Euklidiska rum (inre- produktrum)

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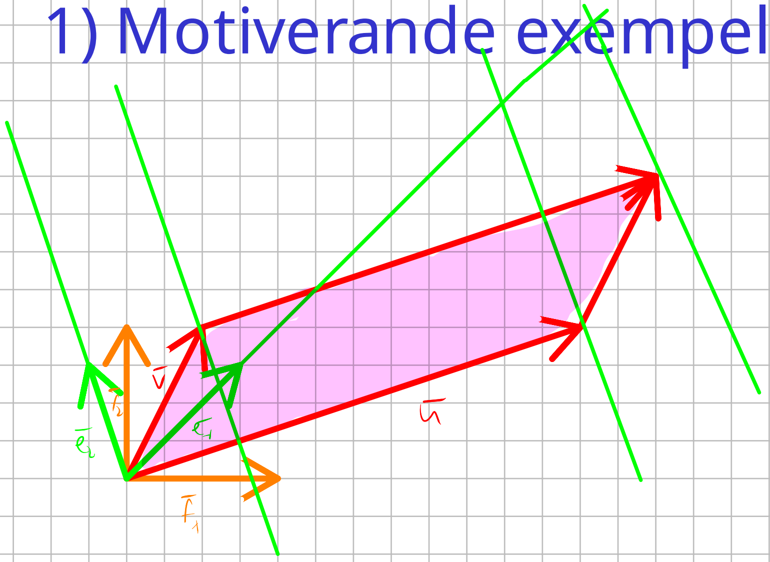
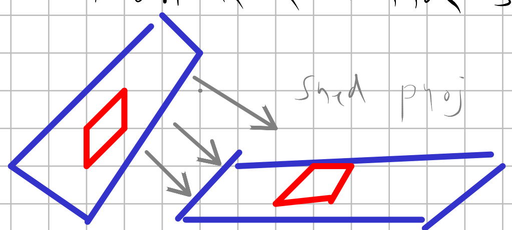
1) Motiverande exempel



- Givet figuren ovan, beräkna
  - längden $\|\bar{u}\|$, $\|\bar{v}\|$
  - arean parallellogram$(\bar{u},\bar{v})$
  - vinkel $\bar{u},\bar{v}$
- Olika resultat om vi antar $\underline{f}$ ON resp $\underline{e}$ ON
- Om $\underline{f}$ ON, kan räkna med $\underline{e}$-koord

  men använda annan skalärprodukt



$\bar{u} \approx \underline{f}\begin{pmatrix}3\\1\end{pmatrix}$, $\bar{v} \approx \underline{f}\begin{pmatrix}1/2\\1\end{pmatrix}$

Om $\underline{f}$ ON så $\|\bar{u}\| = \sqrt{10}$, $\|\bar{v}\| = \frac{1}{2}\sqrt{5}$

$\|\bar{u}\times\bar{v}\| = \frac{5}{2}$ = area, $\cos(\alpha) = \frac{\bar{u}\cdot\bar{v}}{\|\bar{u}\|\|\bar{v}\|} = \frac{5/2}{\frac{1}{2}\cdot 5\cdot\sqrt{2}} = \frac{1}{\sqrt{2}}$, $\alpha = 45°$

Om $\underline{e}$ ON så, eftersom $\bar{u} \approx \underline{e}\begin{pmatrix}3\\-2\end{pmatrix}$, $\bar{v} \approx \underline{e}\begin{pmatrix}1\\1/2\end{pmatrix}$

så $\|\bar{u}\| \approx \sqrt{13}$, $\|\bar{v}\| \approx \frac{1}{2}\sqrt{5}$

$\|\bar{u}\times\bar{v}\| \approx 7/2 \approx$ area

- Antag: $\underline{e}$ ON.

$$\begin{aligned}
\bar{e}_1 &\approx \tfrac{3}{4}\bar{f}_1 + \tfrac{3}{4}\bar{f}_2\\
\bar{e}_2 &\approx -\tfrac{1}{4}\bar{f}_1 + \tfrac{3}{4}\bar{f}_2\\
\bar{f}_1 &\approx \bar{e}_1 - \bar{e}_2\\
\bar{f}_2 &\approx \tfrac{1}{3}\bar{e}_1 + \bar{e}_2
\end{aligned}$$

$$\underline{f}\begin{pmatrix}x_1\\x_2\end{pmatrix}\cdot\underline{f}\begin{pmatrix}y_1\\y_2\end{pmatrix} = \left(x_1(\bar{e}_1-\bar{e}_2) + x_2(\tfrac{1}{3}\bar{e}_1+\bar{e}_2)\right)\cdot\left(y_1(\bar{e}_1-\bar{e}_2) + y_2(\tfrac{1}{3}\bar{e}_1+\bar{e}_2)\right)$$

$$= \underline{e}\begin{pmatrix}x_1+\frac{1}{3}x_2\\-x_1+x_2\end{pmatrix}\cdot\underline{e}\begin{pmatrix}y_1+\frac{1}{3}y_2\\-y_1+y_2\end{pmatrix}$$

