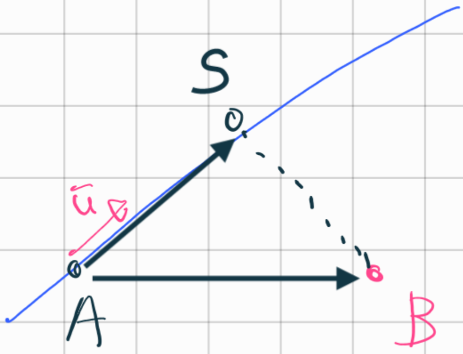
Föreläsning 11 — Repetition

- Avst. linje-pkt
- Avst. plan-pkt
- Avst. linje-linje
- Matrisinvers

- Matrisekv
-
⋮

Avst. pkt - linje



$$l: \overline{OP}(t) = \overline{OA} + t\bar{u}$$

$$\overline{AS} = \overline{AB}_{//\bar u}, \quad \overline{OS} = \overline{OA} + \overline{AS}$$

Ex $\quad l: \underline{e}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underline{e}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \quad B = (3,4,10)$

$$\overline{AB} = \underline{e}\begin{pmatrix} 2 \\ 3 \\ 9 \end{pmatrix}, \quad \overline{AB}_{//\bar u} = \frac{\underline{e}\begin{pmatrix} 2 \\ 3 \\ 9 \end{pmatrix} \circ \underline{e}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}}{\underline{e}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \circ \underline{e}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}} \underline{e}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$$= \frac{-6}{2}\underline{e}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \underline{e}\begin{pmatrix} 0 \\ -3 \\ 3 \end{pmatrix} = \overline{AS}$$

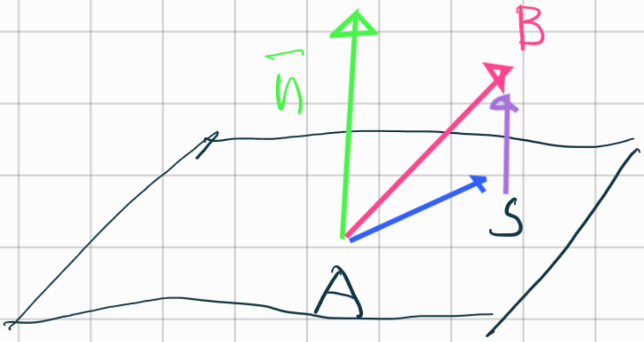
$$\overline{OS} = \overline{OA} + \overline{AS} = \underline{e}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \underline{e}\begin{pmatrix} 0 \\ -3 \\ 3 \end{pmatrix} = \underline{e}\begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$$

$$\|\overline{AS}\| = 3\left\|\underline{e}\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}\right\| = 3\sqrt{2}$$

Kontroll: $\overline{BS} = \underline{e}\begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} - \underline{e}\begin{pmatrix} 3 \\ 4 \\ 10 \end{pmatrix} = \underline{e}\begin{pmatrix} -2 \\ -6 \\ -6 \end{pmatrix}$

$\bar u \circ \overline{BS} = \underline{e}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \circ \underline{e}\begin{pmatrix} -2 \\ -6 \\ -6 \end{pmatrix} = 0$ ok!

Avst. pkt - plan



projicera på normalen!

$$\overline{SB} = \overline{AB}_{/\!/ \bar n}$$

Ex

Planet $x - 2y + 5z = 7$

$B = (3,3,3)$

$A = (7,0,0)$, t.ex.

