Exempel, Föreläsning 18.

Determinanter: det för $n \times n$-matriser till reella tal.

$\det A \neq 0 \Leftrightarrow A$ är inverterbar.

Ex: Beräkna

$$\det\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 2 & 0 & 2 \end{pmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 2 & 0 & 2 \end{vmatrix}$$

Lösning:

Metod 1:

$$\begin{vmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 2 & 0 & 2 \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & 3 \\ 0 & 2 \end{vmatrix} - 2 \begin{vmatrix} 0 & 3 \\ 2 & 2 \end{vmatrix} + 1 \cdot \begin{vmatrix} 0 & 1 \\ 2 & 0 \end{vmatrix} =$$

$$= 1 \cdot (1 \cdot 2 - 3 \cdot 0) - 2(0 \cdot 2 - 3 \cdot 2) + 1 \cdot (0 \cdot 0 - 1 \cdot 2) =$$

$$= 2 + 12 - 2 = \underline{\underline{12}}.$$

Metod 2:

$$\begin{vmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 2 & 0 & 2 \end{vmatrix} \begin{matrix} \\ \\ \leftarrow (1/2) \end{matrix} = 2 \begin{vmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 1 & 0 & 1 \end{vmatrix} \begin{matrix} \leftarrow \\ \\ (-1) \end{matrix} = 2 \begin{vmatrix} 0 & 2 & 0 \\ 0 & 1 & 3 \\ 1 & 0 & 1 \end{vmatrix} \begin{matrix} \leftarrow (1/2) \\ \\ \\ \end{matrix} =$$

$$= 4 \begin{vmatrix} 0 & 1 & 0 \\ 0 & 1 & 3 \\ 1 & 0 & 1 \end{vmatrix} \begin{matrix} (-1) \\ \leftarrow \\ \\ \end{matrix} = 4 \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 3 \\ 1 & 0 & 1 \end{vmatrix} \begin{matrix} \\ \leftarrow (1/3) \\ \\ \end{matrix} = 12 \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} \updownarrow$$

$$= -12 \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} \updownarrow = 12 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \underline{\underline{12}}.$$

Bas, koordinater:

Kom ihåg att en bas $\underline{u} = (\bar{u}_1\ \bar{u}_2 \cdots \bar{u}_n)$ till vektorrum $\mathbb{U}$ är en uppsättning vektorer sådana att varje vektor $\bar{u} \in \mathbb{U}$ på entydigt sätt kan skrivas p.f.

$$\bar{u} = x_1\bar{u}_1 + x_2\bar{u}_2 + \dots + x_n\bar{u}_n = \underline{u}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \underline{u}X .$$

När vi behöver indikera basen skriver vi t.ex. X_{u}.

<u>Skalärprodukt och bas:</u> Om E Euklidiskt rum och $\underline{e} = (\bar{e}_1\ \bar{e}_2 \dots \bar{e}_n)$ ON bas, d.v.s.

$$(\bar{e}_i | \bar{e}_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} ,$$

och $\bar{u} = \underline{e}X$, $\bar{v} = \underline{e}Y$ då gäller att

$(\bar{u}|\bar{v}) = X^t Y$. T.ex. Om $n=2$ $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

$$\left(\underline{e}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \middle| \underline{e}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) = (x_1\bar{e}_1 + x_2\bar{e}_2 | y_1\bar{e}_1 + y_2\bar{e}_2) =$$

$$= \underbrace{(x_1\bar{e}_1 | y_1\bar{e}_1)}_{= x_1 y_1} + \underbrace{(x_1\bar{e}_1 | y_2\bar{e}_2)}_{= 0} + \underbrace{(x_2\bar{e}_2 | y_1\bar{e}_1)}_{= 0} + \underbrace{(x_2\bar{e}_2 | y_2\bar{e}_2)}_{= x_2 y_2}$$

$$= x_1 y_1 + x_2 y_2 = (x_1\ \ x_2)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = X^t Y.$$

$F: \mathbb{U} \to \mathbb{V}$ kallas linjär om

- $F(\bar{u}_1 + \bar{u}_2) = F(\bar{u}_1) + F(\bar{u}_2)$
- $F(\lambda \bar{u}) = \lambda F(\bar{u})$.