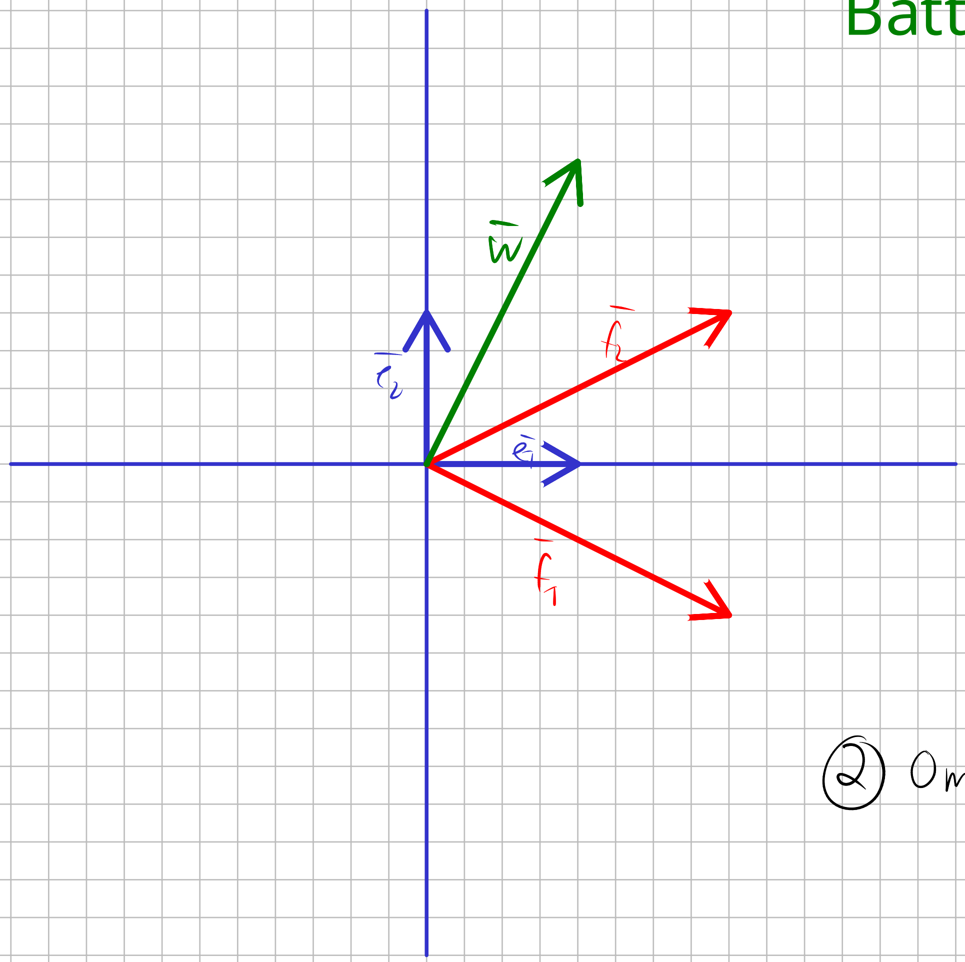
$$= (x_1+\tfrac{1}{3}x_2)(y_1+\tfrac{1}{3}y_2) + (-x_1+x_2)(-y_1+y_2)$$

$$= (1+1)x_1y_1 + (\tfrac{1}{3}-1)x_1y_2 + (\tfrac{1}{3}-1)x_2y_1 + (\tfrac{1}{9}+1)x_2y_2$$

$$= 2x_1y_1 - \tfrac{2}{3}x_1y_2 - \tfrac{2}{3}x_2y_1 + \tfrac{10}{9}x_2y_2$$

Formel för modifierad skalärprod.

Bättre exempel



$\bar{f}_1 = \underline{e}\begin{pmatrix}2\\-1\end{pmatrix}$, $\bar{f}_2 = \underline{e}\begin{pmatrix}2\\1\end{pmatrix}$

$\underline{f} = \underline{e}\begin{pmatrix}2 & 2\\-1 & 1\end{pmatrix}$, $\underline{e} = \underline{f}\,\frac{1}{4}\begin{pmatrix}1 & -2\\1 & 2\end{pmatrix} = \underline{f}\begin{pmatrix}\frac{1}{4} & -\frac{1}{2}\\ \frac{1}{4} & \frac{1}{2}\end{pmatrix}$

$\bar{w} = \underline{e}\begin{pmatrix}1\\2\end{pmatrix} = \underline{f}\,\frac{1}{4}\begin{pmatrix}1 & -2\\1 & 2\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix} = \underline{f}\,\frac{1}{4}\begin{pmatrix}-3\\5\end{pmatrix} = \underline{f}\begin{pmatrix}-3/4\\5/4\end{pmatrix}$