$$\overline{AB} =_e \begin{pmatrix}-4\\3\\3\end{pmatrix}$$

$$\bar n =_e \begin{pmatrix}1\\-2\\5\end{pmatrix}$$

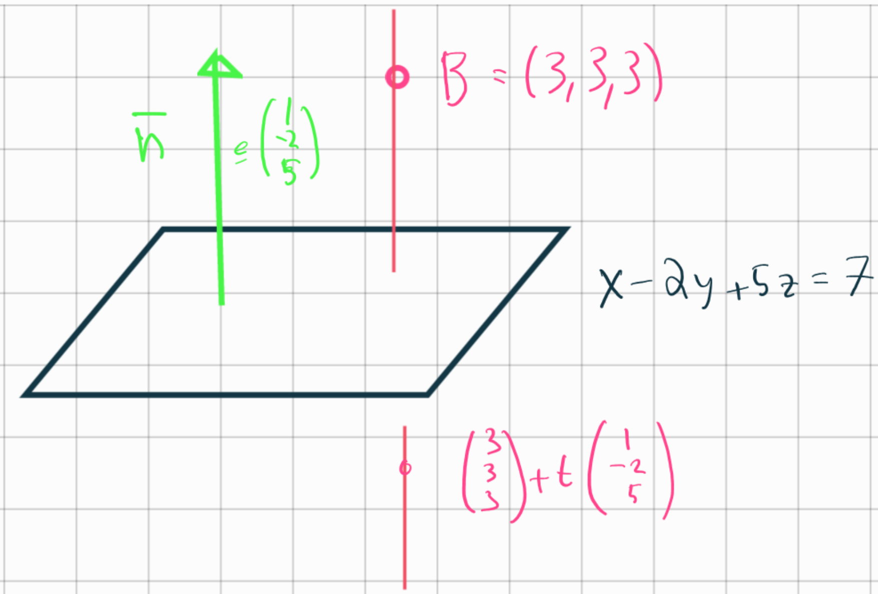
$$AB_{/\!/ \bar n} = \frac{\begin{pmatrix}-4\\3\\3\end{pmatrix} \circ \begin{pmatrix}1\\-2\\5\end{pmatrix}}{\begin{pmatrix}1\\-2\\5\end{pmatrix} \circ \begin{pmatrix}1\\-2\\5\end{pmatrix}} \begin{pmatrix}1\\-2\\5\end{pmatrix} = \frac{+5}{30} \begin{pmatrix}1\\-2\\5\end{pmatrix}$$

$$= \frac{+1}{6} \begin{pmatrix}1\\-2\\5\end{pmatrix} \qquad d = \frac{1}{6} \left\| \begin{pmatrix}1\\-2\\5\end{pmatrix} \right\| = \frac{1}{6}\sqrt{30} = \overline{SB}$$

$$\overline{OS} = \overline{OB} - \overline{SB}$$

$$= \underline{e}\begin{pmatrix}3\\3\\3\end{pmatrix} - \frac{1}{6}\underline{e}\begin{pmatrix}1\\-2\\5\end{pmatrix} = \underline{e}\,\frac{1}{6}\begin{pmatrix}17\\20\\13\end{pmatrix}$$

Alt:



$$\overset{x}{(3+t)} - 2\overset{y}{(3-2t)} + 5\overset{z}{(3+5t)} = 7$$

$$3 + t - 6 + 4t + 15 + 25t = 7$$

$$(1+4+25)t = 7 - 3 + 6 - 15 = -5$$

$$t = -\frac{5}{30} = -\frac{1}{6}$$

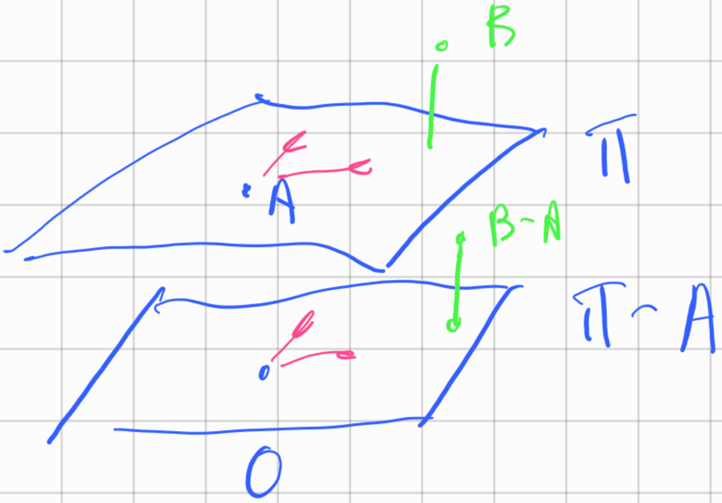
$$d = |t| \cdot \|\bar{n}\| = \frac{1}{6} \cdot \sqrt{30}$$

$$t = -\frac{1}{6} \text{ gir } \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}3\\3\\3\end{pmatrix} - \frac{1}{6}\begin{pmatrix}1\\-2\\5\end{pmatrix} = \frac{1}{6}\begin{pmatrix}17\\20\\13\end{pmatrix}$$

Alt: ORTO-bas för planet: $x-2y+5z=7$

$$\left(\begin{array}{ccc|c} x & y & z & \\ 1 & -2 & 5 & 7 \end{array}\right) \qquad y=s,\ z=t,\ x=7+2s-5z$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix}$$



Translatera först!

