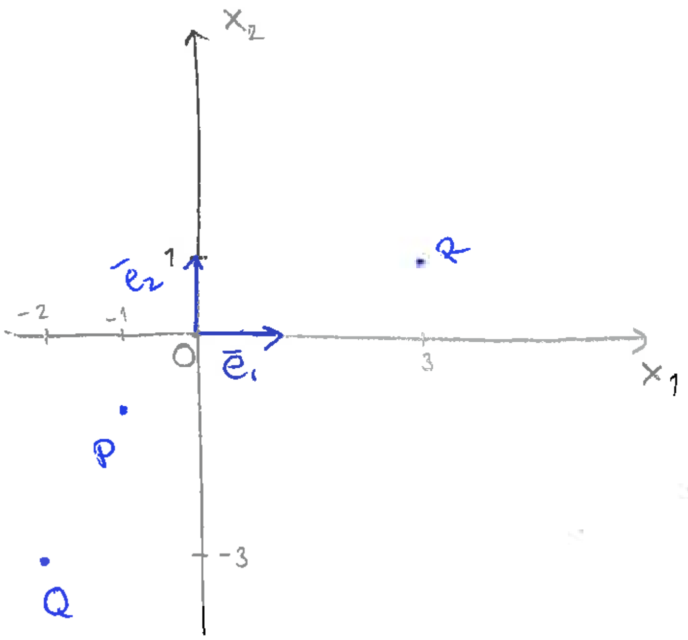
Exempel m.m. Föreläsning 3

Repetition: I $\mathbb{R}^2$ låter vi $\bar{e}_1 = (1,0)$, $\bar{e}_2 = (0,1)$ beteckna standardbasen och $\underline{e} = (\bar{e}_1\ \bar{e}_2)$.

Låt $P = (-1,-1)$, $Q = (-2,-3)$ och $R = (3,1)$.

Kom ihåg att $\overrightarrow{PQ}$ betecknar den riktade sträckan mellan P och Q, och $[\overrightarrow{PQ}]$ motsvarande vektor. $|[\overrightarrow{PQ}]|$ betecknar längden av $[\overrightarrow{PQ}]$.



Vi har nu $[\overrightarrow{OP}] = \underline{e}\begin{pmatrix}-1\\-1\end{pmatrix}$, $[\overrightarrow{OQ}] = \underline{e}\begin{pmatrix}-2\\-3\end{pmatrix}$, $[\overrightarrow{OR}] = \underline{e}\begin{pmatrix}3\\1\end{pmatrix}$

$$[\overrightarrow{PQ}] = [\overrightarrow{OQ}] - [\overrightarrow{OP}] = \underline{e}\begin{pmatrix}-2\\-3\end{pmatrix} - \underline{e}\begin{pmatrix}-1\\-1\end{pmatrix} = \underline{e}\begin{pmatrix}-1\\-2\end{pmatrix}$$

$$[\overrightarrow{PQ}] + 2[\overrightarrow{OQ}] = \underline{e}\begin{pmatrix}-1\\-2\end{pmatrix} + 2\,\underline{e}\begin{pmatrix}-2\\-3\end{pmatrix} = \underline{e}\begin{pmatrix}-1\\-2\end{pmatrix} + \underline{e}\begin{pmatrix}-4\\-6\end{pmatrix} = \underline{e}\begin{pmatrix}-5\\-8\end{pmatrix}.$$

Lösning (Exempel 1):

$$e\begin{pmatrix}3\\2\\0\end{pmatrix}\cdot e\begin{pmatrix}0\\1\\-4\end{pmatrix} = 3\cdot 0 + 2\cdot 1 + 0\cdot(-4) = \underline{\underline{2}}.$$

Motivering av $\bar{u}\cdot(\lambda\bar{v}) = \lambda(\bar{u}\cdot\bar{v})$:

$$(u_1, u_2, \dots, u_n)\cdot(\lambda(v_1, v_2, \dots, v_n)) = (u_1, u_2, \dots, u_n)\cdot(\lambda v_1, \lambda v_2, \dots, \lambda v_n)$$

$$= u_1(\lambda v_1) + u_2(\lambda v_2) + \dots + u_n(\lambda v_n) = \lambda(u_1v_1 + u_2v_2 + \dots + u_nv_n) =$$

$$= \lambda((u_1, u_2, \dots, u_n)\cdot(v_1, v_2, \dots, v_n)).$$

Schwarz olikhet (2d):