(1) Om $\underline{e}$ ON:

$\|\bar{f}_1\| = \|\bar{f}_2\| = \sqrt{5}$, $\|\bar{e}_1\| = \|\bar{e}_2\| = 1$

$\|\bar{w}\| = \sqrt{5}$

$\bar{f}_1 \circ \bar{w} = \underline{e}\begin{pmatrix}2\\-1\end{pmatrix} \circ \underline{e}\begin{pmatrix}1\\2\end{pmatrix} = 0$ rät vinkel

(2) Om $\underline{f}$ ON: $\|\bar{f}_1\| = \|\bar{f}_2\| = 1$, $\|\bar{e}_1\| = \frac{1}{4}\sqrt{2}$, $\|\bar{e}_2\| = \frac{1}{2}\sqrt{2}$

$\|\bar{w}\| = \frac{1}{4}\sqrt{3^2+5^2} = \frac{1}{4}\sqrt{34}$

$\bar{f}_1 \circ \bar{w} = \underline{f}\begin{pmatrix}1\\0\end{pmatrix} \circ \underline{f}\,\frac{1}{4}\begin{pmatrix}-3\\5\end{pmatrix} = -\frac{3}{4} \neq 0$ trubbig vinkel.

2) Def av inre produkt

Definition 6.2.1. En *skalärprodukt* på ett vektorrum $\mathbb{V}$ är en funktion som till varje par av vektorer $\mathbf{u}, \mathbf{v} \in \mathbb{V}$ ordnar ett reellt tal. Vi betecknar talet $(\mathbf{u}|\mathbf{v})$ och följande villkor skall vara uppfyllda för alla $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{V}$ och $\lambda \in \mathbb{R}$:

(i) $(\mathbf{u}|\mathbf{v}) = (\mathbf{v}|\mathbf{u})$ (Kommutativa lagen)

(ii) $(\mathbf{u}|\mathbf{v}+\mathbf{w}) = (\mathbf{u}|\mathbf{v}) + (\mathbf{u}|\mathbf{w})$ (Distributiva lagen)

(iii) $(\mathbf{u}|\lambda\mathbf{v}) = \lambda(\mathbf{u}|\mathbf{v})$

(iv) $(\mathbf{u}|\mathbf{u}) \geq 0$

(v) $(\mathbf{u}|\mathbf{u}) = 0 \Longrightarrow \mathbf{u} = \mathbf{0}$

Ett vektorrum försett med en skalärprodukt kallas ett *euklidiskt rum.*

Jag kommer att skriva $\langle \bar{u}, \bar{v} \rangle$
(och kalla det för en inre produkt i det allmänna fallet).

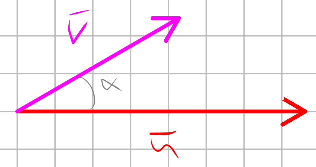
Givet $\langle \cdot, \cdot \rangle$ definierar vi

1) $\|\bar{u}\| = \langle \bar{u}, \bar{u} \rangle^{1/2}$

2) $\cos(\alpha) = \dfrac{\langle \bar{u}, \bar{v} \rangle}{\|\bar{u}\| \, \|\bar{v}\|}$

3) $\bar{u} \perp \bar{v}$ om $\langle \bar{u}, \bar{v} \rangle = 0$

Ex 1: Geometriska vektorer i rummet (el planet).



$\langle \bar{u}, \bar{v} \rangle = \bar{u} \circ \bar{v} = \|\bar{u}\| \|\bar{v}\| \cdot \cos(\alpha)$

Givna på förhand

Ex $\mathbb{R}^n$. Standardbas (ordnad) $\underline{e} = (\bar{e}_1 \cdots \bar{e}_n)$

$\bar{e}_1 = (1, 0, \ldots, 0)$, $\bar{e}_2 = (0, 1, \ldots, 0)$

$\bar{u} = (x_1, \ldots, x_n) = \underline{e}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \underline{e}X$

$\bar{v} = (y_1, \ldots, y_n) = \underline{e}\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \underline{e}Y$

$\langle \bar{u}, \bar{v} \rangle = \bar{u} \circ \bar{v} = \sum_{j=1}^{n} x_j y_j \text{ "="} X^t Y$

Ex i $\mathbb{R}^4$: $(1, 0, 3, 7) \circ (5, 1, 0, 1) = 1 \cdot 5 - 7 \cdot 1 = -2$

3) **Sats** 1) Om A $n\times n$, inverterbar så
$$\langle \underline{e}X, \underline{e}Y\rangle = (AX)^t(AY) \quad \text{inre prod}$$
på $\mathbb{R}^n$.