$\pi - A$ har bas

$$\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix}$$

GS: $\bar{f}_1 = (2,1,0) = \bar{g}_1$

$$\bar{f}_{2\parallel \bar{g}_1} = \frac{(2,1,0)\circ(-5,0,1)}{(2,1,0)\circ(2,1,0)}(2,1,0)$$

$$= \frac{-10}{5}(2,1,0) = (-4,-2,0)$$

så $\bar{g}_2 = \bar{f}_2 - \bar{f}_{2\parallel \bar{g}_1}$

$$= (-5,0,1) - (-4,-2,0) = (-1,2,1)$$

Sv: ORTO-bas $\left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \right\}$

$$B-A = \begin{pmatrix}3\\3\\3\end{pmatrix} - \begin{pmatrix}7\\0\\0\end{pmatrix} = \begin{pmatrix}-4\\3\\3\end{pmatrix}$$

Translaterat problem: avst $\begin{pmatrix}-4\\3\\3\end{pmatrix}$ till $x-2y+5z=0$

proj på trns plan: $\dfrac{\begin{pmatrix}-4\\3\\3\end{pmatrix}\circ\begin{pmatrix}2\\1\\0\end{pmatrix}}{5}\begin{pmatrix}2\\1\\0\end{pmatrix} + \dfrac{\begin{pmatrix}-4\\3\\3\end{pmatrix}\circ\begin{pmatrix}-1\\2\\1\end{pmatrix}}{6}\begin{pmatrix}-1\\2\\1\end{pmatrix}$

$$= \frac{-5}{5}\begin{pmatrix}2\\1\\0\end{pmatrix} + \frac{13}{6}\begin{pmatrix}-1\\2\\1\end{pmatrix} = \frac{1}{6}\begin{pmatrix}-12\\-6\\0\end{pmatrix} + \frac{1}{6}\begin{pmatrix}-13\\26\\13\end{pmatrix}$$

$$= \frac{1}{6}\begin{pmatrix}-25\\20\\13\end{pmatrix}$$

Addera A: $\begin{pmatrix}7\\0\\0\end{pmatrix} + \frac{1}{6}\begin{pmatrix}-25\\20\\13\end{pmatrix} = \frac{1}{6}\begin{pmatrix}42\\0\\0\end{pmatrix} + \frac{1}{6}\begin{pmatrix}-25\\20\\13\end{pmatrix} = \frac{1}{6}\begin{pmatrix}17\\20\\13\end{pmatrix}$

Alt: MK

$$\begin{pmatrix}7\\0\\0\end{pmatrix} + s\begin{pmatrix}2\\1\\0\end{pmatrix} + t\begin{pmatrix}-5\\0\\1\end{pmatrix} = \begin{pmatrix}3\\3\\3\end{pmatrix}$$ olösligt!

MK-lös

$$A = \begin{pmatrix}2 & -5\\1 & 0\\0 & 1\end{pmatrix} \quad X = \begin{pmatrix}s\\t\end{pmatrix} \quad B = \begin{pmatrix}-4\\3\\3\end{pmatrix}$$

$$A^t A = \begin{pmatrix}2 & 1 & 0\\-5 & 0 & 1\end{pmatrix}\begin{pmatrix}2 & -5\\1 & 0\\0 & 1\end{pmatrix} = \begin{pmatrix}5 & -10\\-10 & 26\end{pmatrix}$$

$$A^t B = \begin{pmatrix}2 & 1 & 0\\-5 & 0 & 1\end{pmatrix}\begin{pmatrix}-4\\3\\3\end{pmatrix} = \begin{pmatrix}-5\\23\end{pmatrix}$$

