5) Ekvationssystem och matriser

Exempel:

$$\begin{aligned} 3x_1 - 2x_2 + 5x_3 &= 7 \\ -x_1 + 2x_2 + x_3 &= 5 \end{aligned}$$

Koefficientmatris: $A = \begin{pmatrix} 3 & -2 & 5 \\ -1 & 2 & 1 \end{pmatrix}$

totalmatris

$$\left(\begin{array}{ccc|c} 3 & -2 & 5 & 7 \\ -1 & 2 & 1 & 5 \end{array}\right)$$

Högerleds-kolonnmatris $B = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$

Variabler: $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

Matrisform: $AX = B$

VL: $\begin{pmatrix} 3 & -2 & 5 \\ -1 & 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3x_1 - 2x_2 + 5x_3 \\ -x_1 + 2x_2 + x_3 \end{pmatrix}$

HL: $\begin{pmatrix} 7 \\ 5 \end{pmatrix}$

Homogena och inhomogena system

inhomogent system

$$(*)\quad AX = B \qquad \begin{pmatrix} 3 & -2 & 5 \\ -1 & 2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix} \qquad \begin{aligned} 3x_1 - 2x_2 + 5x_3 &= 7 \\ -x_1 + 2x_2 + x_3 &= 5 \end{aligned}$$

homogents system

$$(h)\quad AX = 0 \qquad \begin{pmatrix} 3 & -2 & 5 \\ -1 & 2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \begin{aligned} 3x_1 - 2x_2 + 5x_3 &= 0 \\ -x_1 + 2x_2 + x_3 &= 0 \end{aligned}$$

löser båda parallellt

$$\left(\begin{array}{ccc|cc} 3 & -2 & 5 & 7 & 0 \\ -1 & 2 & 1 & 5 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|cc} 1 & -2 & -1 & -5 & 0 \\ 0 & 4 & 8 & 22 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|cc} 1 & 0 & 3 & 6 & 0 \\ 0 & 1 & 2 & 11/2 & 0 \end{array}\right)$$

$$\begin{aligned} z &= t \\ y &= \tfrac{11}{2} - 2t \\ x &= 6 - 3t \end{aligned} \quad \text{resp} \quad \begin{aligned} z &= t \\ y &= -2t \\ x &= -3t \end{aligned}$$

$$X_h = t\begin{pmatrix} -3 \\ -2 \\ 0 \end{pmatrix}, \quad X_p = \begin{pmatrix} 6 \\ 11/2 \\ 0 \end{pmatrix}, \quad X = X_p + X_h$$

Sats: a) Om $AX_1 = AX_2 = B$ så $A(X_1 - X_2) = 0$

b) Om $AX_p = B$, $AX_H = 0$ så $A(X_p + X_h) = B$

c) Låt $AX_p = B$.

Då:

Varje lösning X till $AX = B$ kan skrivas $X = X_p + X_h$, där $AX_h = 0$

Exempel

* $x + y = 1$

$$\begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \end{pmatrix}$$

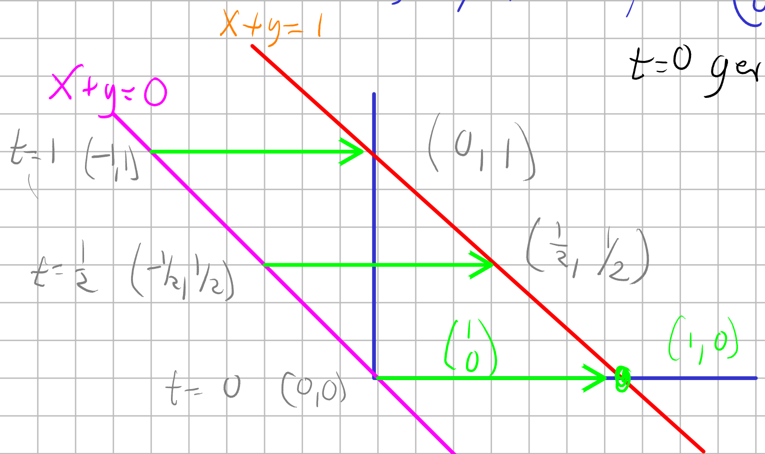
$y = t$, $x = 1 - t$, $X = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$t = 0$ ger $X_p = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

