# 1 Euklidiske rom, rep.

$V$ vr

$(\cdot|\cdot)$ indre prod

1) $(\bar{u}_1+\bar{u}_2|\bar{v}) = (\bar{u}_1|\bar{v})+(\bar{u}_2|\bar{v})$

2) $(c\bar{u}|\bar{v}) = c(\bar{u}|\bar{v})$

3) $(\bar{u}|\bar{v}) = (\bar{v}|\bar{u})$

4) $(\bar{u}|\bar{u}) \geq 0$,

og likhet omm $\bar{u}=\bar{0}$

Inkl: $\|\bar{u}\| = \sqrt{(\bar{u}|\bar{u})}$

$$d(\bar{u},\bar{v}) = \|\bar{u}-\bar{v}\| = \|\bar{v}-\bar{u}\|$$

$\bar{u} \perp \bar{v}$ omm $(\bar{u}|\bar{v}) = 0$

---

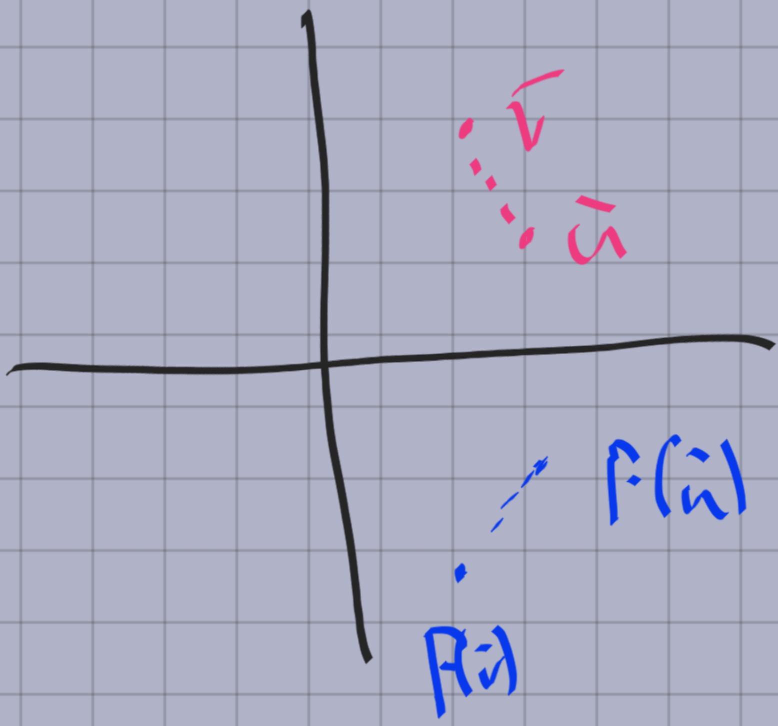
$V$ Euklidisk,

$F : V \to V$ linj. avb.

**Def** $F$ isometri om

$$d(F(\bar{u}), F(\bar{v})) = d(\bar{u},\bar{v})$$

alla $\bar{u},\bar{v} \in V$




Spegling funksjon

**Sats** Følgande likn.

1) $F$ iso
2) $\|F(\bar{u})\| = \|\bar{u}\|$ alle $\bar{u}$
3) $(F(\bar{u}) \mid F(\bar{v})) = (\bar{u} \mid \bar{v})$ alle $\bar{u}, \bar{v}$

B)

1) ⇒ 2):

$$\|F(\bar{u})\| = d(F(\bar{u}), \bar{0})$$

$$\overset{a|1)}{=} d(\bar{u}, \bar{0}) = \|\bar{u}\|$$

2) ⇒ 3):

$$\|F(\bar{u}) + F(\bar{v})\|^2 = \|F(\bar{u}+\bar{v})\|^2$$

$$\overset{2)}{=} \|\bar{u}+\bar{v}\|^2$$

$$VL = (F(\bar{u}) + F(\bar{v}) \mid F(\bar{u}) + F(\bar{v}))$$

$$= \|F(\bar{u})\|^2 + \|F(\bar{v})\|^2 + 2(F(\bar{u}) \mid F(\bar{v}))$$

$$HL = (\bar{u}+\bar{v} \mid \bar{u}+\bar{v}) =$$

$$\|\bar{u}\|^2 + \|\bar{v}\|^2 + 2(\bar{u}\mid\bar{v})$$

si $\quad 2\,(F(\bar{u}) \mid F(\bar{v})) = 2(\bar{u}\mid\bar{v})$

3) $\Rightarrow$ 1) $\quad d(F(\bar{u}), F(\bar{v}))^2$

$$= (F(\bar{u}) - F(\bar{v}) \mid F(\bar{u}) - F(\bar{v}))$$

$$= (F(\bar{u}-\bar{v}) \mid F(\bar{u}-\bar{v}))$$

$$\overset{3)}{=} (\bar{u}-\bar{v} \mid \bar{u}-\bar{v})$$

$$= d(\bar{u},\bar{v})^2.$$

Sats Om $\underline{e} = (\bar{e}_1 \dots \bar{e}_n)$ ON-bas för $V$, $F(\underline{e}X) = \underline{e}AX$

så $A^t A = I$.