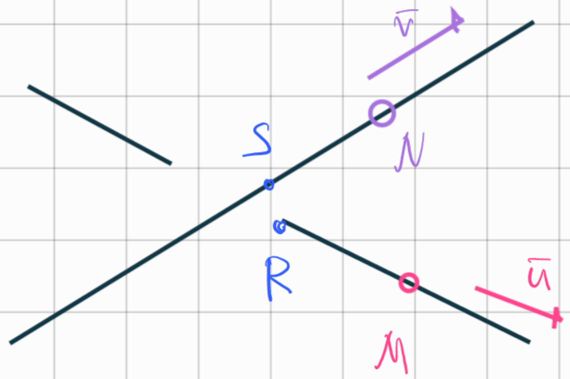
$$\left(\begin{array}{cc|c}5 & -10 & -5\\-10 & 26 & 23\end{array}\right) \sim \left(\begin{array}{cc|c}1 & -2 & -1\\0 & 6 & 13\end{array}\right) \sim \left(\begin{array}{cc|c}1 & 0 & 20/6\\0 & 1 & 13/6\end{array}\right)$$

$$X_p = \frac{1}{6}\begin{pmatrix}20\\13\end{pmatrix}$$

$$\begin{pmatrix}7\\0\\0\end{pmatrix} + \frac{20}{6}\begin{pmatrix}2\\1\\0\end{pmatrix} + \frac{13}{6}\begin{pmatrix}-5\\0\\1\end{pmatrix}$$

$$= \frac{1}{6}\begin{pmatrix}42 + 40 - 65\\0 + 20 + 0\\0 + 0 + 13\end{pmatrix} = \frac{1}{6}\begin{pmatrix}17\\20\\13\end{pmatrix}$$

Avst. linje–linje.



$l_1 : \overline{OP}(s) = \overline{OM} + s\bar u$, $s = s_0$ ger R

$l_2 : \overline{OQ}(t) = \overline{ON} + t\bar v$, $t = t_0$ ger S

$d = \|RS\|$

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1) MK-lös $\overline{OP}(s) = \overline{OQ}(t)$ dvs $s\bar u - t\bar v = \overline{ON} - \overline{OM}$

$$A = \begin{pmatrix} | & | \\ \bar u & -\bar v \\ | & | \end{pmatrix}, \quad X = \begin{pmatrix} s \\ t \end{pmatrix}, \quad B = \overline{ON} - \overline{OM}$$

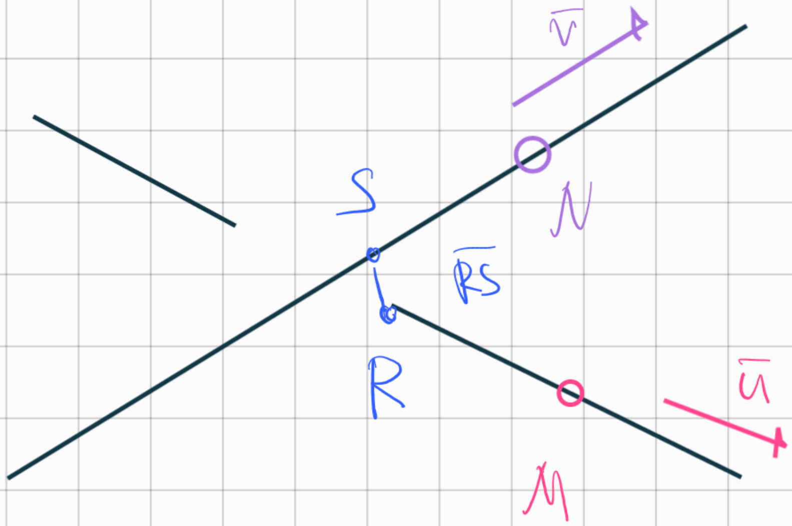
$$A^t A = \begin{pmatrix} \bar u \circ \bar u & -\bar u \circ \bar v \\ -\bar u \circ \bar v & \bar v \circ \bar v \end{pmatrix} \quad A^t B = \begin{pmatrix} \bar u \circ (\overline{ON} - \overline{OM}) \\ -\bar v \circ (\overline{ON} - \overline{OM}) \end{pmatrix}$$