2) Varje inre prod på $\mathbb{R}^n$ kan ges av inverterbar A $n\times n$, som ovan.

B 1) Vi kontrollerar $(\bar{u}=\underline{e}X, \bar{v}=\underline{e}Y, \bar{w}=\underline{e}Z)$

$$\langle \bar{v},\bar{u}\rangle = (AY)^tAX = Y^tA^tAX \overset{1\times1}{=} (Y^tA^tAX)^t$$
$$= X^tA^tAY = (AX)^tAY = \langle \bar{u},\bar{v}\rangle$$
$$\langle \bar{u},\bar{v}+\bar{w}\rangle = X^tA^tA(Y+Z) = X^tA^tAY + X^tA^tAZ$$
$$\langle \bar{u},\lambda\bar{v}\rangle = X^tA^tA(\lambda Y) = \lambda X^tA^tAY$$
$$\langle \bar{u},\bar{u}\rangle = (\underset{n\times1}{AX})^t(\underset{n\times1}{AX}) = (\text{längd } AX)^2 \geq 0$$
$$\langle \bar{u},\bar{u}\rangle = 0 \Rightarrow AX = \bar{0} \Rightarrow X = \bar{0}$$
ty A inverterbar

2) Svårare, gör själv!

Ex $V=\mathbb{R}^2$, $\left\langle \underline{e}\begin{pmatrix}x_1\\x_2\end{pmatrix}, \underline{e}\begin{pmatrix}y_1\\y_2\end{pmatrix}\right\rangle =$
$$5x_1y_1 + 4x_2y_1 + 4x_1y_2 + 5x_2y_2$$
$$\left\langle \underline{e}\begin{pmatrix}x_1\\x_2\end{pmatrix}, \underline{e}\begin{pmatrix}x_1\\x_2\end{pmatrix}\right\rangle = 5x_1^2 + 8x_1x_2 + 5x_2^2$$
$$= 5\left(x_1^2 + \tfrac{8}{5}x_1x_2 + x_2^2\right) = 5\left(\left(x_1+\tfrac{4}{5}x_2\right)^2 - \tfrac{16}{25}x_2^2 + x_2^2\right)$$
$$= 5\left(x_1+\tfrac{4}{5}x_2\right)^2 + \tfrac{9}{5}x_2^2 \geq 0, \text{ bara } 0 \text{ om } x_1=x_2=0$$
$$A=\begin{pmatrix}2&1\\1&2\end{pmatrix},\quad A^tA=\begin{pmatrix}2&1\\1&2\end{pmatrix}\begin{pmatrix}2&1\\1&2\end{pmatrix}=\begin{pmatrix}5&4\\4&5\end{pmatrix}$$
$$A^{-1}=\frac{1}{3}\begin{pmatrix}2&-1\\-1&2\end{pmatrix}$$

Ex $\left\langle \underline{e}\begin{pmatrix}x_1\\x_2\end{pmatrix}, \underline{e}\begin{pmatrix}y_1\\y_2\end{pmatrix}\right\rangle = x_1^2+y_1^2$

$$\left\langle \underline{e}\begin{pmatrix}1\\0\end{pmatrix}, \underline{e}\begin{pmatrix}1\\0\end{pmatrix}\right\rangle = 1,\quad \left\langle \underline{e}\begin{pmatrix}1\\0\end{pmatrix}, \lambda\underline{e}\begin{pmatrix}1\\0\end{pmatrix}\right\rangle = 1^2+\lambda^2 \neq \lambda$$
för alla λ

Ex $\left\langle \underline{e}\begin{pmatrix}x_1\\x_2\end{pmatrix}, \underline{e}\begin{pmatrix}y_1\\y_2\end{pmatrix}\right\rangle = x_1y_1 + x_1y_2 + x_2y_1 + x_2y_2$
$$\left\langle \underline{e}\begin{pmatrix}x_1\\x_2\end{pmatrix}, \underline{e}\begin{pmatrix}x_1\\x_2\end{pmatrix}\right\rangle = x_1^2 + 2x_1x_2 + x_2^2 = (x_1+x_2)^2 \geq 0$$

Men: Om $x_1=t$, $x_2=-t$, $t\neq0$ så $=0$
trots ~~att~~ $\underline{e}\begin{pmatrix}t\\-t\end{pmatrix} \neq \bar{0}$

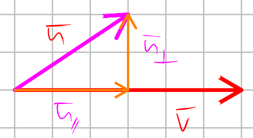
4) Pythagoras, Triangelolikheten, Cauchy-Schwarz

Sats (Pythagoras) Om $\bar{u} \perp \bar{v}$ så $\|\bar{u}+\bar{v}\|^2 = \|\bar{u}\|^2 + \|\bar{v}\|^2$

