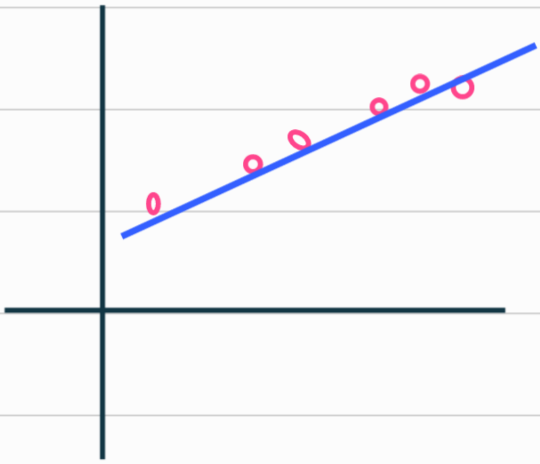
Föreläsning 10

Minsta kvadrat-metoden



Innehåll:

- Kolonnrum, lösbarhet för $AX = B$
- Minstakvadrat-lösning
- Sats om existens/entydighet
- Exempel
- Modellproblem

Sats Studera det linjära ekvationssystemet

$$\boxed{(*)\quad \underset{m\times n}{A}\,\underset{n\times 1}{X} = \underset{m\times 1}{B}} \iff AX-B=\bar{0} \iff \|AX-B\|=0$$

lösbar omm $B \in \text{kolumn}(A)$.

(i) $\boxed{\text{Felvektorn } AX-B}$ minimeras i norm

om X väljs som en lösning till

normalekvationerna $\boxed{A^tAX = A^tB}$

(ii) Normalekvationerna har alltid någon lösning, säg X_{n}. En sådan X_n kallas för minstakvadratlösning till $(*)$

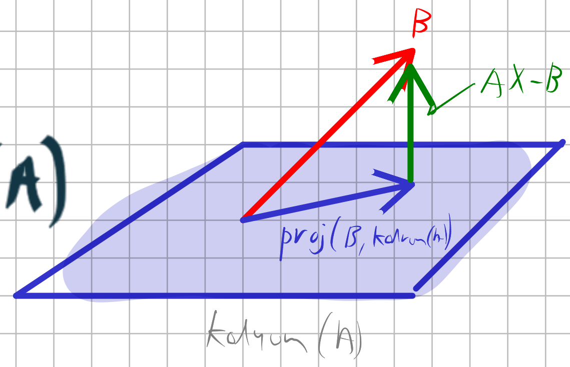
(iii) Om kolonnerna i A är $\boxed{\text{linj. obero.}}$ så är denna lösning unik.

(iv) Om $(*)$ skulle råka ha en "äkta" lösning X så är X även minstakvadratlösningen.

$\|AX-B\|=0$ då

Bevis

- $\{AX : X \in \mathbb{R}^n\} = \text{kolumn}(A) \le \mathbb{R}^m$
- $AX = B$ **lösbar omm** $B \in \text{kolumn}(A)$

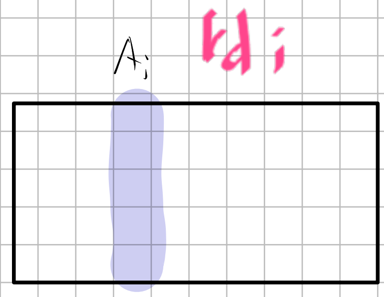


- $\|AX - B\|$ minimeras om

$$AX = \text{proj}(B, \text{kolumn}(A))$$

- Då är $AX - B$ ortogonal mot kolumn(A),

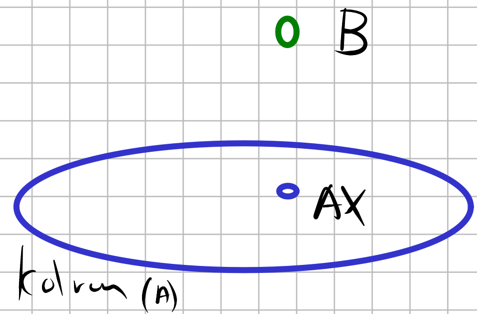
dvs $A_i \circ (AX - B) = 0$ alla kol A_i



- Så $A_i^t (AX - B) = 0$
- Så $A^t (AX - B) = 0$ (alla kol på en gång).
- Om A_i:na linj obero så bas för kolumn(A).