X_p lösn till $A^t A X = A^t B$,

$$\overset{\shortparallel}{\begin{pmatrix} s \\ t \end{pmatrix}}, \quad \underset{\overline{OM} + s\bar u}{\overline{OP}(s)} = \overline{OR}, \quad \underset{\overline{ON} + t\bar v}{\overline{OQ}(t)} = \overline{OS}$$

$$\overline{QP} = \overline{OM} - \overline{ON} + s\bar u - t\bar v$$
$$= \overline{NM} + AX_p$$

2)



Rät vinkel!

$$\overline{RS} \circ \bar{u} = 0$$
$$\overline{RS} \circ \bar{v} = 0$$

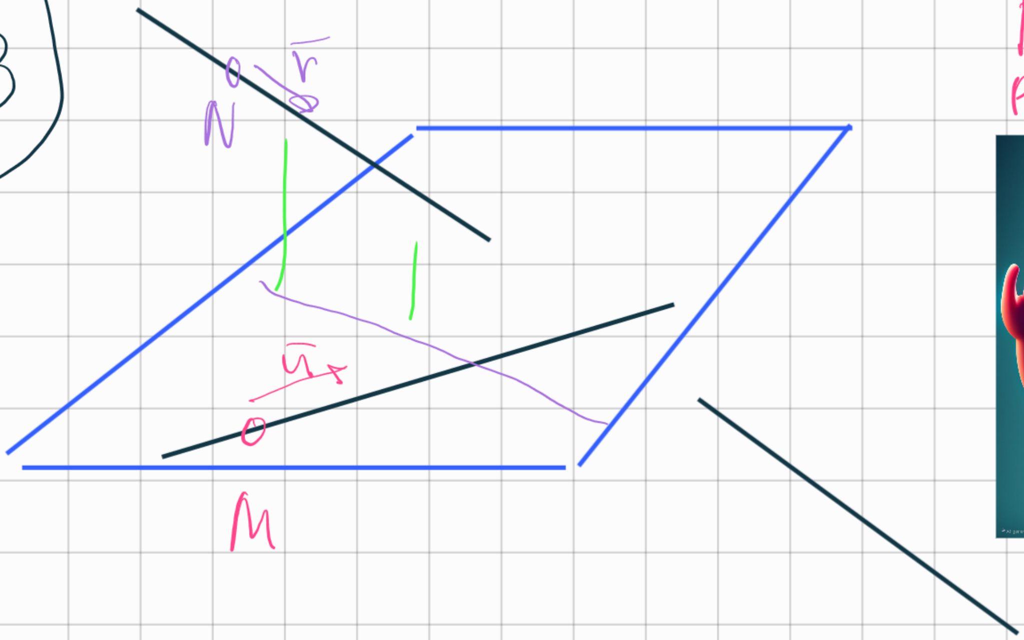
$$\overline{RS} = \overline{ON} + t\bar{v} - \overline{OM} - s\bar{u} = \overline{MN} + t\bar{v} - s\bar{u}$$

$$\overline{RS} \circ \bar{u} = \overline{MN} \circ \bar{u} + t\bar{v} \circ \bar{u} - s\bar{u} \circ \bar{u} = 0$$
$$\overline{RS} \circ \bar{v} = \overline{MN} \circ \bar{v} + t\bar{v} \circ \bar{v} - s\bar{u} \circ \bar{v} = 0$$

$$\begin{pmatrix} -\bar{u} \circ \bar{u} & \bar{u} \circ \bar{v} \\ -\bar{u} \circ \bar{v} & \bar{v} \circ \bar{v} \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} \overline{MN} \circ \bar{u} \\ \overline{MN} \circ \bar{v} \end{pmatrix}$$

3)