B $\|\bar{u}+\bar{v}\|^2 = \langle \bar{u}+\bar{v}, \bar{u}+\bar{v}\rangle = \langle \bar{u},\bar{u}\rangle + \langle \bar{u},\bar{v}\rangle + \langle \bar{v},\bar{u}\rangle + \langle \bar{v},\bar{v}\rangle$

$= \|\bar{u}\|^2 + 0 + 0 + \|\bar{v}\|^2$

Def $\bar{u},\bar{v}$ givna, $\bar{v} \neq 0$. Sätt $\bar{u}_{\parallel \bar{v}} = \dfrac{\langle \bar{u},\bar{v}\rangle}{\|\bar{v}\|^2}\bar{v}$, $\bar{u}_{\perp \bar{v}} = \bar{u} - \bar{u}_{\parallel \bar{v}}$



Sats (i) $\langle \bar{u}_{\parallel \bar{v}}, \bar{u}_{\perp \bar{v}}\rangle = 0$

(ii) $\|\bar{u}_{\parallel \bar{v}}\| \leq \|\bar{u}\|$

B: $\left\langle \dfrac{\langle \bar{u},\bar{v}\rangle}{\|\bar{v}\|^2}\bar{v}, \bar{u} - \dfrac{\langle \bar{u},\bar{v}\rangle}{\|\bar{v}\|^2}\bar{v}\right\rangle$

$= \dfrac{\langle \bar{u},\bar{v}\rangle}{\|\bar{v}\|^2}\langle \bar{v},\bar{u}\rangle - \dfrac{\langle \bar{u},\bar{v}\rangle^2}{\|\bar{v}\|^4}\|\bar{v}\|^2 = 0$

B: $\|\bar{u}\|^2 = \|\bar{u}_\parallel\|^2 + \|\bar{u}_\perp\|^2 \Longrightarrow \|\bar{u}_\parallel\|^2 \leq \|\bar{u}\|^2$

Sats (Cauchy-Schwarz) $|\langle \bar{u},\bar{v}\rangle| \leq \|\bar{u}\|\|\bar{v}\|$

B: $\|\bar{u}_\parallel\| = \left\| \dfrac{\langle \bar{u},\bar{v}\rangle}{\|\bar{v}\|^2}\bar{v}\right\| = \dfrac{|\langle \bar{u},\bar{v}\rangle|}{\|\bar{v}\|} \overset{\text{ovan}}{\leq} \|\bar{u}\|$, så

$|\langle \bar{u},\bar{v}\rangle| \leq \|\bar{v}\|\|\bar{u}\|$

Ex $V = C^\infty(0,2\pi)$

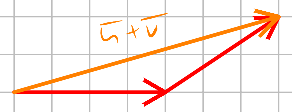
$\langle f,g\rangle = \int_0^{2\pi} f(t)g(t)\,dt$

$\int_0^{2\pi} \sin(t)\sin(5t)\,dt = 0$, så

$\int_0^{2\pi} (\sin(t)+\sin(5t))^2\,dt = \int_0^{2\pi}\sin^2(t)\,dt + \int_0^{2\pi}\sin^2(5t)\,dt = \pi + \pi = 2\pi$

Ex $\underbrace{\left(\int_0^{2\pi} e^t \sin(t)\,dt\right)^2}_{\frac{1}{4}(e^{2\pi}-1)^2 \approx 7\cdot 10^5} \leq \underbrace{\int_0^{2\pi} e^{2t}\,dt \int_0^{2\pi} \sin^2(t)\,dt}_{\frac{1}{2}\pi(e^{4\pi}-1) \approx 5\cdot 10^6}$

Sats: (Triangelolikheten) $\|\bar{u}+\bar{v}\| \leq \|\bar{u}\| + \|\bar{v}\|$



B| $\|\bar{u}+\bar{v}\|^2 = \langle \bar{u}+\bar{v}, \bar{u}+\bar{v} \rangle = \|\bar{u}\|^2 + 2\langle \bar{u}, \bar{v}\rangle + \|\bar{v}\|^2$

$\leq \|\bar{u}\|^2 + 2|\langle \bar{u}, \bar{v}\rangle| + \|\bar{v}\|^2 \leq \|\bar{u}\|^2 + 2\|\bar{u}\|\,\|\bar{v}\| + \|\bar{v}\|^2$

$= \left(\|\bar{u}\| + \|\bar{v}\|\right)^2$

Ex: $V = \mathbb{R}^{\infty}$, $\langle (a_i), (b_i) \rangle = \sum a_i b_i$

$\|(a_i)\|^2 = \sum a_i^2$

$\bar{u} = (\underbrace{1, 1, \dots, 1}_{10}, 0, 0, \dots) \qquad \bar{v} = (\underbrace{1, 1, \dots, 1}_{12}, 0, 0, \dots)$

$\|\bar{u} - \bar{v}\| = \sqrt{2} \leq \|\bar{u}\| + \|\bar{v}\| = \sqrt{10} + \sqrt{12}$

5) ON-baser

Def $S = \{\bar{e}_1, \ldots, \bar{e}_n\} \subseteq V$ (euklidiskt rum)

är en Ortonormerad mängd om

$$\langle \bar{e}_i, \bar{e}_j \rangle = \begin{cases} 1 & \text{om } i = j \\ 0 & \text{annars} \end{cases}$$

Sats ON-mängd S linj. ober.