$AX = \text{proj}(B, \text{kolumn}(A))$ alltid unikt,

i detta fall även X unikt.



Alternativ härledning:

$$F(X) = \|AX-B\|^2 = \langle AX-B, AX-B\rangle = (AX-B)^t(AX-B)$$

$$= X^tA^tAX - X^tA^tB - B^tAX + B^tB$$

1×1

$$= X^tA^tAX - 2X^tA^tB + B^tB$$

$$\text{grad}(F) = 2X^tA^tA - 2X^tA^tB = 0 \text{ i minimum}$$

Vårt ex: $F(X) = \left\| \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} \right\|^2 = \left\| \begin{pmatrix} x+y-3 \\ 2x+y-4 \\ x+2y-2 \end{pmatrix} \right\|^2$

$$= (x+y-3)^2 + (2x+y-4)^2 + (x+2y-2)^2$$

$$= 6x^2 + 10xy + 6y^2 - 26x - 22y + 29$$

$$\text{grad}(F) = (12x + 10y - 26,\ 10x + 12y - 22)$$

$$= 2\,(6x + 5y - 13,\ 5x + 6y - 11)$$

$$AX = B$$

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 1 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}$$

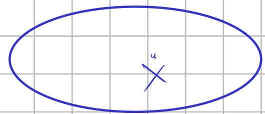
$$A^tA = \begin{pmatrix} 6 & 5 \\ 5 & 6 \end{pmatrix} \quad A^tB = \begin{pmatrix} 13 \\ 11 \end{pmatrix}$$

Exempel

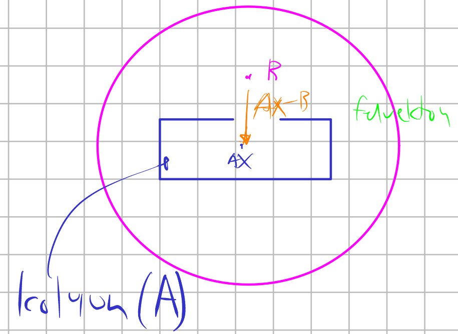
~~Nytt~~ överbestämt ekvationssystem:

$$\left.\begin{aligned} x+y&=3\\ 2x+y&=4\\ x+2y&=2 \end{aligned}\right\}
\qquad
\begin{array}{ccc} A & X & B \\ \begin{pmatrix}1&1\\2&1\\1&2\end{pmatrix} & \begin{pmatrix}x\\y\end{pmatrix} = & \begin{pmatrix}3\\4\\2\end{pmatrix}\end{array}$$

1) Vill minimera $\|AX-B\|$



2) Välj $\hat{X}$ så $A\hat{X} = \text{proj}(B, \text{kolumn}(A))$



$$\text{proj}\left(\begin{pmatrix}3\\4\\2\end{pmatrix}, \left[\begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}1\\1\\2\end{pmatrix}\right]\right) = \begin{pmatrix}24/11\\47/11\\25/11\end{pmatrix}$$

3) Då är $AX-B \perp$ varje kol i A

och $\|AX-B\|$ minimerad

4) Likvärdigt med att lösa $A^t(AX-B)=0$

5) Lös $A^tAX=A^tB$ $\quad \left(\begin{array}{cc|c} 6 & 5 & 13 \\ 5 & 6 & 11 \end{array}\right)$

felvektor

Får $X = \frac{1}{11}\begin{pmatrix}23\\1\end{pmatrix}$, $AX = \frac{1}{11}\begin{pmatrix}24\\47\\25\end{pmatrix}$, $AX-B = \frac{1}{11}\begin{pmatrix}-9\\3\\3\end{pmatrix}$ norm $\frac{1}{11}\sqrt{(-9)^2+3^2+3^2}$

proj (B, kolumn (A))