Plan innehållande l_1, parallell med l_2



$$\bar{n} = \bar{u} \times \bar{v}$$

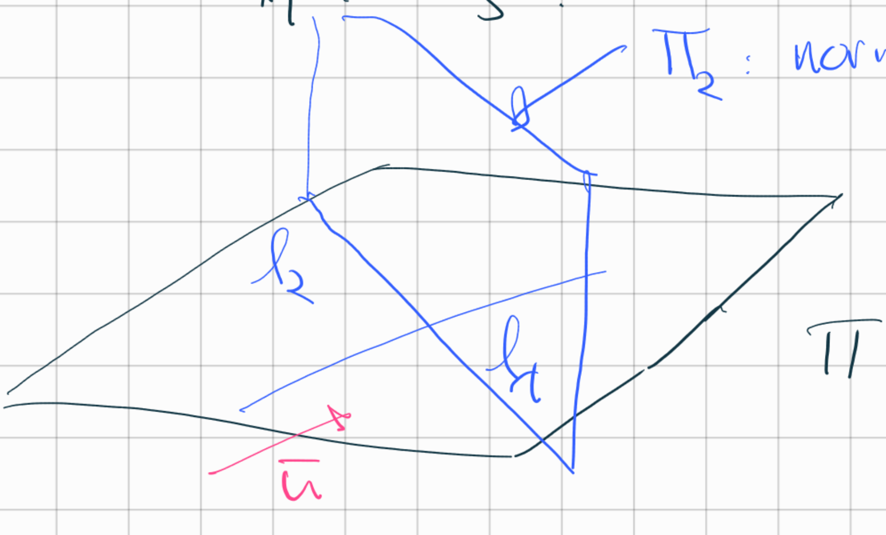
$$\Pi : \bar{n} \cdot \left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \overline{OM} \right) = 0$$

$$d = d(N, \Pi) = \|H\|, \quad H = \overline{MN}_{//\bar{n}}$$

$$l_3 : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \overline{ON} - H + t\bar{r}$$ linje i Π

Bestäm $l_1 \cap l_3 = l_1 \cap \Pi_2$

Π_2 : normal $\bar{u}$, innehåller l_3



Matrisinvers

$$I = I_n = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix},$$

$AI_n = I_nA = A$

alla A $n \times n$

$AA^{-1} = A^{-1}A = I$, men ett av villkoren räcker

1) $(a)^{-1} = (1/a)$

2) $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Byt plats på diag

byt tecken på anti-diag

3) $\left(A \mid I\right) \sim \dots \sim \left(I \mid A^{-1}\right)$

Ex

$$\left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right) \sim \begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{array} \sim \begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 1 \end{array}$$

kontroll: $\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ OK!

Ex: $a \in \mathbb{R}$, $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ a & 1 & 3 \end{pmatrix}$. A^{-1} ?

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ a & 1 & 3 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 1-a & 3-a & -a & 0 & 1 \end{array}\right) \sim_{a-3} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 2 & -1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 1-a & 0 & 3-2a & a-3 & 1 \end{array}\right)$$

$-a+(3-a)=3-2a$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 2 & -1 & 0 \\ 0 & 1-a & 0 & 3-2a & a-3 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right)$$

Fall 1: $a=1$, invers saknas!

Fall 2: $a \neq 1$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2-\frac{3-2a}{1-a} & -1-\frac{a-3}{1-a} & \frac{-1}{1-a} \\ 0 & 1 & 0 & \frac{3-2a}{1-a} & \frac{a-3}{1-a} & \frac{1}{1-a} \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right)$$

Ex $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 4 \\ 0 & 0 & 4 \end{pmatrix}$ $B = \begin{pmatrix} 1 & c & d \\ 0 & 1/4 & e \\ 0 & 0 & 1/4 \end{pmatrix}$

Bestäm c, d, e så att $B = A^{-1}$.

$$AB = \left(\begin{array}{c|c|c} 1 & c+\frac{1}{4} & d+2e+\frac{3}{4} \\ 0 & 1 & 4e+1 \\ 0 & 0 & 1 \end{array}\right)$$

Så ekv. sys.
$$\begin{aligned} c \qquad\quad &= -\frac{1}{4} \\ d+2e &= -3/4 \\ 4e &= -1 \end{aligned}$$

$c = -\frac{1}{4}$, $e = -\frac{1}{4}$, $d = -\frac{3}{4} + \frac{2}{4} = -\frac{1}{4}$

$$B = \frac{1}{4}\begin{pmatrix} 4 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