Om vi valt baser $\underline{u}$, $\underline{v}$ så finns unik matris A sådan att

$$F(\underline{u}X) = \underline{v}AX \quad .$$

Ex: Låt $F: \mathbb{R}^3 \to \mathbb{R}^2$ via

$$F(x_1, x_2, x_3) = (2x_1 + x_3, x_2 - 4x_1).$$

a) Visa att F är linjär

b) Bestäm F:s matris i standardbaserna

c) Låt $\underline{u} = ((2,0,0)\ (0,3,0)\ (0,1,2))$

$\underline{v} = ((1,2)\ (2,0))$ vara baser till $\mathbb{R}^3$ resp. $\mathbb{R}^2$

Bestäm F:s matris relativt dessa baser

Lösning:

a) $F(x_1+y_1, x_2+y_2, x_3+y_3) = (2(x_1+y_1) + (x_3+y_3), (x_2+y_2) - 4(x_1+y_1)) =$

$= (2x_1 + x_3, x_2 - 4x_1) + (2y_1 + y_3, y_2 - 4y_1) =$

$= F(x_1, x_2, x_3) + F(y_1, y_2, y_3)$

$F(\lambda x_1, \lambda x_2, \lambda x_3) = (2(\lambda x_1) + (\lambda x_3), (\lambda x_2) - 4(\lambda x_1)) =$

$= \lambda(2x_1 + x_3, x_2 - 4x_1) = \lambda F(x_1, x_2, x_3)$.

b) $$F(x_1, x_2, x_3) = F\left(\underline{e}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = (2x_1 + x_3, x_2 - 4x_1) =$$

$$= \underline{e}\begin{pmatrix} 2x_1 + x_3 \\ x_2 - 4x_1 \end{pmatrix} = \underline{e}\underbrace{\begin{pmatrix} 2 & 0 & 1 \\ -4 & 1 & 0 \end{pmatrix}}_{A}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

(OBS! $F(1,0,0) = (2,-4)$, $F(0,1,0) = (0,1)$, $F(0,0,1) = (1,0)$.)

c)

$$F(2,0,0) = (4,-8) = \underline{e}\begin{pmatrix}4\\-8\end{pmatrix} = \underline{v}\begin{pmatrix}a_1\\b_1\end{pmatrix}$$

$$F(0,3,0) = (0,3) = \underline{e}\begin{pmatrix}0\\3\end{pmatrix} = \underline{v}\begin{pmatrix}a_2\\b_2\end{pmatrix}$$

$$F(0,1,2) = (2,1) = \underline{e}\begin{pmatrix}2\\1\end{pmatrix} = \underline{v}\begin{pmatrix}a_3\\b_3\end{pmatrix} \qquad \left(A = \begin{pmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{pmatrix}\right).$$

$$a_i\bar{V}_1 + b_i\bar{V}_2 = a_i(1,2) + b_i(2,0) = \underline{e}\begin{pmatrix}1 & 2\\ 2 & 0\end{pmatrix}\begin{pmatrix}a_i\\ b_i\end{pmatrix}$$

$$\left(\begin{array}{cc|ccc}1 & 2 & 4 & 0 & 2\\ 2 & 0 & -8 & 3 & 1\end{array}\right) \leftarrow \textcircled{$\tfrac{1}{2}$} \sim \left(\begin{array}{cc|ccc}1 & 2 & 4 & 0 & 2\\ 1 & 0 & -4 & 3/2 & 1/2\end{array}\right) \textcircled{-1}$$