B $\bar{0} = \sum_{j=1}^{n} c_j \bar{e}_j$ med för $0 = \langle \bar{e}_1, \bar{0} \rangle =$

$$= \langle \bar{e}_1, \sum_{j=1}^{n} c_j \bar{e}_j \rangle = c_1 \langle \bar{e}_1, \bar{e}_1 \rangle + c_2 \langle \bar{e}_1, \bar{e}_2 \rangle + \cdots + c_n \langle \bar{e}_1, \bar{e}_n \rangle$$

$= c_1$, så $c_1 = 0$, osv. $\square$

Följd ON-mängd S bas för $[S]$. Om $n = \dim V$ så S bas för V (ON-bas)

Def Ortogonal bas om $\langle \bar{e}_i, \bar{e}_j \rangle = 0$ omm $i = j$

Ex $V = \mathbb{R}^4$. $\bar{f}_1 = (1,1,1,1)$, $\bar{f}_2 = (1,-1,0,0)$

$\bar{f}_3 = (0,0,1,-1)$.

- Hitta $\bar{f}_4$ så $\{\bar{f}_1, \bar{f}_2, \bar{f}_3, \bar{f}_4\}$ Ortogonal bas för $\mathbb{R}^4$
- Gör om till ON-bas

Ansätt $f_4 = (a,b,c,d)$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 0 \\ 0 & -2 & -1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & -2 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right) \sim \begin{array}{c} \begin{array}{cccc|c} & & & d & \end{array} \\ \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right) \end{array}$$

$d = t,\ c = t,\ b = -t,\ a = -t$

$t = -1$ så $(a,b,c,d) = (1,1,-1,-1)$

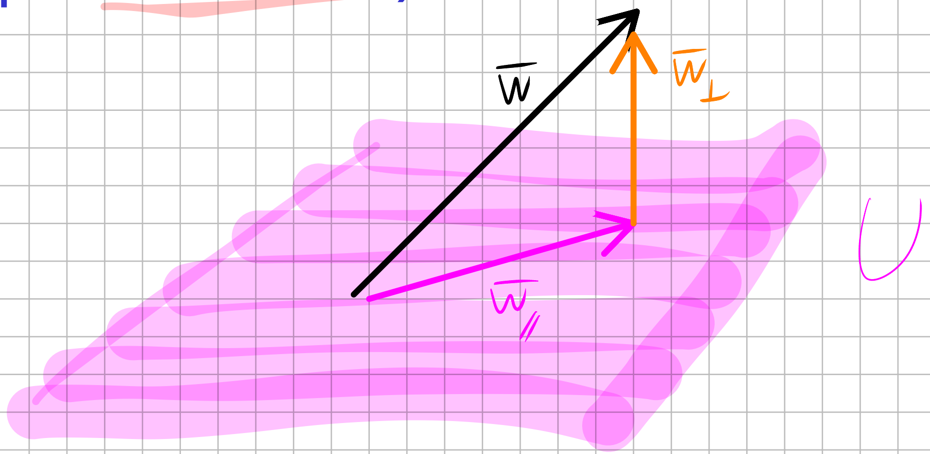
Tar $\bar{f}_4 = (1,1,-1,-1)$

- Gör om till ON-bas (från ortogonal dito) genom att normera:

$$\hat{f}_j = \frac{\bar{f}_j}{\|\bar{f}_j\|}, \quad \hat{f}_j \parallel \bar{f}_j, \quad \|\hat{f}_j\| = 1$$

Dela med 2 resp $\sqrt{2}$

6) Ortogonal projektion (på underrum)



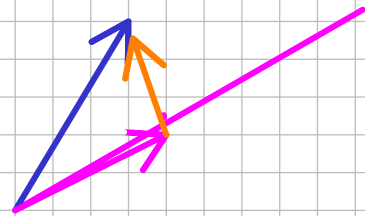
- $U \leq V$, V euklidiskt
- $\bar{w} \in V$
- komposantuppdela $\bar{w} = \bar{w}_{//U} + \bar{w}_{\perp U}$

  med $\bar{w}_{//} = \bar{w}_{//U} \in U$ och $w_{\perp U} = w_{\perp}$ ortogonal

  mot varje $\bar{u} \in U$
- Unikt sätt att göra detta!