Exempel: Anpassa parabel

$$p(x) = ax^2 + bx + c \quad \text{till}$$

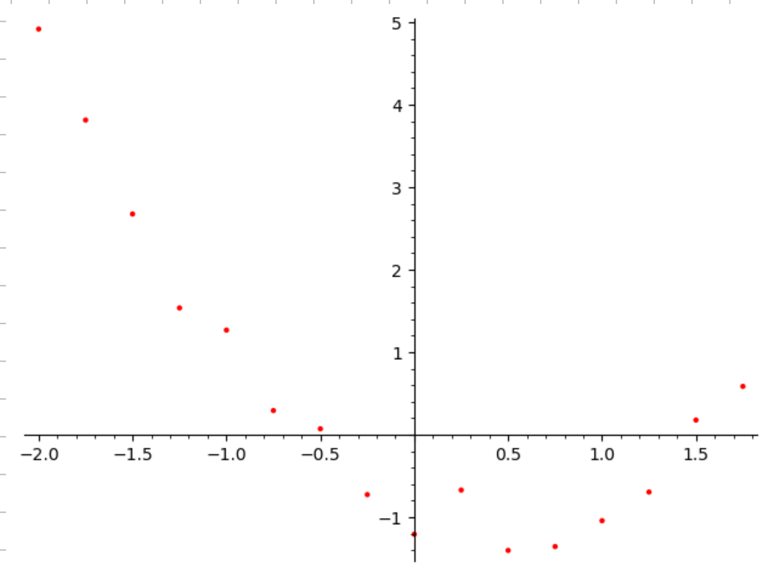
Lista $(x_0, y_0), \dots, (x_n, y_n)$ av punkter

Varje (x_j, y_j) ger en linjär ekv för a, b, c:

$$ax_j^2 + bx_j + c = y_j$$

skriven $\begin{bmatrix} x_j^2 & x_j & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} y_j \end{bmatrix}$

Alla ekv: $\underbrace{\begin{pmatrix} x_0^2 & x_0 & 1 \\ x_1^2 & x_1 & 1 \\ \vdots & & \\ x_n^2 & x_n & 1 \end{pmatrix}}_{(n+1)\times 3} \underbrace{\begin{pmatrix} a \\ b \\ c \end{pmatrix}}_{3\times 1} = \underbrace{\begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{pmatrix}}_{(n+1)\times 1}$



$$\left(\begin{array}{rrr|r}
4.00000000000000 & -2.00000000000000 & 1.00000000000000 & 4.91692245777862 \\
3.06250000000000 & -1.75000000000000 & 1.00000000000000 & 3.81368021738102 \\
2.25000000000000 & -1.50000000000000 & 1.00000000000000 & 2.67456307050707 \\
1.56250000000000 & -1.25000000000000 & 1.00000000000000 & 1.53579962318383 \\
1.00000000000000 & -1.00000000000000 & 1.00000000000000 & 1.26726179851753 \\
0.562500000000000 & -0.750000000000000 & 1.00000000000000 & 0.292027043486136 \\
0.250000000000000 & -0.500000000000000 & 1.00000000000000 & 0.0704248184617894 \\
0.0625000000000000 & -0.250000000000000 & 1.00000000000000 & -0.727467702650252 \\
0.000000000000000 & 0.000000000000000 & 1.00000000000000 & -1.20797850362489 \\
0.0625000000000000 & 0.250000000000000 & 1.00000000000000 & -0.672085903631460 \\
0.250000000000000 & 0.500000000000000 & 1.00000000000000 & -1.40403204350903 \\
0.562500000000000 & 0.750000000000000 & 1.00000000000000 & -1.35685528683648 \\
1.00000000000000 & 1.00000000000000 & 1.00000000000000 & -1.04318467810628 \\
1.56250000000000 & 1.25000000000000 & 1.00000000000000 & -0.696095509714119 \\
2.25000000000000 & 1.50000000000000 & 1.00000000000000 & 0.176611418975385 \\
3.06250000000000 & 1.75000000000000 & 1.00000000000000 & 0.584948432708328
\end{array}\right)$$

