Föreläsning 21

Andragradskurvor:

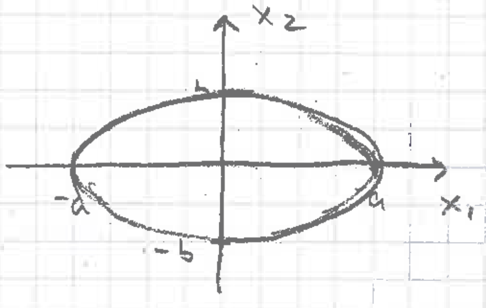
Om $g(x_1, x_2) = a_1 x_1 + a_2 x_2 + a_{11} x_1^2 + a_{12} x_1 x_2 + a_{22} x_2^2$

så kallas

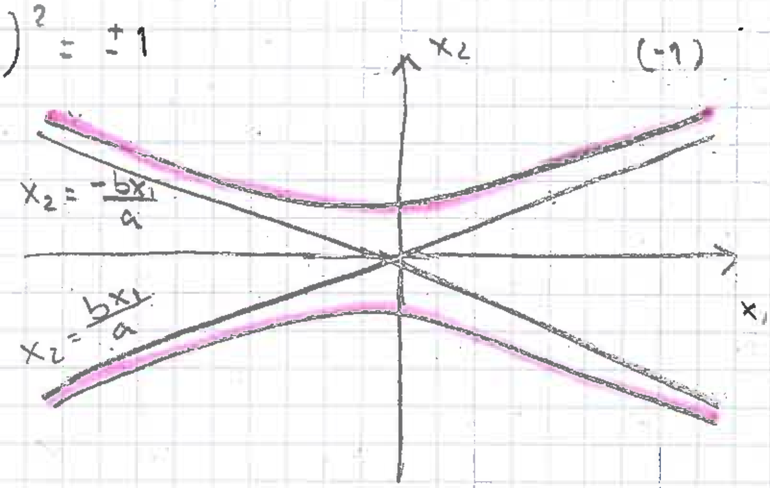
$"g(x_1, x_2) = C" = \{(x_1, x_2) : g(x_1, x_2) = C\}$ en andragradskurva.

Vi har tre huvudtyper med x_1, x_2 som symmetriaxlar

Ellips: $\left(\frac{x_1}{a}\right)^2 + \left(\frac{x_2}{b}\right)^2 = 1$



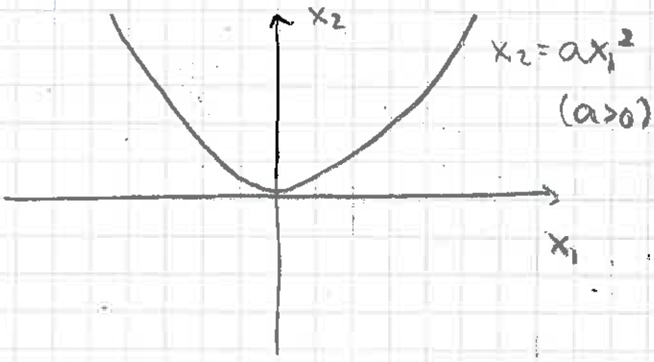
Hyperbel: $\left(\frac{x_1}{a}\right)^2 - \left(\frac{x_2}{b}\right)^2 = \pm 1$



$$\left(\frac{x_2}{b}\right)^2 = 1 + \left(\frac{x_1}{a}\right)^2 \iff x_2 = \pm b\sqrt{1 + \left(\frac{x_1}{a}\right)^2}$$

Notera att $\approx \pm b\sqrt{\left(\frac{x_1}{a}\right)^2} = \pm b\frac{x_1}{a}$ för stora värden på $\left(\frac{x_1}{a}\right)^2$.