(h) $x + y = 0$

$$\begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \end{pmatrix}$$

$y = t$, $x = -t$, $X_h = t\begin{pmatrix} -1 \\ 1 \end{pmatrix}$



Flera högerled

$$\begin{aligned} x+y&=1 \qquad &&=0 \qquad &&=a \\ x-y&=2 \qquad &&=0 \qquad &&=b \end{aligned}$$

$$\begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \left[\begin{array}{cc|ccc} 1 & 1 & 0 & 1 & a \\ 1 & -1 & 0 & 2 & b \end{array}\right] \end{array} \sim \begin{array}{cc|ccc} 1 & 1 & 0 & 1 & a \\ 0 & -2 & 0 & 1 & b-a \end{array} \sim \begin{array}{cc|ccc} 1 & 1 & 0 & 1 & a \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{a-b}{2} \end{array} \sim \begin{array}{cc|ccc} 1 & 0 & 0 & 3/2 & \frac{a+b}{2} \\ 0 & 1 & 0 & -1/2 & \frac{a-b}{2} \end{array}$$

Obs: $a=1$, $b=2$ ger $\frac{a+b}{2} = \frac{3}{2}$

$\frac{a-b}{2} = -\frac{1}{2}$

6) Kvadratiska matriser

$A_{n \times n}$

$A^2 = AA$

$A^3 = AA^2 = A(AA)$

$A^4 = AA^3 = A(A^3)$

$\vdots$

$A^n = AA^{n-1}$

$A^0 = I$

$A^{m+n} = A^m A^n$ om $m, n \geq 0$
heltal

"iterand dubbling"

7) Matrisinvers

A $n\times n$. B $n\times n$ är inversen till A, $B = A^{-1}$, om

$$AB = I \text{ och } BA = I$$

Sats: Om $AB = BA = I$, $AC = CA = I$ så $B = C$

Sats: Om A, B $n\times n$, $AB = I$, så $BA = I$

A 1×1: $(c)^{-1} = (1/c)$ $c \neq 0$

A 2×2: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ om $ad-bc \neq 0$

Def A inverterbar om A^{-1} finns

Sats Om A inv.bar så $A^{m+n} = A^m A^n$ alla m, n heltal

Elementära matriser

Ex $\quad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$

Radop :

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} d & e & f \\ a & b & c \\ g & h & i \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} 2a & 2b & 2c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} a & b & c \\ d+2a & e+2b & f+2c \\ g & h & i \end{pmatrix}$$

9) Algoritm för beräkning av matrisinvers

Lös $AX = I \iff X = A^{-1}$

då $XA = I \iff X = A^{-1}$

$(E_m \cdots E_2 E_1) A = I \iff E_n \cdots E_2 E_1 I = A^{-1}$

$$(A|I) \sim (E_1 A | E_1 I) \sim \cdots \sim (E_m \cdots E_2 E_1 A | E_n \cdots E_2 I) = (I \mid A^{-1})$$

exempel

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 3/2 & -1/2 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & 3/2 & -1/2 \end{array}\right)$$

$1 - 2 \cdot 3/2 = -2$

Kontroll: $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ OK!

10) Kvadratiska ekvationssystem

A $n \times n$. • Om A^{-1} finns så har

$AX = B$ unik lösning $X = A^{-1}B$

• $AX = 0$ har alltid lösn $X = 0$,

unik lösn om A^{-1} finns.

