2 \Rightarrow 2\times 2$-matrices!

Ex.: spin $\frac{1}{2}$ particle; $j \equiv s = 1/2$; $m \equiv m_s = \pm 1/2$

$\Rightarrow$ Two eigenstates $|s, m_s\rangle$ of commuting operators S^2 and S_z:

$|1/2, +1/2\rangle$ "spin up"; $|1/2, -1/2\rangle$ "spin down"

If system is invariant under rotation, these states are indistinguishable!

Connected by "raising/lowering" operators $S_\pm = S_x \pm i S_y$

Write eigenstates as column vectors: $\underbrace{\begin{pmatrix}1\\0\end{pmatrix}}_{\text{"spin up"}}, \underbrace{\begin{pmatrix}0\\1\end{pmatrix}}_{\text{"spin down"}}$

$\Rightarrow S_+ = \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}$; $S_- = \begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}$ (Exercise: show that this choice fulfills $S_\pm|s, m_s\rangle = \sqrt{s(s+1) - m_s(m_s \pm 1)}\,|s, m_s \pm 1\rangle$)

$\Rightarrow$ spin component matrices:

$$S_x = \begin{pmatrix}0 & 1/2\\1/2 & 0\end{pmatrix}; \quad S_y = \begin{pmatrix}0 & -i/2\\i/2 & 0\end{pmatrix}; \quad S_z = \begin{pmatrix}1/2 & 0\\0 & -1/2\end{pmatrix}$$

Pauli matrices σ divided by 2 (in units of $\hbar$)

The group SU(3) (D.J. Sec. 10.1)

(14:2)

In general, we may define groups $SU(n)$ as the symmetry groups of all unitary $n\times n$-matrices with determinant 1. (cf. SU(2)!)

An arbitrary matrix U in $SU(n)$ can be represented as:

$$U = e^{-i\theta\sum_{j=1}^{n^2-1} n_j F_j} \equiv e^{-i\theta\hat{n}\cdot\bar{F}}$$

($\hat{n} = (n_1, \ldots, n_{n^2-1})$: (n^2-1)-dim. unit vector)

(cf. SU(2): $U = e^{-i\theta\hat{n}\cdot\bar{S}}$; $\bar{S} = (\frac{\tau_1}{2}, \frac{\tau_2}{2}, \frac{\tau_3}{2})$ Pauli matrices ($\tau_i \equiv \sigma_i$ before!) $(2^2-1=3)$)

The (n^2-1) $(n\times n)$-dimensional matrices F_j are the generators of the group $SU(n)$. They are linearly independent and span the space of all Hermitian traceless $(n\times n)$-matrices.

For particular case of $SU(3)$, we have 8 generators, and in analogy with Pauli matrices we introduce the 8 Gell-Mann matrices $\lambda_j \equiv 2F_j$: (D.J. Sec. 10.1.1)

$$\lambda_1 = \begin{pmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 0\end{pmatrix} \quad \lambda_2 = \begin{pmatrix}0 & -i & 0\\i & 0 & 0\\0 & 0 & 0\end{pmatrix} \quad \lambda_3 = \begin{pmatrix}1 & 0 & 0\\0 & -1 & 0\\0 & 0 & 0\end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix}0 & 0 & 1\\0 & 0 & 0\\1 & 0 & 0\end{pmatrix} \quad \lambda_5 = \begin{pmatrix}0 & 0 & -i\\0 & 0 & 0\\i & 0 & 0\end{pmatrix} \quad \lambda_6 = \begin{pmatrix}0 & 0 & 0\\0 & 0 & 1\\0 & 1 & 0\end{pmatrix}$$

$$\lambda_7 = \begin{pmatrix}0 & 0 & 0\\0 & 0 & -i\\0 & i & 0\end{pmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & -2\end{pmatrix} \qquad (10.3)$$

Note: $\lambda_i = \begin{pmatrix}\tau_i & \begin{matrix}0\\0\end{matrix}\\ \begin{matrix}0 & 0\end{matrix} & 0\end{pmatrix}$; $(i = 1,2,3)$

$\Rightarrow$ I spin operators form SU(2) subgroup of SU(3)!