Omvänt, $\underline{e}X \longmapsto \underline{e}AX$

med $A^t A = I$ är iso.

**B** $(\underline{e}X \mid \underline{e}Y) = X^t Y$

$= (\underline{e}AX \mid \underline{e}AY) =$

$= (AX)^t (AY) = X^t A^t A Y$

För alla X, Y så

$$X^t I Y = X^t (A^t A) Y$$

så $I = A^t A$

<u>Lemma</u> $A^t A = I$

omm $A^{-1} = A^t$

omm kol i A bildar ON-bas

omm rad i A bildar ON-bas

Ex $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

$$A^t A = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = 0$$

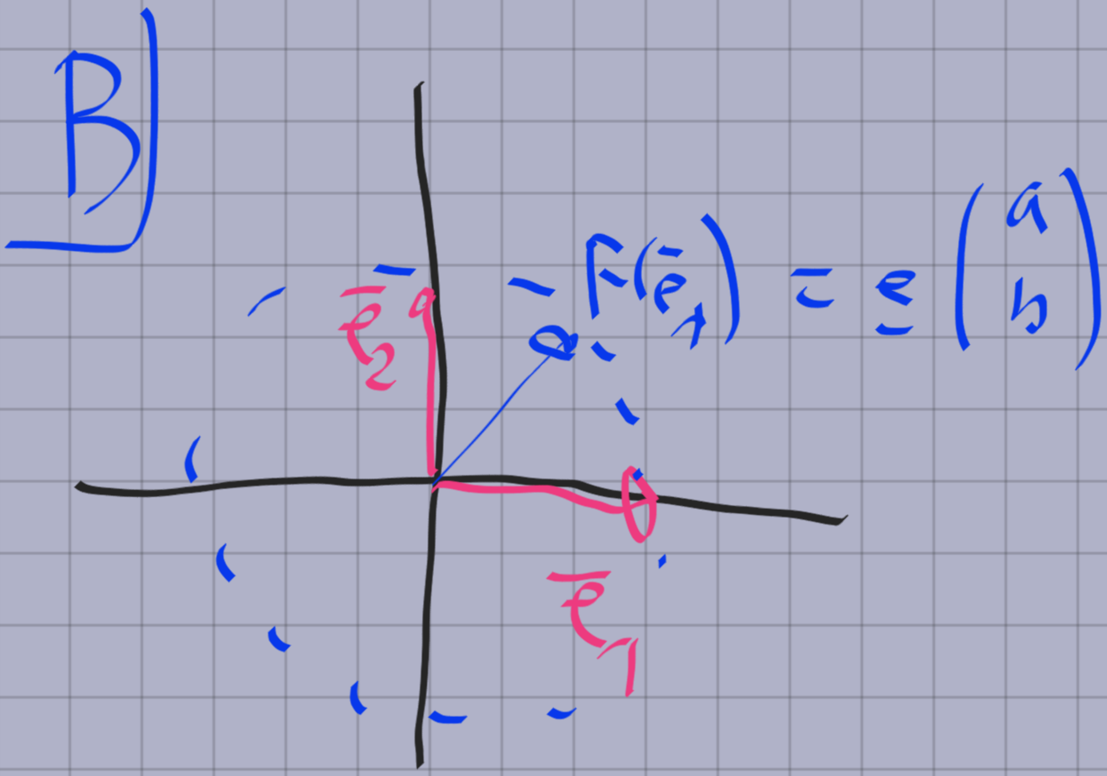
row 1 ; $A^t$ col2 ; $A$ $A_1 \perp A_2$