Parabel: $x_2 = a x_1^2$ eller $x_1 = a x_2^2$



Man kan även få

- tomma mängden : $\emptyset = \{(x_1, x_2) : x_1^2 + x_2^2 = -1\}$
- en punkt : $\{(0,0)\} = \{(x_1, x_2) : x_1^2 + x_2^2 = 0\}$
- två räta linjer : $\{(x_1, x_2) : x_2 = \pm x_1\} = \{(x_1, x_2) : x_1^2 - x_2^2 = 0\}$.

Ex 1: Avgör vilken typ av kurva

$$g(x_1, x_2) = 4x_1^2 + 2\sqrt{2}\, x_1 x_2 + 3x_2^2 = 1$$

utgör, samt ange dess symmetriaxlar.

Lösning:

$$g(x_1,x_2) = (x_1 \;\; x_2)\begin{pmatrix} 4 & \sqrt{2} \\ \sqrt{2} & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{vmatrix} 4-\lambda & \sqrt{2} \\ \sqrt{2} & 3-\lambda \end{vmatrix} = (4-\lambda)(3-\lambda) - 2 = \lambda^2 - 7\lambda + 10 = 0$$

$$\Leftrightarrow \quad \lambda = \frac{7}{2} \pm \sqrt{\frac{49}{4} - 10} = \frac{7}{2} \pm \frac{3}{2}$$

$$\lambda_1 = 5, \quad \lambda_2 = 2.$$

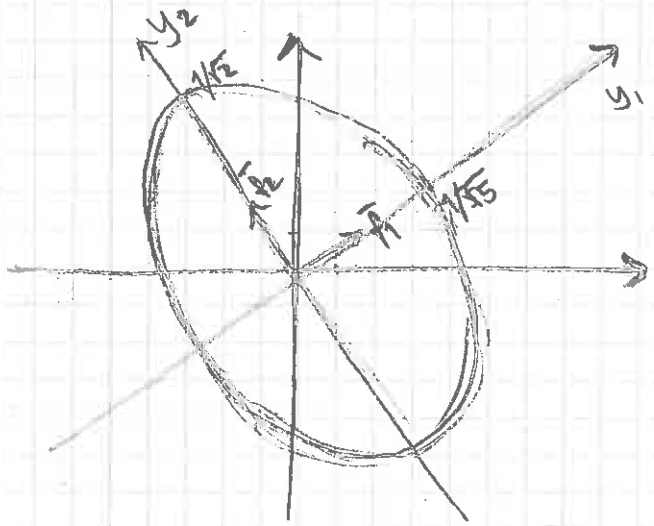
$$\lambda_1 = 5: \quad \left(\begin{array}{cc|c} 4-5 & \sqrt{2} & 0 \\ \sqrt{2} & 3-5 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} -1 & \sqrt{2} & 0 \\ \sqrt{2} & -2 & 0 \end{array}\right) \sqrt{2} \sim \left(\begin{array}{cc|c} -1 & \sqrt{2} & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$\begin{cases} x_1 = \sqrt{2}\,t \\ x_2 = t \end{cases} \qquad \bar{f}_1 = \frac{1}{\sqrt{3}}(\sqrt{2}, 1).$$

$$\lambda_2 = 2: \quad \left(\begin{array}{cc|c} 4-2 & \sqrt{2} & 0 \\ \sqrt{2} & 3-2 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 2 & \sqrt{2} & 0 \\ \sqrt{2} & 1 & 0 \end{array}\right) -\sqrt{2} \sim \left(\begin{array}{cc|c} 0 & 0 & 0 \\ \sqrt{2} & 1 & 0 \end{array}\right)$$

$$\begin{cases} x_1 = -t \\ x_2 = +\sqrt{2}\,t \end{cases} \qquad \bar{f}_2 = \frac{1}{\sqrt{3}}(-1, +\sqrt{2})$$

$$g\left(e\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = g\left(f\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) = (y_1 \;\; y_2)\begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} =$$



$$= 5y_1^2 + 2y_2^2 = 1$$

Max/min till $5y_1^2 + 2y_2^2$ på enhetscirkeln ges av

max $\;5\;$ i $\;y_1 = \pm 1 \quad \left((x_1,x_2) = \pm\frac{1}{\sqrt{3}}(\sqrt{2},1)\right)$

min $\;2\;$ i $\;y_2 = \pm 1. \quad \left((x_1,x_2) = \pm\frac{1}{\sqrt{3}}(-1,\sqrt{2})\right)$

Andragradsytor:

$$g(x_1,x_2,x_3) = (x_1\ x_2\ x_3)\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + (a_1\ a_2\ a_3)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = C$$

kallas andragradsyta.