Note also: Two diagonal operators: λ_3 and λ_8

I_3 ; Y, hypercharge!

Explicit calculation shows that generators satisfy commutation relation:

$$[F_i, F_j] = i f_{ijk} F_k$$

structure constants, antisymmetric under interchange of any two indices

Table 10.1 The structure constants of SU(3)

$f_{123} = 1$
$f_{147} = f_{246} = f_{257} = f_{345} = f_{516} = f_{637} = \frac{1}{2}$
$f_{458} = f_{678} = \sqrt{3}/2$

(2:2)

$\therefore$ Introduction of weak hypercharge makes a link between electromagnetic current and weak currents!

$\Rightarrow$ Unification of weak and electromagnetic interactions has revealed a larger underlying symmetry group, $SU(2)_L \otimes U(1)_Y$

Table 13.2 Weak isospin and weak hypercharge assignments for fermions

	I	I_3	Q	Y	
ν_e, ν_μ, ν_τ	$\frac{1}{2}$	$+\frac{1}{2}$	0	-1	3 left-handed lepton doublets
e_L^-, μ_L^-, τ_L^-	$\frac{1}{2}$	$-\frac{1}{2}$	-1	-1	
e_R^-, μ_R^-, τ_R^-	0	0	-1	-2	3 right-handed lepton singlets
u_L, c_L, t_L	$\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{2}{3}$	$+\frac{1}{3}$	3 left-handed quark doublets
d'_L, s'_L, b'_L	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{3}$	$+\frac{1}{3}$	
u_R, c_R, t_R	0	0	$+\frac{2}{3}$	$+\frac{4}{3}$	2x3 right-handed quark singlets
d'_R, s'_R, b'_R	0	0	$-\frac{1}{3}$	$-\frac{2}{3}$	

$\Rightarrow$ New quantum numbers: ($Q = I_3 + \frac{1}{2}Y$)

Important note about the quark states d', s', b':

The weak interaction quark eigenstates are not the same as the mass or strong interaction eigenstates!!

Cabibbo (1963): First generation weak isospin doublet is:

$$\begin{pmatrix}u\\d'\end{pmatrix} = \begin{pmatrix}u\\d\cdot\cos\theta_c + s\cdot\sin\theta_c\end{pmatrix}$$

θ_c: Cabibbo angle (D.J. Sec. 11.10)

d': Cabibbo "rotated" quark

$\therefore$ Quark mixing between charge $-\frac{1}{3}$-quarks d and s! (Experiments: $\Rightarrow \theta_c \approx 13°$)

Later, second generation doublet: $\begin{pmatrix}c\\s'\end{pmatrix} = \begin{pmatrix}c\\s\cos\theta_c - d\sin\theta_c\end{pmatrix}$ (GIM 1970)

$$\Rightarrow \begin{pmatrix}d'\\s'\end{pmatrix} = \begin{pmatrix}\cos\theta_c & \sin\theta_c\\-\sin\theta_c & \cos\theta_c\end{pmatrix}\begin{pmatrix}d\\s\end{pmatrix}$$

(D.J. Sec. 11.11)

Finally (?), third generation $\begin{pmatrix}t\\b'\end{pmatrix}$, where $\begin{pmatrix}d'\\s'\\b'\end{pmatrix} = \underbrace{\begin{pmatrix}V_{ud} & V_{us} & V_{ub}\\V_{cd} & V_{cs} & V_{cb}\\V_{td} & V_{ts} & V_{tb}\end{pmatrix}}_{\text{"K-M mixing matrix"}}\begin{pmatrix}d\\s\\b\end{pmatrix}$

(Kobayashi-Maskawa, 1973)

(D.J. Sec. 11.12)

$$V = \begin{pmatrix}c_{12}c_{13} & s_{12}c_{13} & s_{13}\exp(-i\delta_{13})\\-s_{12}c_{23} - c_{12}s_{23}s_{13}\exp(i\delta_{13}) & c_{12}c_{23} - s_{12}s_{23}s_{13}\exp(i\delta_{13}) & s_{23}c_{13}\\s_{12}s_{23} - c_{12}c_{23}s_{13}\exp(i\delta_{13}) & -c_{12}s_{23} - s_{12}c_{23}s_{13}\exp(i\delta_{13}) & c_{23}c_{13}\end{pmatrix}$$