col 1 ; $A$

# Sats Isometrier

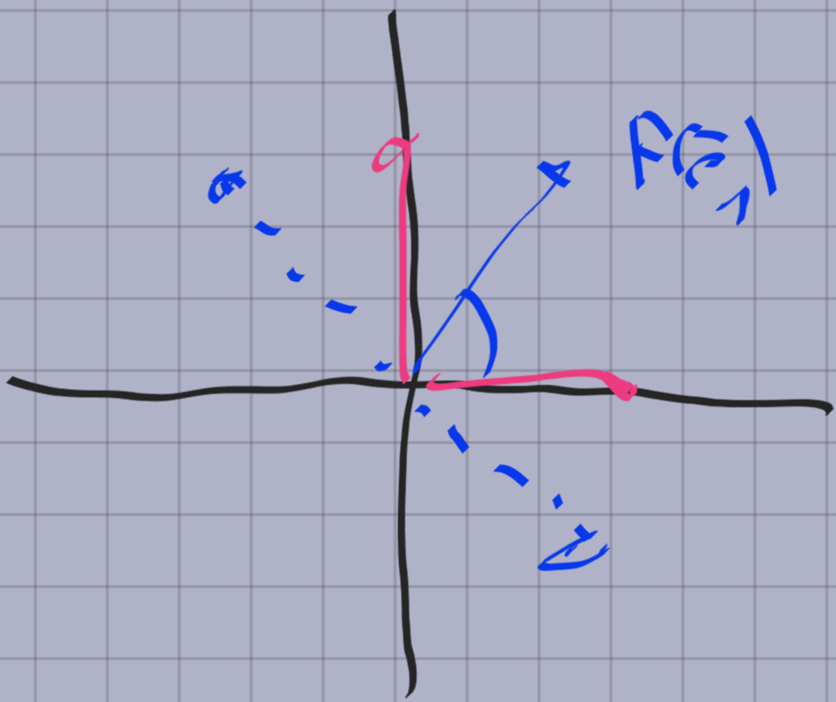
i planet är vridningar

el speglingar.



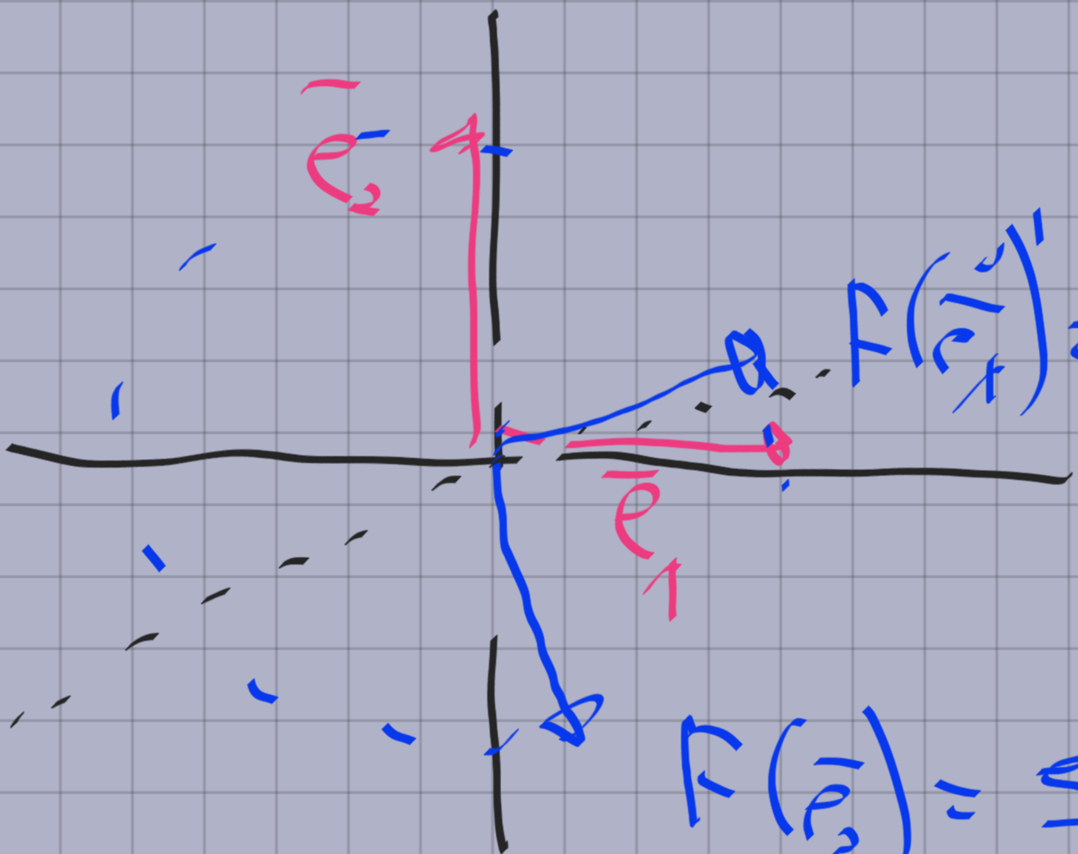

$F(\bar{e}_2) \perp F(\bar{e}_1)$ så

$$F(\vec{e}_2) = \pm c \begin{pmatrix} -b \\ a \end{pmatrix}$$




Fall 1: Drehung (um $\alpha$ rad,
$\cos(\alpha) = a$
$\sin(\alpha) = b$ )