Vi skriver vårt överbestämda linjära ekv.sys som $AX = B$

$$X = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad A = \begin{pmatrix} x_0^2 & x_0 & 1 \\ \vdots & \vdots & \vdots \\ x_n^2 & x_n & 1 \end{pmatrix}, \quad B = \begin{pmatrix} y_0 \\ \vdots \\ y_n \end{pmatrix}$$

Systemet olösligt!

Idé: Om $AX - Y = 0$ inte går, hitta X_m så att $\|AX_m - Y\|$ så liten som möjl.

Ges av normalekv $\underbrace{\underset{3\times\tilde{n}}{A^t}\,\underset{\tilde{n}\times 3}{A}}_{3\times 3}\,\underset{3\times 1}{X} = \underset{3\times\tilde{n}}{A^t}\,\underset{\tilde{n}\times 1}{B}$

Lösning X_m till detta är

$$X_m = (A^tA)^{-1}A^tB$$

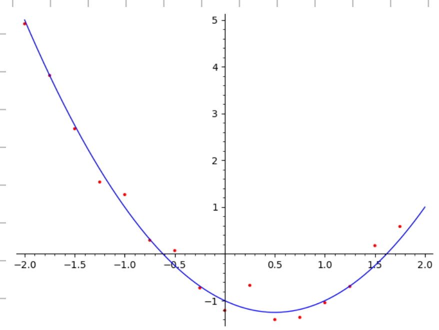
(men eliminera $A^tA \mid A^tB$ bättre)

$$\begin{array}{cccc} & A & & B \\ \end{array}$$
$$\begin{pmatrix}
4.00000000000000 & -2.00000000000000 & 1.00000000000000 & 4.91692245777862 \\
3.06250000000000 & -1.75000000000000 & 1.00000000000000 & 3.81368021738102 \\
2.25000000000000 & -1.50000000000000 & 1.00000000000000 & 2.67456307050707 \\
1.56250000000000 & -1.25000000000000 & 1.00000000000000 & 1.53579962318383 \\
1.00000000000000 & -1.00000000000000 & 1.00000000000000 & 1.26726179851753 \\
0.562500000000000 & -0.750000000000000 & 1.00000000000000 & 0.292027043486136 \\
0.250000000000000 & -0.500000000000000 & 1.00000000000000 & 0.0704248184617894 \\
0.0625000000000000 & -0.250000000000000 & 1.00000000000000 & -0.727467702650252 \\
0.000000000000000 & 0.000000000000000 & 1.00000000000000 & -1.20797850362489 \\
0.0625000000000000 & 0.250000000000000 & 1.00000000000000 & -0.672085903631460 \\
0.250000000000000 & 0.500000000000000 & 1.00000000000000 & -1.40403204350903 \\
0.562500000000000 & 0.750000000000000 & 1.00000000000000 & -1.35685528683648 \\
1.00000000000000 & 1.00000000000000 & 1.00000000000000 & -1.04318467810628 \\
1.56250000000000 & 1.25000000000000 & 1.00000000000000 & -0.696095509714119 \\
2.25000000000000 & 1.50000000000000 & 1.00000000000000 & 0.176611418975385 \\
3.06250000000000 & 1.75000000000000 & 1.00000000000000 & 0.584948432708328
\end{pmatrix}$$

$$\underset{A^tA}{\begin{pmatrix}
52.5312500000000 & -8.00000000000000 & 21.5000000000000 \\
-8.00000000000000 & 21.5000000000000 & -2.00000000000000 \\
21.5000000000000 & -2.00000000000000 & 16.0000000000000
\end{pmatrix}} \underset{X}{\begin{pmatrix} a \\ b \\ c \end{pmatrix}} = \underset{A^tB}{\begin{pmatrix} 14.7285523463470 \\ 37.7924460060656 \\ 1.40243803585753 \end{pmatrix}}$$