Complex phase $\delta_{13} \neq 0$ gives rise to CP-violation!! (D.J. Sec. 14.3.10)

Experimental values of moduli of K-M matrix elements:

$$|V| = \begin{array}{c} \begin{array}{ccc}(d) & (s) & (b)\end{array} \\ \begin{pmatrix}0.9754 \pm 0.004 & 0.2206 \pm 0.0018 & 0.0045 \pm 0.0010\\0.2203 \pm 0.0019 & 0.9743 \pm 0.0005 & 0.045 \pm 0.005\\0.0101 \pm 0.0086 & 0.0449 \pm 0.0062 & 0.9990 \pm 0.0002\end{pmatrix} \begin{array}{c}(u)\\(c)\\(t)\end{array}\end{array}$$

Föreläsningsanteckningar

Ekonomisk analys: Besluts- och finansiell metodik

Tentamenslösningar

Uppgift 4

a)
Imperfekt (ty alla beslutspunkter formar inte sitt egna informationsrum)
Säker (eftersom naturen inte drar alls)
Asymmetrisk (då informationen spelarna har att tillgå förändras under spelets gång)
Fullständig (båda spelarnas information styrs inte av naturen)

b) Formulera spelets på **normalform** genom att ta fram spelarnas **strategier** och bilda **utdelningsmatrisen** enligt:
a1 = A3 b1 = B1
a2 = A1, C1 b2 = B2
a3 = A1, C2 b3 = B3
a4 = A2, D1
a5 = A2, D2

$$\begin{array}{c|c} & \begin{array}{ccc} b1 & b2 & b3 \end{array} \\ \begin{array}{c} a1 \\ a2 \\ a3 \\ a4 \\ a5 \end{array} & \begin{bmatrix} 3 & 3 & 3 \\ 12 & -2 & 7 \\ 8 & 4 & 6 \\ 4 & 12 & -7 \\ 2 & 10 & 0 \end{bmatrix} \end{array}$$

c) Spelets **jämviktslösning** fås genom följande analys:

$$\begin{array}{cccc} & \begin{array}{ccc} b1 & b2 & b3 \end{array} & & rad\ \text{min} \\ \begin{array}{c} a1 \\ a2 \\ a3 \\ a4 \\ a5 \end{array} & \begin{bmatrix} 3 & 3 & 3 \\ 12 & -2 & 7 \\ 8 & 4 & 6 \\ 4 & 12 & -7 \\ 2 & 10 & 0 \end{bmatrix} & & \begin{array}{c} 3 \\ -2 \\ 4 \\ -7 \\ 0 \end{array} \\ kol\ \text{max} & \begin{array}{ccc} 12 & 12 & 7 \end{array} & & \end{array}$$

Ingen sadelpunkt då max radmin = 4 är skiljt från min kolmax = 7.
Sök därför blandade strategier.
a1 domineras av a3 och b1 domineras av b3.
Detta leder till att B endast har två strategier att välja bland.

$$\begin{array}{cccc} & \begin{array}{cc} b2 & b3 \end{array} & & rad\ \text{min} \\ \begin{array}{c} a2 \\ a3 \\ a4 \\ a5 \end{array} & \begin{bmatrix} -2 & 7 \\ 4 & 6 \\ 12 & -7 \\ 10 & 0 \end{bmatrix} & & \begin{array}{c} -2 \\ 4 \\ -7 \\ 0 \end{array} \\ kol\ \text{max} & \begin{array}{cc} 12 & 7 \end{array} & & \end{array}$$

Säg att B väljer b2 med sannolikhet p.