SE KURSBOKEN FÖR DE 9 HUVUDTYPERNA.

Ex 2:

$$g(x_1,x_2,x_3) = 4x_1^2 + 2\sqrt{2}\,x_1x_2 + 3x_2^2 + x_3^2 = 1$$

$$g(x_1,x_2,x_3) = (x_1\ x_2\ x_3)\begin{pmatrix} 4 & \sqrt{2} & 0 \\ \sqrt{2} & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\begin{vmatrix} 4-\lambda & \sqrt{2} & 0 \\ \sqrt{2} & 3-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1-\lambda \\ 4-\lambda & \sqrt{2} & 0 \\ \sqrt{2} & 3-\lambda & 0 \end{vmatrix} = (1-\lambda)\begin{vmatrix} 4-\lambda & \sqrt{2} \\ \sqrt{2} & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad \lambda_1 = 5,\ \lambda_2 = 2,\ \lambda_3 = 1$$

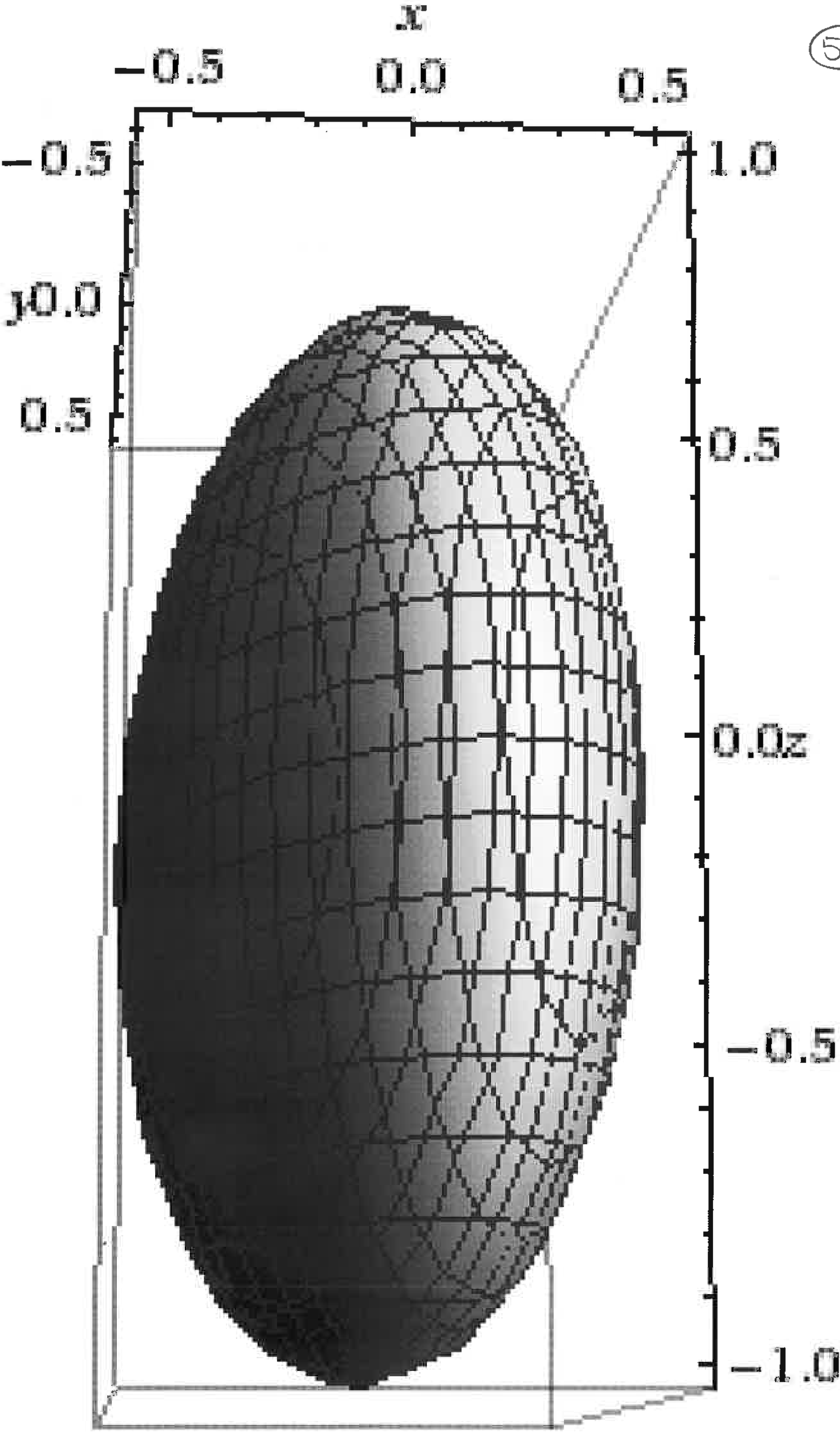
$$\lambda_1 = 5: \quad \left(\begin{array}{ccc|c} -1 & \sqrt{2} & 0 & 0 \\ \sqrt{2} & -2 & 0 & 0 \\ 0 & 0 & -4 & 0 \end{array}\right) \Leftrightarrow \begin{cases} x_1 = \sqrt{2}\,t \\ x_2 = t \\ x_3 = 0 \end{cases} \qquad \bar{f}_1 = \frac{1}{\sqrt{3}}(\sqrt{2},1,0)$$