① Om $U = [f]$ så

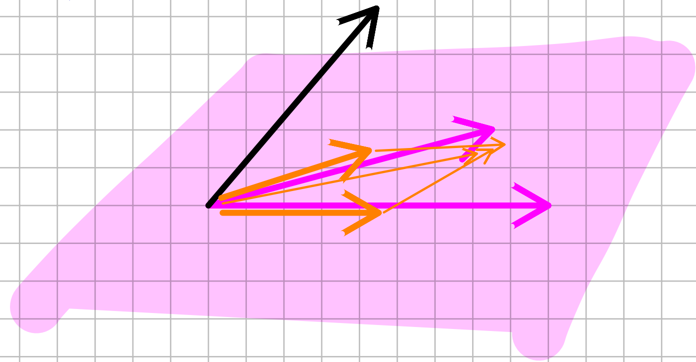
$$\bar{w}_{//\bar{f}} = \frac{\langle \bar{w}, f \rangle}{\|\bar{f}\|^2} \bar{f}$$



② Om $U = [\bar{f}_1, \bar{f}_2]$

Så $\bar{w}_{//\bar{f}_1} + \bar{w}_{//\bar{f}_2} \in U$, men

bara $\bar{w}_{//}$ om $\bar{f}_1 \perp \bar{f}_2$



③ Om $U = [\bar{f}_1, \dots, \bar{f}_n]$ och $\{\bar{f}_1, \dots, \bar{f}_n\}$ ortog.

Så $$\bar{w}_{//U} = \sum_{j=1}^{n} \bar{w}_{//\bar{f}_j} = \sum_{j=1}^{n} \frac{\langle \bar{w}, \bar{f}_j \rangle}{\langle \bar{f}_j, \bar{f}_j \rangle} \bar{f}_j$$

"allmänna projektionsformeln"

Exempel: $V=\mathbb{R}^3$, $\bar f_1=(1,1,1)$, $\bar f_2=(1,-1,1)$

$\bar w=(1,2,3)$. $U=[\bar f_1,\bar f_2]$.

$$\bar w_{/\!/\bar f_1} = \frac{\bar w\circ\bar f_1}{\bar f_1\circ \bar f_1}\bar f_1 = \frac{6}{3}\bar f_1 = (2,2,2)$$

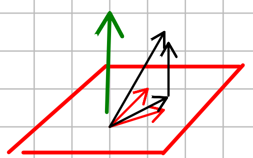
$$\bar w_{/\!/\bar f_2} = \frac{\bar w\circ \bar f_2}{\bar f_2\circ\bar f_2} = \frac{2}{3}\bar f_2 = \left(\tfrac23,-\tfrac23,\tfrac23\right)$$

$$\bar u = \bar w_{/\!/\bar f_1}+\bar w_{/\!/\bar f_2} = \left(\tfrac83,\tfrac43,\tfrac83\right), \quad \bar w-\bar u = \left(-\tfrac53,\tfrac23,\tfrac13\right)$$

$$(\bar w-\bar u)\circ\bar f_1 = -\tfrac23 \neq 0$$

U plan, normal $\bar f_1\times\bar f_2 = e\begin{pmatrix}1\\1\\1\end{pmatrix}\times_e\begin{pmatrix}1\\-1\\1\end{pmatrix} = e\begin{pmatrix}2\\0\\-2\end{pmatrix} /\!/ e\begin{pmatrix}1\\0\\-1\end{pmatrix}=\bar n$

Så $\bar w_{\perp U} = \bar w_{/\!/\bar n} = \dfrac{(1,2,3)\circ(1,0,-1)}{(1,0,-1)\circ(1,0,-1)} = \dfrac{-2}{2}(1,0,-1) = (-1,0,1)$



$$\bar w_{/\!/U} = \bar w - \bar w_{\perp U} = (1,2,3)-(-1,0,1) = (2,2,2)$$

Välj Ortogonalbas $\bar f_1, \bar f_3$: $\bar f_3 = c\,\bar f_1\times\bar n = \begin{pmatrix}1\\1\\1\end{pmatrix}\times\begin{pmatrix}1\\0\\-1\end{pmatrix} = \begin{pmatrix}-1\\2\\-1\end{pmatrix}$

$\bar w_{/\!/\bar f_1} = (2,2,2)$, $\quad \bar w_{/\!/\bar f_3} = \dfrac{(1,2,3)\circ(-1,2,-1)}{6}(-1,2,-1) = 0\cdot(-1,2,-1) = \bar 0$

Så $\bar w_{/\!/U} = \bar w_{/\!/\bar f_1}$ i detta fall!!