# Fall 2: Spegling!




$$F(\bar{e}_1) = \begin{pmatrix} a \\ b \end{pmatrix}$$

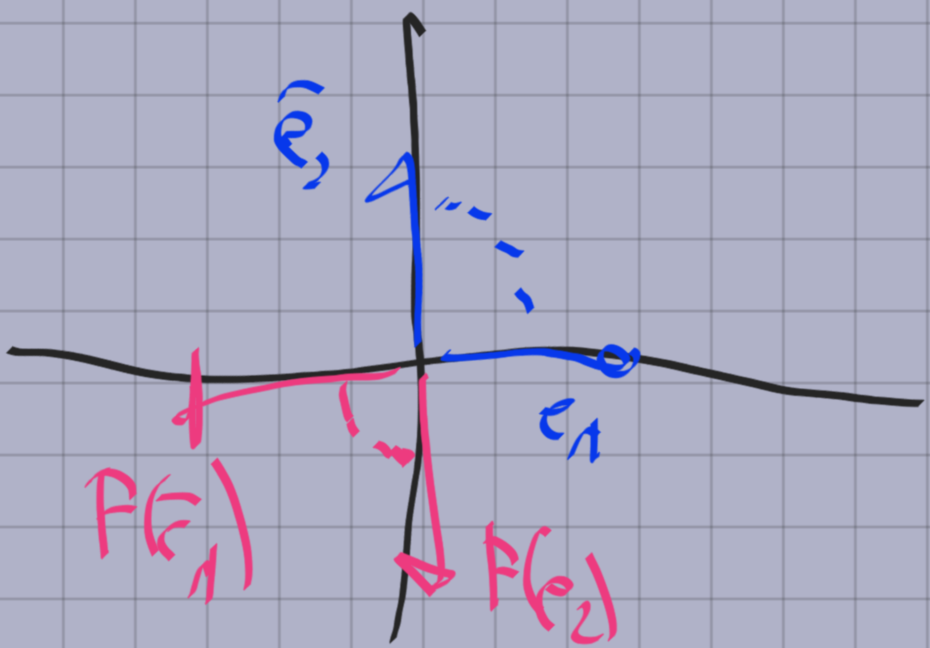
$$F(\bar{e}_2) = \begin{pmatrix} b \\ -a \end{pmatrix}$$

$$\ell = \left[ \frac{1}{2}\bar{e}_1 + \frac{1}{2}F(\bar{e}_1) \right]$$

Ex $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

ku l ON, si ISO.

Vad?




$F(\vec{e}_2)$ "Rice", si vridning.

180°

**Sats** Iso i rummet

är antingen

1) Vridning (runt axel)

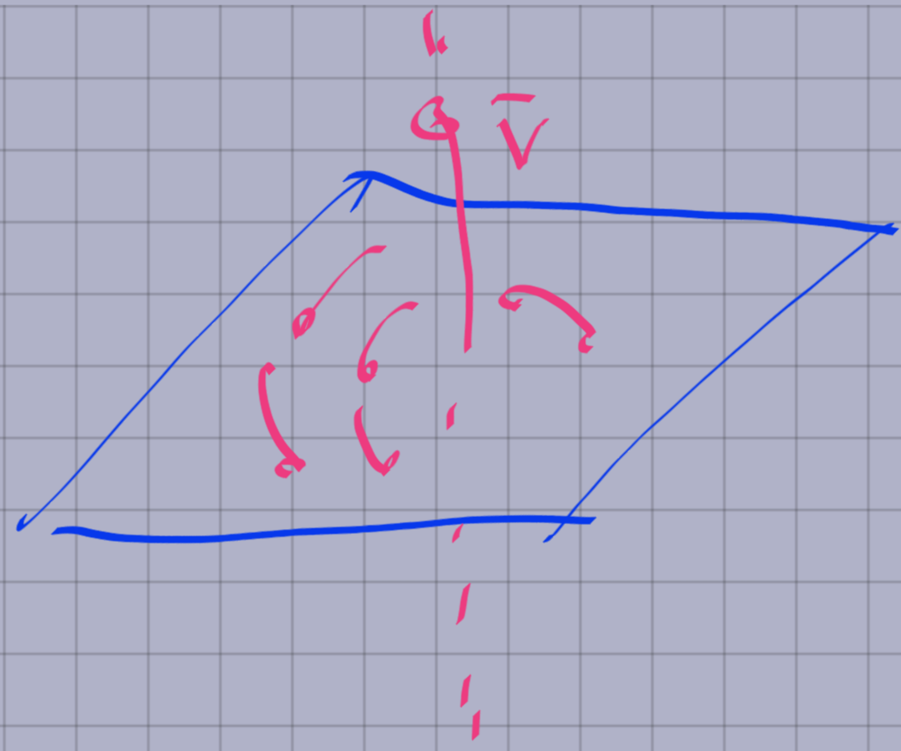
2) Spegling (i plan)

eller

3) Vridspegling (kombo av 1 o 2).