1) 1. Newton's equation:

$$m\ddot{z}=-k\left(z-z_0\right)\Rightarrow z=z_0+A\cos(\omega t+\delta)\ \ ;\ \ \omega=\sqrt{k/m}\Rightarrow \ddot{z}=-A\omega^2\cos(\omega t+\delta)$$

$$\mathbf{p}=qz\hat{\mathbf{z}}\Rightarrow \ddot{\mathbf{p}}=q\ddot{z}\hat{\mathbf{z}}=-\hat{\mathbf{z}}qA\omega^2\cos(\omega t+\delta)\Rightarrow \ddot{p}^2=q^2A^2\omega^4\cos^2(\omega t+\delta)$$

a) $$P=\frac{2\left[\ddot{p}^2\right]}{3c^3}=\frac{2q^2A^2\omega^4}{3c^3}\cos^2(\omega t'+\delta)\ \ ;\ \ t'=t-r/c$$

b) $$\langle P\rangle=\frac{2q^2A^2\omega^4}{3c^3}\left\langle\cos^2(\omega t'+\delta)\right\rangle=\frac{2q^2A^2\omega^4}{3c^3}\frac{\omega}{2\pi}\int_0^{2\pi/\omega}dt'\cos^2(\omega t'+\delta)$$

$$=\frac{2q^2A^2\omega^4}{3c^3}\frac{\omega}{2\pi}\frac{1}{2}\int_0^{2\pi/\omega}dt'\underbrace{\left[\cos^2(\omega t'+\delta)+\sin^2(\omega t'+\delta)\right]}_{1}=\frac{2q^2A^2\omega^4}{3c^3}\frac{\omega}{2\pi}\frac{1}{2}\frac{2\pi}{\omega}=\underline{\frac{q^2A^2\omega^4}{3c^3}}$$

2a) Monopole moment: 0 (total charge is zero)

Dipole moment: 0 (inversion symmetry)

Quadrupole tensor: Symmetry gives

$$\tilde{Q}=\begin{pmatrix}Q_{11}&0&0\\0&-Q_{11}&0\\0&0&0\end{pmatrix}$$

$$Q_{ij}=\int_{-\infty}^{+\infty}dx_1\int_{-\infty}^{+\infty}dx_2\int_{-\infty}^{+\infty}dx_3\rho(\mathbf{r})\left(3x_ix_j-r^2\delta_{ij}\right)$$

$$Q_{11}=\int_{-a}^{a}dx\,\tfrac{-q}{2a}\left(3x^2-x^2\right)+\int_{-a}^{a}dy\,\tfrac{q}{2a}\left(3\cdot0^2-y^2\right)$$

$$=\tfrac{-q}{2a}\left(4\frac{a^3}{3}+2\frac{a^3}{3}\right)=-a^2q$$

2b) $$\Phi^{(4)}=-\tfrac{1}{6}Tr\left[\tilde{Q}\cdot\tilde{\varphi}\right]$$

$$\tilde{\varphi}=\frac{1}{r^5}\begin{pmatrix}r^2-3x^2&-3xy&-3xz\\-3xy&r^2-3y^2&-3yz\\-3xz&-3yz&r^2-3z^2\end{pmatrix}$$

$$\tilde{Q}\cdot\tilde{\varphi}=\frac{Q_{11}}{r^5}\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix}\begin{pmatrix}r^2-3x^2&-3xy&-3xz\\-3xy&r^2-3y^2&-3yz\\-3xz&-3yz&r^2-3z^2\end{pmatrix}$$

$$=\frac{Q_{11}}{r^5}\begin{pmatrix}r^2-3x^2&-3xy&-3xz\\3xy&-r^2+3y^2&3yz\\0&0&0\end{pmatrix}$$

$$\Phi^{(4)}=-\tfrac{1}{6}Tr\left[\tilde{Q}\cdot\tilde{\varphi}\right]=-\tfrac{1}{6}\frac{-qa^2}{r^5}\left(r^2-3x^2-r^2+3y^2+0\right)$$

$$=\tfrac{1}{2}\frac{qa^2}{r^5}\left(y^2-x^2\right)=\tfrac{1}{2}\frac{qa^2}{r^5}r^2\sin^2\theta\left(\sin^2\varphi-\cos^2\varphi\right)$$

$$=\frac{qa^2}{r^3}\sin^2\theta\left(\sin^2\varphi-\tfrac{1}{2}\right)$$

Elektromagnetisk fälteori och vågutbredning

Tentamenslösningar

Numerisk linjär algebra

Example: Matrix–Matrix multiply (cont.)