Ex $V=\mathbb{R}^4$, $U=[(1,0,0,1),(0,1,1,0),(0,1,-1,0)]$

- Redan ortogonalbas!
- $\bar w = (a,b,c,d)$
- $\bar w_{/\!/U} = \dfrac{\langle\bar w,\bar f_1\rangle}{2}\bar f_1 + \dfrac{\langle \bar w,\bar f_2\rangle}{2}\bar f_2 + \dfrac{\langle\bar w,\bar f_3\rangle}{2}\bar f_3$

$$= \frac12\big((a+d)(1,0,0,1) + (b+c)(0,1,1,0) + (b-c)(0,1,-1,0)\big)$$

$$= \frac12(a+d,\ 2b,\ 2c,\ a+d)$$

$$\bar w_{\perp U} = (a,b,c,d) - \bar w_{/\!/U}$$

7) Gram-Schmidt, algoritm för att producera ortogonalbas

indata: $\bar f_1, \dots, \bar f_m \in V$, euklidiskt rum

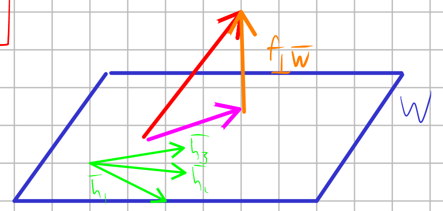
utdata: $\bar h_1, \dots, \bar h_r$ ortogonalbas för $U = [\bar f_1, \dots, \bar f_m]$

① $h_1 = \bar f_1$, $W = [\bar h_1]$

⋮

① : Tag nästa $\bar f$ som är obehandlad

- Beräkna $\bar f_{//W}$, $\bar f_{\perp W}$
- Om $\bar f_{\perp W} = \bar 0$
- Annars $\bar h_* = -\bar f_{//W} + \bar f = \bar f_{\perp W}$, $\bar W = \bar W \oplus [\bar f_{\perp W}]$



Ex $\bar f_1 = (0,1,1,1)$, $\bar f_2 = (3,2,1,0)$, $\bar f_3 = (3,1,0,-1)$

① $\bar h_1 = \bar f_1$

② $\text{proj}(\bar f_2, \bar h_1) = \frac{\bar f_2 \cdot \bar h_1}{\bar h_1 \cdot \bar h_1} \bar h_1 = \frac{3}{3} \bar h_1 = \bar h_1$

$\bar f_2 - \text{proj}(\bar f_2, \bar h_1) = (3,2,1,0) - (0,1,1,1) = (3,1,0,-1)$ $= \bar h_2$

③ $\text{proj}(\bar f_3, \bar h_1) + \text{proj}(\bar f_3, \bar h_2) = \text{proj}((3,1,0,-1),(0,1,1,1))$

$+ \text{proj}((3,1,0,-1),(3,1,0,-1)) = \bar 0 + \bar f_3 = \bar f_3$

$\bar f_{3\perp} = \bar 0$

④ Klart, $[\bar f_1, \bar f_3]$ ortogonalbas

⑤ Normera: $\hat f_1 = \frac{1}{\sqrt{3}}(0,1,1,1)$

$\hat f_3 = \frac{1}{\sqrt{11}}(3,1,0,-1)$

Ex $V = P^2$, $\langle f,g\rangle = \int_0^1 f(t)g(t)\,dt$

$1, x, x^2$ bas, gör om till ON-bas.

$\bar f_1 = 1$, $\|\bar f_1\| = 1$, $\bar h_1 = \bar f_1$

$\langle \bar f_1, \bar f_2\rangle = \int_0^1 t\,dt = \frac{1}{2}$

$\bar f_{2 \parallel \bar f_1} = \frac{\langle \bar f_1, \bar f_2\rangle}{\langle \bar f_1, \bar f_1\rangle}\bar f_1 = \frac{1}{2} f_1 = \frac{1}{2}$

$\bar f_{2\perp} = \bar f_2 - \bar f_{2 \parallel \bar f_1} = -\frac{1}{2} + t = \bar h_2$

$\|\bar h_2\|^2 = \langle \bar h_2, \bar h_2\rangle = \int_0^1 \left(-\frac{1}{2} + t\right)^2 dt$

$= \int_0^1 \left(t^2 - t + \frac{1}{4}\right) dt = \frac{1}{3} - \frac{1}{2} + \frac{1}{4} = \frac{1}{12}$

$\bar f_3 - \bar f_{3 \parallel \bar h_1} - \bar f_{3 \parallel \bar h_2}$

$= t^2 - \frac{\langle t^2, 1\rangle}{1} 1 - \frac{\langle t^2, t - \frac{1}{2}\rangle}{1/12}\left(t - \frac{1}{2}\right)$

$= t^2 - \frac{1}{3} - \frac{1/12}{1/12}\left(t - \frac{1}{2}\right)$

$= t^2 - \frac{1}{3} - t + \frac{1}{2} = t^2 - t + \frac{1}{6}$

Kontroll: $\langle \bar h_3, \bar h_2\rangle$

$= \int_0^1 \left(t^2 - t + \frac{1}{6}\right)\left(t - \frac{1}{2}\right) dt = 0$