B) Kursboken!

1)




$F(\bar{v}) = \bar{v}$

$F(\text{plan}) = \text{plan}$

2)


$F(\bar{v}) = -\bar{v}$

$F(\bar{u}) = \bar{u}$

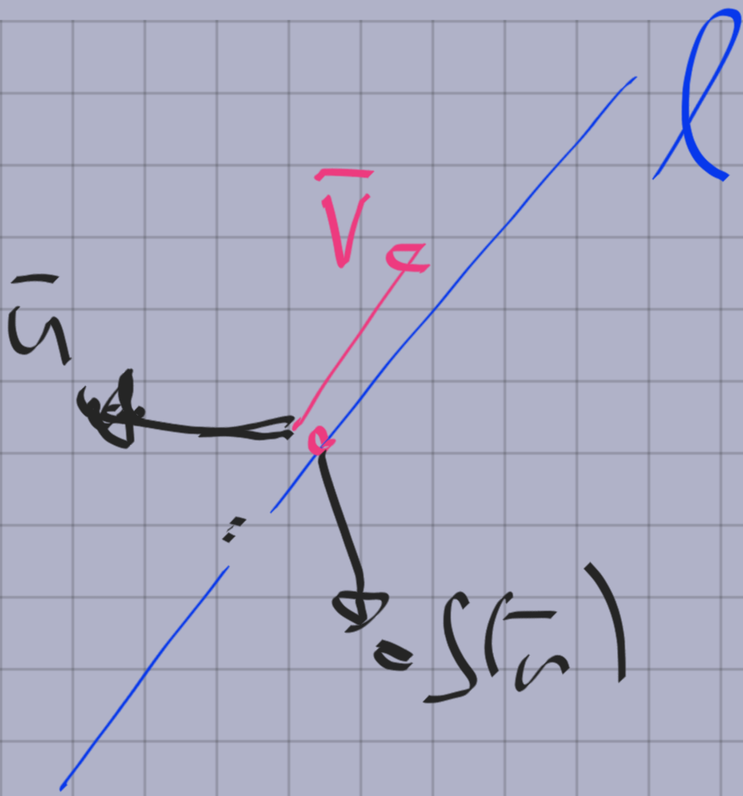
alle $\bar{u} \subset \text{plan}$

3)

$F(\bar{v}) = -\bar{v}$

$F(\text{plan}) = \text{plan}.$

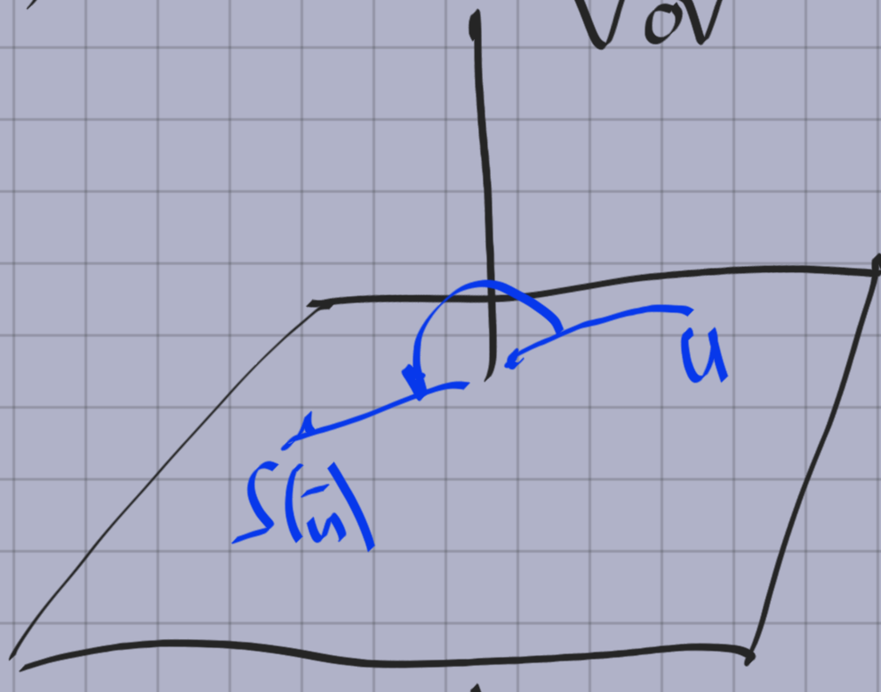
**Ex**




Speiling i linje, i rommet.