$$\lambda_2 = 2: \quad \left(\begin{array}{ccc|c} 2 & \sqrt{2} & 0 & 0 \\ \sqrt{2} & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right) \Leftrightarrow \begin{cases} x_1 = -t \\ x_2 = \sqrt{2}\,t \\ x_3 = 0 \end{cases} \qquad \bar{f}_2 = \frac{1}{\sqrt{3}}(-1,\sqrt{2},0)$$

$$\lambda_3 = 1: \quad \left(\begin{array}{ccc|c} 3 & \sqrt{2} & 0 & 0 \\ \sqrt{2} & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Leftrightarrow \begin{cases} x_1 = 0 \\ x_2 = 0 \\ x_3 = t \end{cases} \qquad \bar{f}_3 = (0,0,1)$$

$$g\left(\underline{e}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = g\left(\underline{f}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}\right) = 5y_1^2 + 2y_2^2 + y_3^2 = 1$$

SVAR: Ytan är en ellipsoid med huvudaxlar $\bar{f}_1, \bar{f}_2, \bar{f}_3$.


x
−0.5
0.0
0.5
−0.5
y0.0
0.5
1.0
0.5
0.0z
−0.5
−1.0

Ex 3:

$$X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}.$$

Med $A = \begin{pmatrix} 4 & \sqrt{2} \\ \sqrt{2} & 3 \end{pmatrix}$ gäller

$$X'(t) = AX(t).$$

Enligt ex 1 har A egenvärden / egenvektorer

$$\lambda_1 = 5, \quad X_1 = \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}, \quad \lambda_2 = 2, \quad X_2 = \begin{pmatrix} -1 \\ \sqrt{2} \end{pmatrix}$$

Så

$$X(t) = C_1 X_1 e^{\lambda_1 t} + C_2 X_2 e^{\lambda_2 t}:$$

SVAR: $\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = C_1 \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix} e^{5t} + C_2 \begin{pmatrix} -1 \\ \sqrt{2} \end{pmatrix} e^{2t}$.