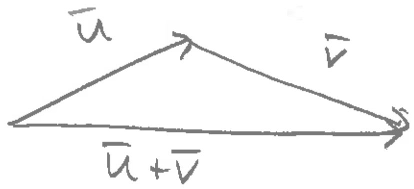
$$|(a,b)\cdot(c,d)| \le |(a,b)||(c,d)| \Leftrightarrow ((a,b)\cdot(c,d))^2 \le |(a,b)|^2|(c,d)|^2$$

$$\Leftrightarrow (ac+bd)^2 \le (a^2+b^2)(c^2+d^2)$$

$$\Leftrightarrow a^2c^2 + 2acbd + b^2d^2 \le a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2$$

$$\Leftrightarrow a^2d^2 + b^2c^2 - 2acbd = (ad-bc)^2 \ge 0.$$

Triangelolikheten:



Lösning (Exempel 2):

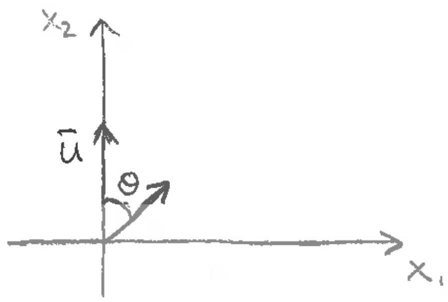
$$\bar{u}\cdot\bar{v} = \underline{e}\begin{pmatrix}0\\2\end{pmatrix}\cdot\underline{e}\begin{pmatrix}1\\1\end{pmatrix} = 0\cdot 1 + 2\cdot 1 = 2$$

$$|\bar{u}| = \sqrt{\bar{u}\cdot\bar{u}} = \sqrt{0^2+2^2} = \sqrt{4} = 2$$

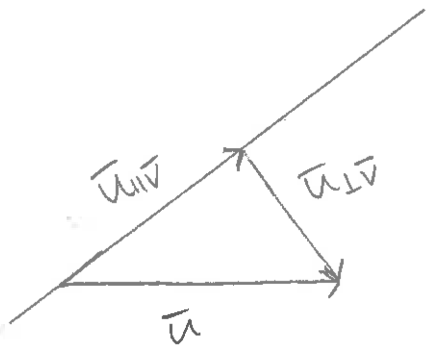
$$|\bar{v}| = \sqrt{\bar{v}\cdot\bar{v}} = \sqrt{1^2+1^2} = \sqrt{2}$$

$\bar{u}\cdot\bar{v} = |\bar{u}||\bar{v}|\cos\theta$ ger $\cos\theta = \dfrac{\bar{u}\cdot\bar{v}}{|\bar{u}||\bar{v}|} = \dfrac{2}{2\sqrt{2}} = \dfrac{1}{\sqrt{2}}$

Så $\theta = \dfrac{\pi}{4}$.



Lösning (Exempel 3):



$$\bar{u}_{\parallel\bar{v}} = \frac{\bar{u}\cdot\bar{v}}{|\bar{v}|^2}\bar{v} = \frac{2}{(\sqrt{2})^2}\underline{e}\begin{pmatrix}1\\1\end{pmatrix} =$$

$$= \underline{e}\begin{pmatrix}1\\1\end{pmatrix} \quad (=\bar{v})$$

Lösning (Exempel 4):

$$\bar{u} \times \bar{v} = \underline{e}\begin{pmatrix}1\\2\\1\end{pmatrix} \times \underline{e}\begin{pmatrix}2\\1\\3\end{pmatrix} = \left/ \begin{array}{cc} 1 & 2 \\ 2 & 1 \\ 1 & 3 \\ 1 & 2 \\ 2 & 1 \end{array} \right/ = \underline{e}\begin{pmatrix}2\cdot 3 - 1\cdot 1\\ 1\cdot 2 - 1\cdot 3\\ 1\cdot 1 - 2\cdot 2\end{pmatrix} = \underline{e}\begin{pmatrix}5\\-1\\-3\end{pmatrix}$$

$|\bar{u}\times\bar{v}| = |\bar{u}||\bar{v}|\sin\theta =$ "area av parallellogram"

$$= \left|\underline{e}\begin{pmatrix}5\\-1\\-3\end{pmatrix}\right| = \sqrt{5^2+(-1)^2+(-3)^2} = \underline{\underline{\sqrt{35}}}.$$

(OBS! $\bar{u}\times\bar{v} = -\bar{v}\times\bar{u}$.)