$$\sim \left(\begin{array}{cc|ccc}0 & 2 & 8 & -3/2 & 3/2\\ 1 & 0 & -4 & 3/2 & 1/2\end{array}\right) \leftarrow \textcircled{$1/2$} \circlearrowright \sim \left(\begin{array}{cc|ccc}1 & 0 & -4 & 3/2 & 1/2\\ 0 & 1 & 4 & -3/4 & 3/4\end{array}\right)$$

$$A = \begin{pmatrix}-4 & 3/2 & 1/2\\ 4 & -3/4 & 3/4\end{pmatrix}$$

<u>Test:</u>

$$\underline{v}\cdot\begin{pmatrix}-4\\4\end{pmatrix} = -4\bar{V}_1 + 4\bar{V}_2 = -4(1,2) + 4(2,0) = (4,-8) \quad \text{o.k.}$$

$$\underline{v}\begin{pmatrix}3/2\\-3/4\end{pmatrix} = \frac{3}{2}(1,2) - \frac{3}{4}(2,0) = (0,3) \quad \text{o.k.}$$

$$\underline{v}\begin{pmatrix}1/2\\3/4\end{pmatrix} = \frac{1}{2}(1,2) + \frac{3}{4}(2,0) = (2,1) \quad \text{o.k.}$$

$F: \mathbb{U} \to \mathbb{V}$ linjär, $\dim \mathbb{U} = n$.

$N(F) = \{\bar{u} \in \mathbb{U} : F(\bar{u}) = \bar{0}\} \subset \mathbb{U}$ nollrummet till F

$V(F) = \{F(\bar{u}) : \bar{u} \in \mathbb{U}\} \subset \mathbb{V}$ värderum

$$\boxed{\dim N(F) + \dim V(F) = n}$$

Ex: Låt $F: \mathbb{R}^3 \to \mathbb{R}^3$ som i std.basen har matris

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 4 & 0 \\ 1 & 2 & 0 \end{pmatrix}$$

Bestäm baser till $N(F)$, $V(F)$ samt $N(F) \cap V(F)$.

Lösning:

$$F(x,y,z) = \underline{e} \begin{pmatrix} 1 & 0 & 2 \\ 2 & 4 & 0 \\ 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \underline{e} \left(x \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + y \begin{pmatrix} 0 \\ 4 \\ 2 \end{pmatrix} + z \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \right)$$

Så $V(F)$ "=" kolonnrummet till A, eller mer precis

$$V(F) = [(1,2,1), (0,4,2), (2,0,0)].$$

$$\left(\begin{array}{ccc|cc} 1 & 0 & 2 & 0 & x \\ 2 & 4 & 0 & 0 & y \\ 1 & 2 & 0 & 0 & z \end{array}\right) \overset{(-2)}{} \updownarrow \sim \left(\begin{array}{ccc|cc} 1 & 0 & 2 & 0 & x \\ 1 & 2 & 0 & 0 & z \\ 0 & 0 & 0 & 0 & y-2z \end{array}\right) \overset{(-1)}{} \sim \left(\begin{array}{ccc|cc} 1 & 0 & 2 & 0 & x \\ 0 & 2 & -2 & 0 & z-x \\ 0 & 0 & 0 & 0 & y-2z \end{array}\right)$$

$V(F) = [\underbrace{(1,2,1), (0,4,2)}_{bas}] = \{(x,y,z) : y - 2z = 0\}$

Lösning till $A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$: $\begin{cases} x = -2t \\ y = t \\ z = t \end{cases} \iff (x,y,z) = t(-2,1,1)$

Så $N(F) = [\underbrace{(-2,1,1)}_{bas}]$

$N(F) \cap V(F) = \{t(-2,1,1) : t - 2t = 0\} = \{(0,0,0)\}$. (bas $= \emptyset$).