$$S(\bar{u}) = \bar{u} - 2\,\frac{\bar{u}\circ\bar{v}}{\bar{v}\circ\bar{v}}\,\bar{v}$$

Men:




Vridning 180° rundt aksen

**Sats** $F: V \to V$ linj.,

$$F(\underline{e}X) = \underline{e}AX$$
$$F(\underline{f}Y) = \underline{f}BY$$

Då: $\det(A) = \det B$

$(= \det F)$.

Visuell tolkning förklaring:

**Sats** $F$ iso $\Rightarrow \det F \in \{-1, 1\}$

i planet: +1 vridning, −1 spegling

i rummet: +1 vridning

−1 spegel el vrid.spegl.

B]

$$A^t A = I$$

$\Longrightarrow \det(A^t A) = \det(I) = 1$

mais $\det(A^t A) = \det(A^t)\det(A)$

$= \det(A)^2$

Si $\det(A)^2 = 1$

- $\begin{vmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{vmatrix} = \cos^2\alpha + \sin^2\alpha = 1$

- $\begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} = -1$

Jan Snellman

TEKNISKA HÖGSKOLAN
LINKÖPINGS UNIVERSITET



## Exempel

Vad är det för avbildning som (map någon ON-bas) har avbildningsmatris

$$M = \frac{1}{3}\begin{pmatrix} 1 & 2 & -2 \\ 2 & -2 & -1 \\ 2 & 1 & 2 \end{pmatrix} ?$$

1. $M^t M = I$, så isometri
2. $\det(M) = -1$ så vridspegling eller spegling
3. $MX = -X$ kan skrivas $(M+I)X = 0$, nollrumm till $\frac{1}{3}\begin{pmatrix} 4 & 2 & -2 \\ 2 & 1 & -1 \\ 2 & 1 & 5 \end{pmatrix}$ är $\operatorname{span}\left(\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}\right)$
4. Så linjen $\ell$ är $\operatorname{span}\left(\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}\right)$ och planet $\Pi$ är $x - 2y = 0$.

Jan Snellman

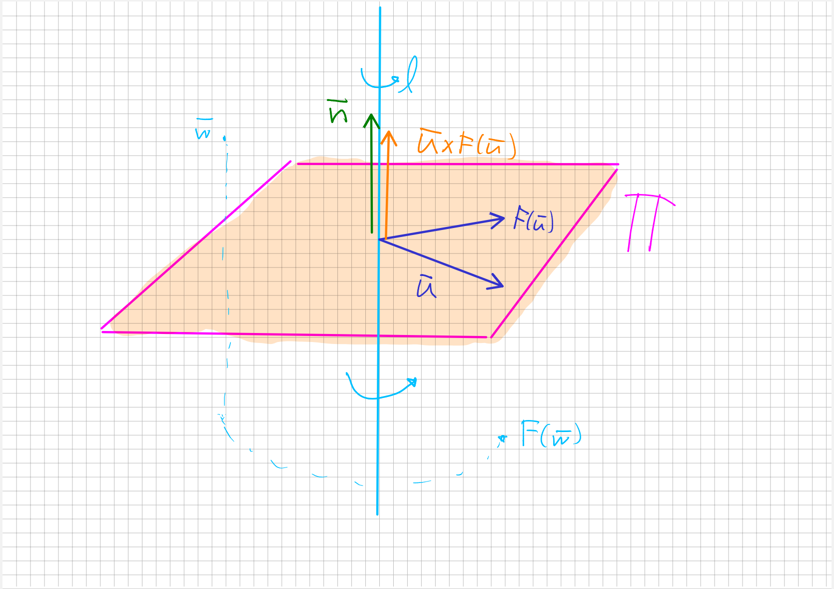
TEKNISKA HÖGSKOLAN
LINKÖPINGS UNIVERSITET



## Exempel (forts)

5. Ta vektor av längd ett i planet, tex $\bar{u} = \underline{e}\begin{bmatrix}0\\0\\1\end{bmatrix}$
6. $F(\bar{u}) = \underline{e}M\begin{bmatrix}0\\0\\1\end{bmatrix} = \underline{e}\begin{bmatrix}-2/3\\-1/3\\2/3\end{bmatrix} = \bar{v}$ har också längd ett, så vi mäter vinkeln mellan $\bar{u}$ och $\bar{v}$ genom $\cos(\alpha) = \bar{u} \cdot \bar{v} = 2/3$
7. Så avbildningen är en vridning med $\arccos(2/3)$ runt $\ell$, följt av en spegling i $\Pi$.
8. $$\bar{u} \times F(\bar{u}) = \underline{e}\begin{bmatrix}0\\0\\1\end{bmatrix} \times \underline{e}\begin{bmatrix}-2/3\\-1/3\\2/3\end{bmatrix} = \underline{e}\begin{bmatrix}1/3\\-2/3\\0\end{bmatrix}$$
   så vridningen sker *moturs* sett från spetsen av $\begin{bmatrix}1\\-2\\0\end{bmatrix}$