• Om A^{-1} saknas så $A \sim \begin{pmatrix} 1 & & 0 \\ & 0 & \\ 0 & & \end{pmatrix}$, $AX = \bar{0}$ beror på $n-y$ param

någon nolla,
$y < n$ ettor

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$y = t$
$x = 0$, $z = 0$

I detta fall: $AX = B$ har $(n-y)$-dim lösn

om B = linj komb kol av A,

ingen lösn annars

$$\left(\begin{array}{ccc|cc} 1 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & 0 \end{array}\right)$$

$z = 2$
$y = t$
$x = 3$

$0 = 1$

exempel

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 1 & 2 & 2 & b \\ 1 & 3 & 3 & c \end{array}\right) \sim \begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & 1 & 1 & b-a \\ 0 & 2 & 2 & c-a \end{array} \sim \begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & 1 & 1 & b-a \\ 0 & 0 & 0 & c-b \end{array} \sim \begin{array}{ccc|c} 1 & 0 & 0 & 2a-b \\ 0 & 1 & 1 & b-a \\ 0 & 0 & 0 & c-b \end{array}$$

Fall 1) $c-b \neq 0$ ingen lösning

Fall 2) $c-b=0$

$$\begin{array}{ccc|c} 1 & 0 & 0 & 2a-b \\ 0 & 1 & 1 & b-a \\ 0 & 0 & 0 & c-b \end{array} = \begin{array}{ccc|c} 1 & 0 & 0 & 2a-b \\ 0 & 1 & 1 & b-a \\ 0 & 0 & 0 & 0 \end{array} \sim \begin{array}{ccc|c} 1 & 0 & 0 & 2a-b \\ 0 & 1 & 1 & b-a \end{array}$$

$$z = t$$
$$y = b-a-t$$
$$x = 2a-b-t$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2a-b \\ b-a \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

Partikulärlösn.

Homogen lösning

11) Matrisekvationer

exempel A, B, C $n \times n$, givna

Lös $AX + 3BX = C$

- X måste vara $n \times n$
- Bryt ut X till höger: $(A+3B)X = C$
- Invertera (om möjligt): $X = (A+3B)^{-1} C$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + 3\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad X = \begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 1 & -1 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} -1 & 1 \\ 4 & -1 \end{pmatrix}$$

Kontroll: $\left(\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + 3 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right)\frac{1}{3}\begin{pmatrix} -1 & 1 \\ 4 & -1 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix} \cdot \frac{1}{3}\begin{pmatrix} -1 & 1 \\ 4 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

OK

exempel, ansättning

$$\begin{pmatrix}1&1\\1&0\end{pmatrix}X + X\begin{pmatrix}0&1\\1&0\end{pmatrix} = \begin{pmatrix}1&0\\0&1\end{pmatrix}$$

Ansätt $X = \begin{pmatrix}a&b\\c&d\end{pmatrix}$, $\begin{pmatrix}1&1\\1&0\end{pmatrix}\begin{pmatrix}a&b\\c&d\end{pmatrix} = \begin{pmatrix}a+c&b+d\\a&b\end{pmatrix}$. $\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix} = \begin{pmatrix}b&a\\d&c\end{pmatrix}$

$VL = \begin{pmatrix}a+b+c & a+b+d\\ a+d & b+c\end{pmatrix}$

$HL = \begin{pmatrix}1&0\\0&1\end{pmatrix}$

ekv.sys.

$$\begin{aligned} a+b+c \quad &= 1\\ a+b+\quad d &= 0\\ a \quad + d &= 0\\ b+c \quad &= 1 \end{aligned}$$

$$\begin{array}{cccc} a & b & c & d \end{array}$$

$$\left(\begin{array}{cccc|c} 1&1&1&0&1\\ 1&1&0&1&0\\ 1&0&0&1&0\\ 0&1&0&1&1 \end{array}\right) \sim \cdots \sim \left(\begin{array}{cccc|c} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&1\\ 0&0&0&1&0 \end{array}\right)$$

$$X = \begin{pmatrix}0&0\\1&0\end{pmatrix}$$

$$\begin{pmatrix}1&1\\1&0\end{pmatrix}\begin{pmatrix}0&0\\1&0\end{pmatrix} + \begin{pmatrix}0&0\\1&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix} = \begin{pmatrix}1&0\\0&0\end{pmatrix} + \begin{pmatrix}0&0\\0&1\end{pmatrix} = \begin{pmatrix}1&0\\0&1\end{pmatrix}$$