Lösning (Exempel 5): Volymen ges av

$|(\bar{u}\times\bar{v})\cdot\bar{w}|$ där $(\bar{u}\times\bar{v})\cdot\bar{w}$ är trippelprodukten av $\bar{u},\bar{v},\bar{w}$.

Enligt exempel 4 är $\bar{u}\times\bar{v} = \underline{e}\begin{pmatrix}5\\-1\\-3\end{pmatrix}$

$$(\bar{u}\times\bar{v})\cdot\bar{w} = \underline{e}\begin{pmatrix}5\\-1\\-3\end{pmatrix}\cdot\underline{e}\begin{pmatrix}2\\0\\-1\end{pmatrix} = 5\cdot 2 + (-1)\cdot 0 + (-3)\cdot(-1) = 13$$

$|13| = \underline{\underline{13}}.$

Lösning (Exempel 6):

$$\bar{u}\cdot\bar{v} = |\bar{u}||\bar{v}|\cos\theta = 2\cdot 3\cos 60^\circ = 2\cdot 3\cdot\frac{1}{2} = \underline{\underline{3}}.$$

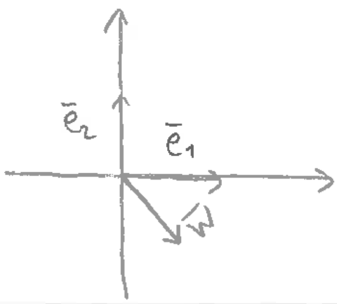
Lösning (Exempel 7):

Notera att i en ON-bas ges koordinaterna av att vi projicerar på axlarna ortogonalt.

Notera också att $\quad \bar{w}_{\parallel \bar{e}_i} = \dfrac{\bar{w}\cdot\bar{e}_i}{|\bar{e}_i|^2}\,\bar{e}_i = (\bar{w}\cdot\bar{e}_i)\,\bar{e}_i$

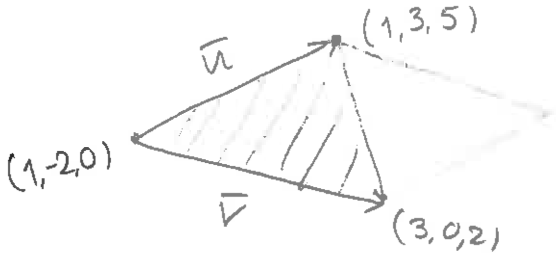
Så $\quad \bar{w} = (\bar{w}\cdot\bar{e}_1)\bar{e}_1 + (\bar{w}\cdot\bar{e}_2)\bar{e}_2 = (|\bar{w}||\bar{e}_1|\cos\pi/4)\,\bar{e}_1 + (|\bar{w}||\bar{e}_2|\cos 3\pi/4)\,\bar{e}_2$

$= \Big/\ |\bar{w}|=1$ är def. av att $\bar{w}$ är enhetsvektor $\Big/ = \dfrac{1}{\sqrt{2}}\,\bar{e}_1 - \dfrac{1}{\sqrt{2}}\,\bar{e}_2 = \underline{e}\underline{\underline{\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}}}.$



Lösning (Exempel 8):

(Schematisk figur)



$\bar{u} = (1,3,5) - (1,-2,0) = (0,5,5)$, $\bar{v} = (3,0,2) - (1,-2,0) = (2,2,2)$

$$\text{Area triangel} = \frac{\text{Area parallellogram}}{2} = \frac{|\bar{u} \times \bar{v}|}{2} =$$

$$= \frac{|(0,5,5) \times (2,2,2)|}{2} = \begin{vmatrix} 0 & 2 \\ 5 & 2 \\ 5 & 2 \\ 0 & 2 \\ 5 & 2 \end{vmatrix} = \frac{|(0,10,-10)|}{2} = \frac{\sqrt{200}}{2} = \underline{\underline{5\sqrt{2}}}.$$