$(\bar{u} \times F(\bar{u})) = +\frac{1}{3}\bar{n}$


ℓ
n̄
w̄
ū × F(ū)
F(ū)
ū
Π
F(w̄)

## Ex  För varje perm $\sigma \in S_n$

Så $F_\sigma : V \to V$

$$F_\sigma(\bar{e}_i) = \bar{e}_{\sigma(i)} \quad \text{Iso}$$

## Ex

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

iso i rummet. Vridning el speglning?

## 2) Symmetriska avbildningar.

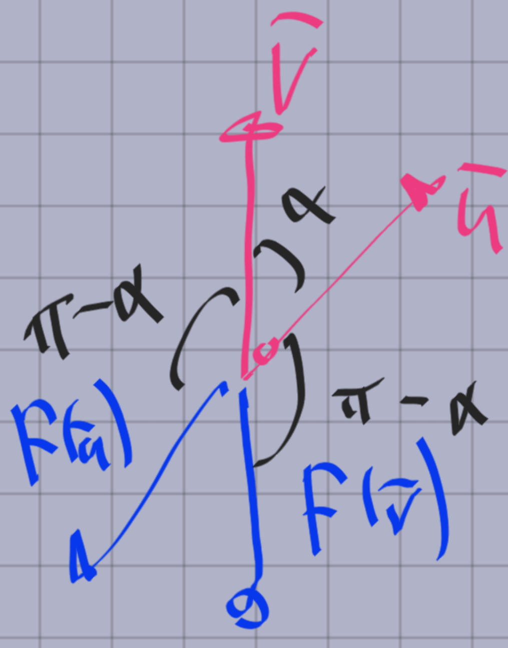
**Def** $F : V \to V$ symm

om $(\bar{u} \mid F(\bar{v})) = (F(\bar{u}) \mid \bar{v})$

alla $\bar{u}, \bar{v}$.

**Ex 1**

vridning 180°

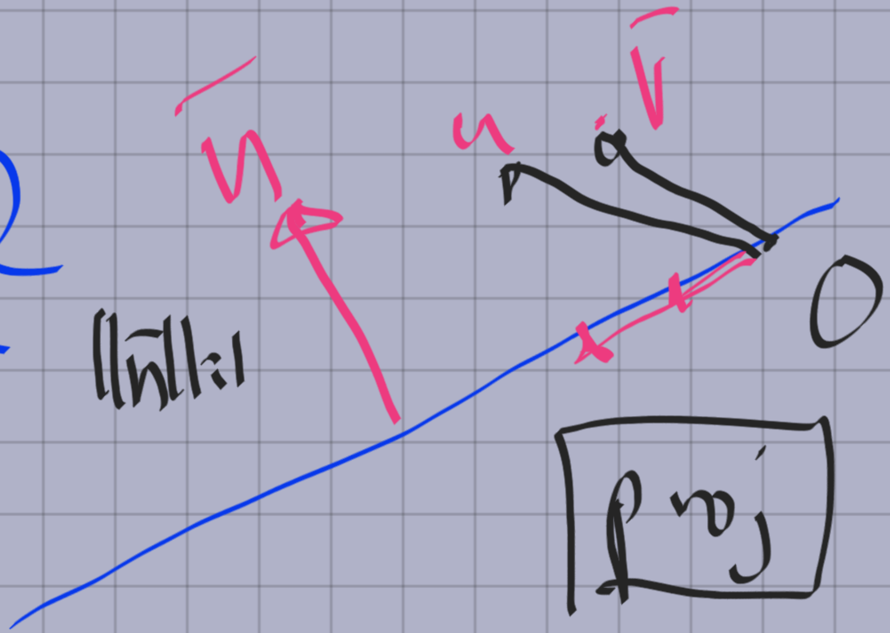



$$\bar{u} \circ F(\bar{v}) = \|\bar{u}\| \, \|F(\bar{v})\| \cdot \cos(\pi - \alpha)$$
$$= \|\bar{u}\| \, \|\bar{v}\|$$

$$\bar{v} \circ F(\bar{u}) = \|\bar{v}\| \, \|F(\bar{u})\| \cos(\pi - \alpha)$$
$$= \|\bar{v}\| \, \|\bar{u}\| \cos(\pi - \alpha)$$