Basbyten :

Ex: Låt $\underline{u} = (\bar{u}_1 \ \bar{u}_2 \ \bar{u}_3) = ((1,0,1) \ \ (0,1,1) \ \ (0,0,1))$

$\underline{v} = (\bar{v}_1 \ \bar{v}_2 \ \bar{v}_3) = ((0,1,0) \ \ (1,0,0) \ \ (0,0,1))$

vara base till $\mathbb{R}^3$.

a) Bestäm transformationsmatrisen från $\underline{u}$ till $\underline{v}$-basen, d.v.s den unika 3x3-matris T s.a

$$\underline{u}X_{\underline{u}} = \underline{v}X_{\underline{v}} \iff X_{\underline{v}} = TX_{\underline{u}}.$$

b) Bestäm transformationsmatrisen från $\underline{v}$ till $\underline{u}$-basen.

c) Låt $\bar{W} = \underline{u}\begin{pmatrix}1\\3\\2\end{pmatrix} = \underline{u}X_{\underline{u}}$ bestäm koordinaten $x_{\underline{v}}$ för $\bar{W}$ i $\underline{v}$-basen.

Lösning:

a) Vi vill alltså att $\underline{v}X_{\underline{v}} = \underline{v}TX_{\underline{u}} = \underline{u}X_{\underline{u}}$

D.v.s $\underline{u} = \underline{v}T$. Så vi har att

$$T = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$$ ska uppfylla att

$$(\bar{u}_1 \ \bar{u}_2 \ \bar{u}_3) = (\bar{v}_1 \ \bar{v}_2 \ \bar{v}_3)\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} =$$

$$= ((a_1\bar{v}_1 + a_2\bar{v}_2 + a_3\bar{v}_3) \ \ (b_1\bar{v}_1 + b_2\bar{v}_2 + b_3\bar{v}_3) \ \ (c_1\bar{v}_1 + c_2\bar{v}_2 + c_3\bar{v}_3))$$

$$\iff \begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}, \quad \begin{pmatrix}0\\1\\1\end{pmatrix} = \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}, \quad \begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 0&1&0&1&0&0\\ 1&0&0&0&1&0\\ 0&0&1&1&1&1 \end{array}\right) \updownarrow \sim \left(\begin{array}{ccc|ccc} 1&0&0&0&1&0\\ 0&1&0&1&0&0\\ 0&0&1&1&1&1 \end{array}\right)$$

$$\underbrace{\begin{matrix}0&1&0\\1&0&0\\1&1&1\end{matrix}}_{T}$$

SVAR: $T = \begin{pmatrix}0&1&0\\1&0&0\\1&1&1\end{pmatrix}$

b) $X_{\underline{v}} = TX_{\underline{u}} \iff T^{-1}X_{\underline{v}} = X_{\underline{u}}$

$$\left(\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \begin{array}{c} (-1) \\ (-1) \\ \end{array} \sim \left(\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right) \updownarrow \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \underbrace{-1 \quad -1 \quad 1}_{T^{-1}} \end{array}\right)$$

SVAR: $T^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & -1 & 1 \end{pmatrix}$

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c)

$$X_{\underline{v}} = TX_{\underline{u}} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 6 \end{pmatrix}$$

$$\left(\begin{array}{ll} \text{TEST:} & \underline{u}\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = 1\cdot(1,0,1) + 3(0,1,1) + 2(0,0,1) = (1,3,6) \\ & \underline{v}\begin{pmatrix} 3 \\ 1 \\ 6 \end{pmatrix} = 3\cdot(0,1,0) + 1\cdot(1,0,0) + 6(0,0,1) = (1,3,6) \quad \text{ok} \end{array}\right)$$

<u>Ex:</u> Låt planet π i $\mathbb{R}^3$ vara givet av $3x_1 + 4x_3 = 0$.

a) Välj en ny ON-bas $\underline{f}$ till $\mathbb{R}^3$ sådan att i denna bas ges π av $y_1 = 0$.

b) Låt F vara spegling genom π, bestäm F:s matris i $\underline{f}$-basen.

c) Bestäm F:s matris i standardbasen $\underline{e}$.

Lösning: a) Normal till π: $\frac{1}{5}(3,0,4)$

Vi vill alltså ha $y_1=(3x_1+4x_3)\cdot c$ $\quad c\neq 0$.