- I dessa problem lönar det sig sällan att normera vektorerna då detta bara motsvarar multiplikation av C_i:a.

- Test:

$$\left(C_1 \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix} e^{5t} + C_2 \begin{pmatrix} -1 \\ \sqrt{2} \end{pmatrix} e^{2t} \right)' =$$

$$= C_1 5 \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix} e^{5t} + C_2 2 \begin{pmatrix} -1 \\ \sqrt{2} \end{pmatrix} e^{2t}$$

$$\begin{pmatrix} 4 & \sqrt{2} \\ \sqrt{2} & 3 \end{pmatrix} \left(C_1 \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix} e^{5t} + C_2 \begin{pmatrix} -1 \\ \sqrt{2} \end{pmatrix} e^{2t} \right) =$$

$$= C_1 \begin{pmatrix} 4 & \sqrt{2} \\ \sqrt{2} & 3 \end{pmatrix} \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix} e^{5t} + C_2 \begin{pmatrix} 4 & \sqrt{2} \\ \sqrt{2} & 3 \end{pmatrix} \begin{pmatrix} -1 \\ \sqrt{2} \end{pmatrix} e^{2t}$$

$$= C_1 5 \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix} e^{5t} + C_2 2 \begin{pmatrix} -1 \\ \sqrt{2} \end{pmatrix} e^{2t}.$$

Ex 4: Lös systemet av differensekvationer

$$\begin{cases} a_n = 4a_{n-1} + \sqrt{2}\, b_{n-1} \\ b_n = \sqrt{2}\, a_{n-1} + 3b_{n-1} \end{cases} \quad \text{med} \quad \begin{cases} a_0 = 1 \\ b_0 = 0 \end{cases}.$$

Lösning:

$$\underbrace{\begin{pmatrix} a_n \\ b_n \end{pmatrix}}_{X_n} = \underbrace{\begin{pmatrix} 4 & \sqrt{2} \\ \sqrt{2} & 3 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} a_{n-1} \\ b_{n-1} \end{pmatrix}}_{X_{n-1}}$$

$$X_n = AX_{n-1} = A^2 X_{n-2} = \dots A^n X_0 = A^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Vi vill alltså hitta en formel för A^n.

Enligt ex 1 gäller

$$A = TDT^t$$

där

$$T = \frac{1}{\sqrt{3}} \begin{pmatrix} \sqrt{2} & -1 \\ 1 & \sqrt{2} \end{pmatrix}, \quad D = \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix}.$$

Så

$$A^n = TD^nT^t = T \begin{pmatrix} 5^n & 0 \\ 0 & 2^n \end{pmatrix} T^t.$$

Alltså gäller

$$\begin{pmatrix} a_n \\ b_n \end{pmatrix} = \frac{1}{3} \begin{pmatrix} \sqrt{2} & -1 \\ 1 & \sqrt{2} \end{pmatrix} \begin{pmatrix} 5^n & 0 \\ 0 & 2^n \end{pmatrix} \begin{pmatrix} \sqrt{2} & 1 \\ -1 & \sqrt{2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \frac{1}{3} \begin{pmatrix} \sqrt{2} & -1 \\ 1 & \sqrt{2} \end{pmatrix} \begin{pmatrix} 5^n & 0 \\ 0 & 2^n \end{pmatrix} \begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} \sqrt{2} & -1 \\ 1 & \sqrt{2} \end{pmatrix} \begin{pmatrix} \sqrt{2} \cdot 5^n \\ -2^n \end{pmatrix} =$$

$$= \frac{1}{3} \begin{pmatrix} 2 \cdot 5^n + 2^n \\ \sqrt{2}\, 5^n - \sqrt{2}\, 2^n \end{pmatrix}$$

D.v.s

$$\begin{cases} a_n = \dfrac{2 \cdot 5^n + 2^n}{3} \\ b_n = \dfrac{\sqrt{2}\, 5^n - \sqrt{2}\, 2^n}{3} \end{cases}$$