## Ex 2

$\|\bar{n}\| = 1$




$$F(\bar{u}) = (\bar{u} \circ \bar{n})\bar{n}$$

$$F(\bar{v}) = (\bar{v} \circ \bar{n})\bar{n}$$

$$\bar{u} \circ F(\bar{v}) = \bar{u} \circ (\bar{v} \circ \bar{n})\bar{n}$$
$$= (\bar{u} \circ \bar{n})(\bar{v} \circ \bar{n})$$

$$F(\bar{u}) \circ \bar{v} = (\bar{u} \circ \bar{n})\bar{n} \circ \bar{v}$$
$$= (\bar{u} \circ \bar{n})(\bar{n} \circ \bar{v})$$

Sats Om $F(\underline{e}X) = \underline{e}AX$

så $F$ sym $\iff A^t = A$.

B] $(F(\bar{u}) \mid \bar{v})$

$= (F(\underline{e}X) \mid \underline{e}Y)$

$= (\underline{e}AX \mid \underline{e}Y)$

$= (AX)^t Y = X^t A^t Y$

$\bar{A}$ andra sidan

$$(\bar{u} \mid f(\bar{v})) = (\underline{e}X \mid \underline{e}AY)$$

$$= X^t A Y$$

si $X^t A^t Y = X^t A Y$

all $X, Y$

si $A^t = A$.

# Ex Diagonalmatr (skalning

av spegling / vrid 180°

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -5 \end{pmatrix}$$

Symm

Sats (bevis kommer senare här)

Om $F: V \to V$ symm

så finns ON-bas $\underline{f}$

s.a. $F(\underline{f}Y) = \underline{f}DY$

$D$ diagonalmat.

**Ex** $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ symm!

$F(\underline{e}X) = \underline{e} AX$ är vad?

$$A^2 = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = 2A$$

Sätt $M = \frac{1}{2} A$

$$M^2 = \frac{1}{4} A^2 = \frac{1}{4} \cdot 2A = \frac{1}{2} A = M$$

$$M = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

$MX = 0 \iff X = t\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

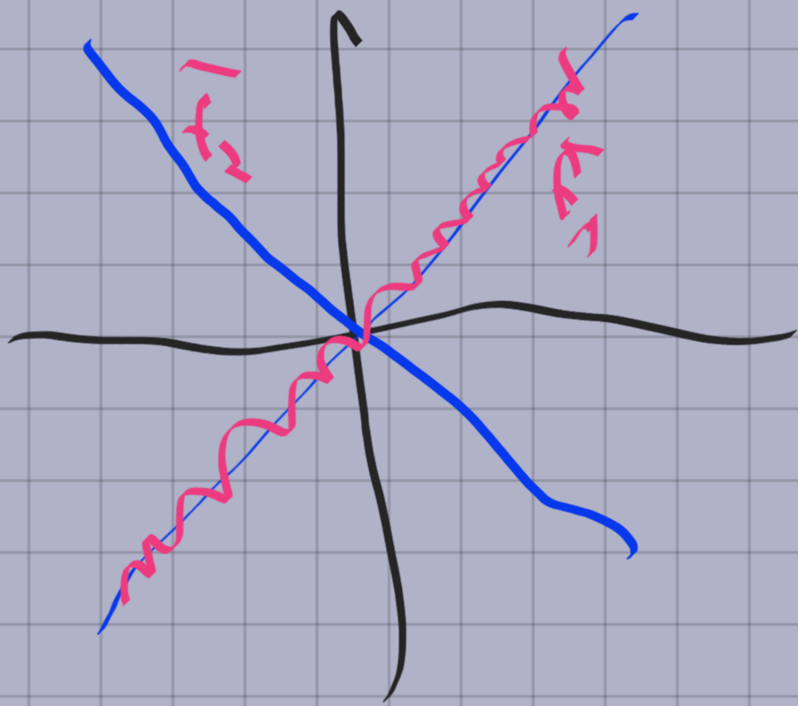
$MX = X$

$$\Longleftrightarrow MX - X = 0$$

$$(M - I)X = 0$$

$$\left(\begin{array}{cc|c} -1/2 & 1/2 & 0 \\ 1/2 & -1/2 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$X = e \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$




$G(\bar{F}_1) = \bar{F}_1$

$G(\bar{F}_2) = 0$

Ort proj ne $[\bar{F}_1]$

$A$ = Ort. proj. ni $[\bar{F}_1]$ [illegible] on uniform sampling

and filter 2.