Matrisekvationer

- Lös ut X om det går
- Annars, ansätt obekanta

Ex $$3X - X\underbrace{A}_{3\times3} = \underbrace{E}_{3\times3} + X\underbrace{C}_{3\times3} \qquad (*)$$

$$X(3I - A - C) = E$$

$$X = E(3I - A - C)^{-1}$$

Ex: $AX = XA$

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

Ansätt $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$AX = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+2c & b+2d \\ 2a+4c & 2b+4d \end{pmatrix}$$

$$XA = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} a+2b & 2a+4b \\ c+2d & 2c+4d \end{pmatrix}$$

$$AX - XA = \begin{pmatrix} 2c-2b & -2a-3b+2d \\ 2a+3c-2d & 2b-2c \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{array}{cccc|c} a & b & c & d & \\ 0 & -1 & 1 & 0 & 0 \\ -2 & -3 & 0 & 2 & 0 \\ 2 & 0 & 3 & -2 & 0 \end{array} \sim \begin{array}{cccc|c} 2 & 0 & 3 & -2 & 0 \\ 0 & -3 & 3 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \end{array} \sim \begin{array}{cccc|c} 1 & 0 & 3/2 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{array}$$

$c = s,\ d = t,\ b = s,\ a = -\frac{3}{2}s + t$

bas $\begin{pmatrix} -3 & 2 \\ 2 & 0 \end{pmatrix}_{s=2,\ t=0},\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{s=0,\ t=1}$

Baser, dim

Givet $S=\{\bar v_1,\dots,\bar v_m\}\subseteq\mathbb{R}^n$, $U=[\bar v_1,\dots,\bar v_n]=[S]$

Hitta

(i) $B\subseteq S$ bas för U

(ii) Linjära former $l_1,\dots,l_k$ på $\mathbb{R}^n$ s.a.

$$U=\{\bar u\in\mathbb{R}^n : l_1(\bar u)=\dots=l_k(\bar u)=0\}$$

(iii) bas för $U^{\perp}$

(iv) bas för rummet av linjära relationer mellan $\bar v_j$,

$$Syz=\{(c_1,\dots,c_m)\in\mathbb{R}^m : \sum c_i\bar v_i=\bar 0\}$$

Ex $S=\{\bar v_1,\bar v_2,\bar v_3\}\subseteq\mathbb{R}^5$

$\bar v_1=(1,0,1,0,1)$, $\bar v_2=(1,1,1,1,1)$, $\bar v_3=(0,1,0,1,0)$

$$\begin{array}{ccc|cc} c_1 & c_2 & c_3 & & \\ 1 & 1 & 0 & 0 & a \\ 0 & 1 & 1 & 0 & b \\ 1 & 1 & 0 & 0 & c \\ 0 & 1 & 1 & 0 & d \\ 1 & 1 & 0 & 0 & e \end{array} \sim \begin{array}{ccc|cc} 1 & 1 & 0 & 0 & a \\ 0 & 1 & 1 & 0 & b \\ 0 & 0 & 0 & 0 & -a+c \\ 0 & 0 & 0 & 0 & -b+d \\ 0 & 0 & 0 & 0 & -a+e \end{array} \sim \begin{array}{ccc|cc} 1 & 0 & -1 & 0 & a-b \\ 0 & 1 & 1 & 0 & b \\ 0 & 0 & 0 & 0 & a-e \\ 0 & 0 & 0 & 0 & b-d \\ 0 & 0 & 0 & 0 & c-e \end{array}$$

(i) $\{\bar{v}_1, \bar{v}_2\}$ bas för U

(ii) U = lösningsrummet
$$\begin{array}{ccccc} a & b & c & d & e \\ \end{array}$$
$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{array}\right)$$

(iii) $\{(1,0,0,0,-1), (0,1,0,-1,0), (0,0,1,0,-1)\}$
bas för $U^\perp$

(iv) $c_3 = t$, $c_2 = -t$, $c_1 = t$ så
$$S_{yz} = \left\{ t\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} : t \in \mathbb{R} \right\}$$

kontroll, $t=1$: $\bar{v}_1 - \bar{v}_2 + \bar{v}_3 = \bar{0}$

$$\underline{e}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} - \underline{e}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \underline{e}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \underline{e}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$ OK