rref

$$\begin{pmatrix}
1.00000000000000 & 0.000000000000000 & 0.000000000000000 & 1.00105160451766 \\
0.000000000000000 & 1.00000000000000 & 0.000000000000000 & 2.03698058397872 \\
0.000000000000000 & 0.000000000000000 & 1.00000000000000 & -1.00288814333217
\end{pmatrix}
\begin{array}{l} a \\ b \\ c \end{array}$$

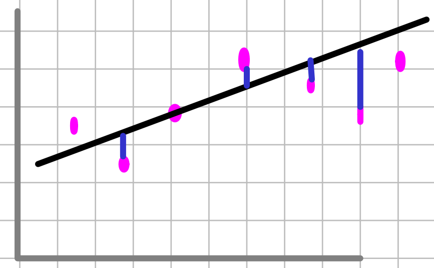


$p(x) \approx 1x^2 + 2x - 1$

Några modellproblem

1. Linjär regression.

- Anpassa $y = {\color{red}k}x + {\color{red}m} = f(x)$
till lista $\{(x_i, y_i) : i = 1..n\}$



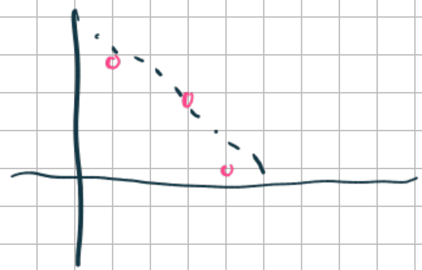
- Minimera vertikalt avst

$$\sum (y_i - f(x_i))^2$$

- Linj. sys

$$\begin{array}{cccc} A & X & & B \\ \begin{pmatrix} x_1 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{pmatrix} & \begin{pmatrix} k \\ m \end{pmatrix} & = & \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \end{array}$$

Ex



$Z^t = (1, 3, 4)$

$Y^t = (3, 2, 0)$

$$A = (Z \mid 1), \quad B = Y = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 3 & 1 \\ 4 & 1 \end{pmatrix}, \quad A^t = \begin{pmatrix} 1 & 3 & 4 \\ 1 & 1 & 1 \end{pmatrix}, \quad A^tA = \begin{pmatrix} 26 & 8 \\ 8 & 3 \end{pmatrix}$$

$$A^tB = \begin{pmatrix} 9 \\ 5 \end{pmatrix}, \quad \begin{pmatrix} k \\ m \end{pmatrix} = \frac{1}{26\cdot 3 - 8^2} \begin{pmatrix} 3 & -8 \\ -8 & 26 \end{pmatrix} \begin{pmatrix} 9 \\ 5 \end{pmatrix}$$

2) Polynomiell växt

- Anpassa $f(x) = c \cdot x^d$ till (x_i, y_i)

- Logaritmera $y = c \cdot x^d$ till $\log(y) = \underbrace{\log(c)}_{m} + \underbrace{d}_{k} \log(x)$

3) Exponentiell

- $y \sim c \exp(dx)$

- $\log y \sim \log c + dx$

- logaritmera y-data, ej x-data

Ex

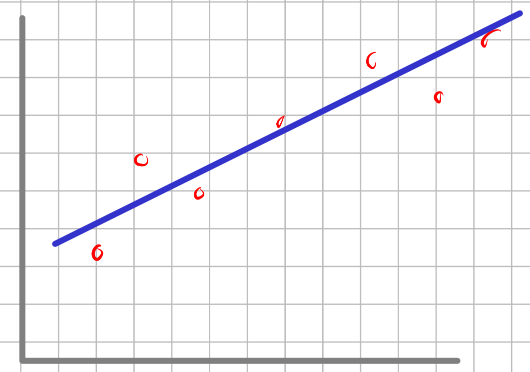
$$A = \begin{pmatrix} \log(x_1) & 1 \\ \vdots & \vdots \\ \log(x_n) & 1 \end{pmatrix}, \quad X = \begin{pmatrix} k \\ m \end{pmatrix}, \quad B = \begin{pmatrix} \log y_1 \\ \vdots \\ \log y_n \end{pmatrix}$$