D.v.s. med $\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix} | & | & | \\ Y_1 & Y_2 & Y_3 \\ | & | & | \end{pmatrix}\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}$

$\underline{e}X=\underline{f}Y=\underline{e}TY$

$\underline{f}=\underline{e}T$

$X=TY$

där $\bar{f}_i=\underline{e}Y_i$

π ges av $0=\frac{1}{5}(3\ \ 0\ \ 4)\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\frac{1}{5}(3\ \ 0\ \ 4)\begin{pmatrix} | & | & | \\ Y_1 & Y_2 & Y_3 \\ | & | & | \end{pmatrix}\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}$

$=\Big/$ om $\frac{1}{5}(3\ 0\ 4)Y_2=\frac{1}{5}(3\ 0\ 4)Y_3=0,\ \frac{1}{5}(3\ 0\ 4)Y_1=1\Big/$

$=\begin{pmatrix}1 & 0 & 0\end{pmatrix}\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}=y_1$.

Vi väljer alltså $\bar{f}_1=\frac{1}{5}(3,0,4)$

Välj $\bar{f}_2$ "i" planet π: $\frac{1}{5}(4,0,-3)$

$\bar{f}_3$ "i" planet π samt ort mot $\bar{f}_2$, t.ex. $\bar{f}_3=(0,1,0)$.

Så $\underline{\underline{\underline{f}=\left(\frac{1}{5}(3,0,4)\ \ \frac{1}{5}(4,0,-3)\ \ (0,1,0)\right).}}$

b)

$F(\bar{f}_1)=-\bar{f}_1=\underline{f}\begin{pmatrix}-1\\0\\0\end{pmatrix}\qquad F(\bar{f}_2)=\bar{f}_2=\underline{f}\begin{pmatrix}0\\1\\0\end{pmatrix}\qquad F(\bar{f}_3)=\bar{f}_3=\underline{f}\begin{pmatrix}0\\0\\1\end{pmatrix}$

Matris $A_f=\begin{pmatrix}-1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}$

c) Transformationsmatris från $\underline{f}$ till $\underline{e}$-basen:

$$T=\frac{1}{5}\begin{pmatrix}3 & 4 & 0\\0 & 0 & 5\\4 & -3 & 0\end{pmatrix}$$

T är en ON-matris (eftersom $\underline{e}$, $\underline{f}$ är ON-baser) så

$$T^{-1}=T^t=\frac{1}{5}\begin{pmatrix}3 & 0 & 4\\4 & 0 & -3\\0 & 5 & 0\end{pmatrix}$$

$$F(\underline{f}X_f) = \underline{f}A_f X_f = \Big/ \; \underline{f} = \underline{e}T, \quad \underline{f}X_f = \underline{e}X_e = \underline{f}T^{-1}X_e \quad \text{så} \quad X_f = T^{-1}X_e \; \Big/$$

$$= \underline{e}TA_fT^{-1}X_e \quad .$$

D.v.s.

$$A_e = TA_fT^{-1} = \frac{1}{5}\begin{pmatrix} 3 & 4 & 0 \\ 0 & 0 & 5 \\ 4 & -3 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\frac{1}{5}\begin{pmatrix} 3 & 0 & 4 \\ 4 & 0 & -3 \\ 0 & 5 & 0 \end{pmatrix} =$$

$$= \frac{1}{25}\begin{pmatrix} 3 & 4 & 0 \\ 0 & 0 & 5 \\ 4 & -3 & 0 \end{pmatrix}\begin{pmatrix} -3 & 0 & -4 \\ 4 & 0 & -3 \\ 0 & 5 & 0 \end{pmatrix} = \frac{1}{25}\begin{pmatrix} 7 & 0 & -24 \\ 0 & 25 & 0 \\ -24 & 0 & -7 \end{pmatrix}$$

(Notera att A_e är en ON-matris, d.v.s.

$A_e^t A_e = I$).