Alternative 2: Block storage.

$$\begin{pmatrix} C_{11} & \dots & C_{1p} \\ \vdots & & \vdots \\ C_{p1} & & C_{pp} \end{pmatrix} = \begin{pmatrix} A_{11} & \dots & A_{1p} \\ \vdots & & \vdots \\ A_{p1} & & A_{pp} \end{pmatrix} \begin{pmatrix} B_{11} & \dots & B_{1p} \\ \vdots & & \vdots \\ B_{p1} & & B_{pp} \end{pmatrix}$$

- Blocks are of size $\sqrt{n} \times \sqrt{n}$. Three blocks fit into cache.
- Want to Compute $C_{ij} = \sum_{k=1}^{p} A_{ik} B_{kj}$, $p = \sqrt{n}$.
  - Keep C_{ij} in Cache. Load A_{ik} and B_{kj} and compute $A_{ik} B_{kj}$.
  - Two main memory calls and $(\sqrt{n})^3$ multiply/additions.

Still need n^3 multiply/additions. But only $2(\sqrt{n})^3 = 2n^{1.5}$ main memory access calls.

Matrix-Matrix multiply

Compute $C = AB$ by

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

Remark This is not a *definition*. It is one possible algorithm for computing the matrix C representing the composite mapping $A \circ B$.

Question The algorithm require n^3 multiplications and $n^2(n-1)$ additions. Is it possible to do better?

Strassen's Matrix-Matrix multiply

Regular matrix-matrix multiply is

$$\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}.$$

This requires 8 multiplications (and 4 additions). An equivalent formula is

$$\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} p_1 + p_4 - p_5 + p_7 & p_3 + p_5 \\ p_2 + p_4 & p_1 + p_3 - p_2 + p_6 \end{pmatrix},$$

where

$$\begin{aligned} &p_1 = (a_{11} + a_{22})(b_{11} + b_{22}), \quad p_2 = (a_{21} + a_{22})b_{11}, \quad p_3 = a_{11}(b_{12} - b_{22}), \\ &p_4 = a_{22}(b_{21} - b_{11}), \quad p_5 = (a_{11} + a_{12})b_{22}, \quad p_6 = (a_{21} - a_{11})(b_{11} + b_{12}), \\ &p_7 = (a_{12} - a_{22})(b_{21} + b_{22}). \end{aligned}$$

Only requires 7 multiplications (and 18 additions). Volker Strassen, 1969.

Strassen's Matrix-Matrix multiply

```
function [C]=strassen(A,B);
 [n,m]=size(A);n=n/2;
 if (n==1/2),      % This is the case A and B are scalars.
     C=A*B;
 else
    A11=A(1:n,1:n);A12=A(1:n,n+1:2*n);A21=A(n+1:2*n,1:n);A22=A(n+1:2*n,n+1:2*n);
    B11=B(1:n,1:n);B12=B(1:n,n+1:2*n);B21=B(n+1:2*n,1:n);B22=B(n+1:2*n,n+1:2*n);
    P1=strassen( A11+A22 , B11+B22 ); P2=strassen( A21+A22 , B11 );
    P3=strassen( A11 , B12-B22 );     P4=strassen( A22 , B21-B11 );
    P5=strassen( A11+A12 , B22 );     P6=strassen( A21-A11 , B11+B12 );
    P7=strassen( A12-A22 , B21+B22 );
    C11=P1+P4-P5+P7;C12=P3+P5;C21=P2+P4;C22=P1+P3-P2+P6;
    C=[ C11 , C12 ; C21 , C22 ];
 end;
```

Remark: Requires $\mathcal{O}(n^{\log_2(7)}) = \mathcal{O}(n^{2.807})$ multiplications/additions.

Föreläsning om matrismultiplikation