MK-lös $AX = B$, $X = \begin{pmatrix} m \\ k \end{pmatrix}$,

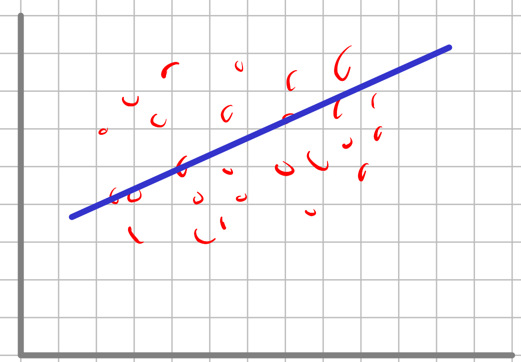
$c = e^m$, $d = e^k$,

Hur bra är kurvanpassningen?

- Anpassa $y = kx + m$ till lista av (x_i, y_i)



- Anpassningens "fel" mäts av $\|AX - B\|^2 = \sum (kx_i + m - y_i)^2$



- Beror av storlek

- Produktmomentkorrelations-koefficient $\quad -1 \le k \le 1$

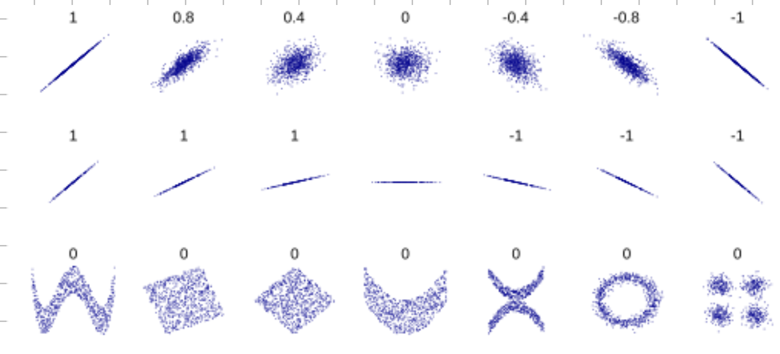
where

- n is sample size
- x_i, y_i are the individual sample points indexed with *i*
- $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ (the sample mean); and analogously for $\bar{y}$.

Rearranging gives us this[10] formula for r_{xy}:

$$r_{xy} = \frac{\sum_i x_i y_i - n\bar{x}\bar{y}}{\sqrt{\sum_i x_i^2 - n\bar{x}^2}\,\sqrt{\sum_i y_i^2 - n\bar{y}^2}},$$

where $n, x_i, y_i, \bar{x}, \bar{y}$ are defined as above.



Ortogonalprojektion via MK

1) $AX = B$ lösbart omm $B \in \text{kolrum}(A) = U$

2) Om $\tilde{X}$ MK-lösn så $A\tilde{X} = \text{proj}(B, U)$

Ex $U = [(1,1,1),(1,1,2)]$, $\bar{v} = (1,2,3)$

Bestäm $\bar{v}_{/\!/U}$.

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$A^t = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}, \quad A^tA = \begin{pmatrix} 3 & 4 \\ 4 & 6 \end{pmatrix}, \quad A^tB = \begin{pmatrix} 6 \\ 9 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 3 & 4 & 6 \\ 4 & 6 & 9 \end{array}\right) \sim \begin{array}{cc|c} 3 & 4 & 6 \\ 1 & 2 & 3 \end{array} \sim \begin{array}{cc|c} 1 & 2 & 3 \\ 0 & -2 & -3 \end{array} \sim \begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 3/2 \end{array}$$

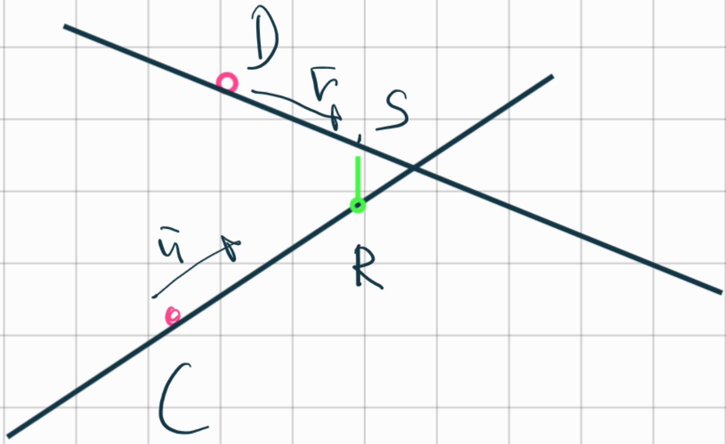
$$A\tilde{X} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 3/2 \end{pmatrix} = \frac{3}{2}\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3/2 \\ 3/2 \\ 3 \end{pmatrix}.$$

Kontroll: $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 3/2 \\ 3/2 \\ 3 \end{pmatrix} = \begin{pmatrix} -1/2 \\ 1/2 \\ 0 \end{pmatrix}$

Fcltvebar

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} -1/2 \\ 1/2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$ OK!

Ex Avst. 2 linjer



$$\overline{RS} = \overline{OS} - \overline{OR} = \overline{OD} + s\bar v - \overline{OC} - u\bar u$$

MK-lös $= \bar 0$

Ex $C=(0,0,0)$, $\bar u = \begin{pmatrix}1\\1\\1\end{pmatrix}$, $D=(1,2,3)$, $\bar v = \begin{pmatrix}1\\-1\\2\end{pmatrix}$

$$\begin{pmatrix}1\\2\\3\end{pmatrix} + s\begin{pmatrix}1\\-1\\2\end{pmatrix} - u\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{array}{cc} u & s \\ \end{array}$$

$$\underbrace{\begin{pmatrix}-1 & 1\\-1 & -1\\-1 & 2\end{pmatrix}}_{A}\underbrace{\begin{pmatrix}u\\s\end{pmatrix}}_{X} = \underbrace{\begin{pmatrix}-1\\-2\\-3\end{pmatrix}}_{B}$$

$$A^t = \begin{pmatrix} -1 & -1 & -1 \\ 1 & -1 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix}$$

$$A^t A = \begin{pmatrix} 3 & -2 \\ -2 & 4 \end{pmatrix} \qquad A^t B = \begin{pmatrix} 6 \\ -5 \end{pmatrix}$$

$$\left.\begin{matrix} 3 & -2 \\ -2 & 4 \end{matrix}\right| \begin{matrix} 6 \\ -5 \end{matrix} \sim \left.\begin{matrix} 1 & -2/3 \\ 0 & 8/3 \end{matrix}\right| \begin{matrix} 2 \\ -1 \end{matrix} \sim \left.\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\right| \begin{matrix} 7/4 \\ -3/8 \end{matrix}$$

$$\overline{OR} = \underline{e} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \frac{7}{4} \underline{e} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \underline{e} \begin{pmatrix} 7/4 \\ 7/4 \\ 7/4 \end{pmatrix}$$

$$\overline{OS} = \underline{e} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \frac{3}{8} \underline{e} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \underline{e} \begin{pmatrix} 5/8 \\ 19/8 \\ 9/4 \end{pmatrix}$$

$$\overline{RS} = \underline{e} \frac{1}{8} \begin{pmatrix} -9 \\ 5 \\ 4 \end{pmatrix} \qquad d = \|\overline{RS}\| = \frac{1}{8} \sqrt{81+25+16}$$