Svar, TATA24, 2019-08-29

1. $2\sqrt{11}$

2. $x_1 = 1/2$, $x_2 = 1/2$

3. $(-1,1,1,0)$, $(1,0,0,1)$

4. $(4,2,0)$

5. -6

6. 5

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7. Sätt $\bar{u}_1 = \underline{e}\begin{pmatrix}3\\1\\-1\\2\end{pmatrix}$, $\bar{u}_2 = \underline{e}\begin{pmatrix}1\\3\\-1\\4\end{pmatrix}$, $\bar{u}_3 = \underline{e}\begin{pmatrix}1\\-1\\0\\-1\end{pmatrix}$, $\bar{u}_4 = \underline{e}\begin{pmatrix}7\\4\\7\\6\end{pmatrix}$.

Gram-Schmidt: $\bar{f}_1 = \bar{u}_1 = \underline{e}\begin{pmatrix}3\\1\\-1\\2\end{pmatrix}$.

$$\bar{f}_2 = \bar{u}_2 - \frac{\bar{u}_2\cdot\bar{f}_1}{\bar{f}_1\cdot\bar{f}_1}\bar{f}_1 = \underline{e}\begin{pmatrix}1\\3\\-1\\4\end{pmatrix} - 1\underline{e}\begin{pmatrix}3\\1\\-1\\2\end{pmatrix} = \underline{e}\begin{pmatrix}-2\\2\\0\\2\end{pmatrix}.$$

Obs att $\bar{u}_3 = -\frac{1}{2}\bar{f}_2$, så $\bar{u}_3 \in [\bar{f}_1, \bar{f}_2]$.

$$\bar{f}_4 = \bar{u}_4 - \frac{\bar{u}_4\cdot\bar{f}_1}{\bar{f}_1\cdot\bar{f}_1}\bar{f}_1 - \frac{\bar{u}_4\cdot\bar{f}_2}{\bar{f}_2\cdot\bar{f}_2}\bar{f}_2 = \underline{e}\begin{pmatrix}7\\4\\7\\6\end{pmatrix} - 2\underline{e}\begin{pmatrix}3\\1\\-1\\2\end{pmatrix} - \frac{1}{2}\underline{e}\begin{pmatrix}-2\\2\\0\\2\end{pmatrix} = \underline{e}\begin{pmatrix}2\\1\\9\\1\end{pmatrix}.$$

Normering ger: <u>Svar:</u> $\underline{e}\frac{1}{\sqrt{15}}\begin{pmatrix}3\\1\\-1\\2\end{pmatrix}$, $\underline{e}\frac{1}{\sqrt{3}}\begin{pmatrix}-1\\1\\0\\1\end{pmatrix}$, $\underline{e}\frac{1}{\sqrt{87}}\begin{pmatrix}2\\1\\9\\1\end{pmatrix}$.

8. $\begin{cases} x_1' = 3x_1 - 2x_2 \\ x_2' = 2x_1 - 2x_2 \end{cases}$ $\quad x_1(0) = 3$, $x_2(0) = 1$

Så $X' = AX$, med $A = \begin{pmatrix}3 & -2\\2 & -2\end{pmatrix}$. Egenvärden till A:

$$\begin{vmatrix}3-\lambda & -2\\ 2 & -2-\lambda\end{vmatrix} = \lambda^2 - \lambda - 2 = 0 \implies \lambda = 2, -1.$$

$\lambda = 2$: $\left(\begin{array}{cc|c}1 & -2 & 0\\ 2 & -4 & 0\end{array}\right)$ ger $s\begin{pmatrix}2\\1\end{pmatrix}$. $\lambda = -1$: $\left(\begin{array}{cc|c}4 & -2 & 0\\ 2 & -1 & 0\end{array}\right)$ ger $s\begin{pmatrix}1\\2\end{pmatrix}$

Detta ger $\begin{pmatrix}x_1(t)\\x_2(t)\end{pmatrix} = C_1e^{2t}\begin{pmatrix}2\\1\end{pmatrix} + C_2e^{-t}\begin{pmatrix}1\\2\end{pmatrix}$.

$t=0$: $\begin{cases}2C_1 + C_2 = 3\\ C_1 + 2C_2 = 1\end{cases} \implies C_1 = \frac{5}{3}$, $C_2 = -\frac{1}{3}$.

<u>Svar:</u> $\begin{cases} x_1(t) = \frac{10}{3}e^{2t} - \frac{1}{3}e^{-t}, \\ x_2(t) = \frac{5}{3}e^{2t} - \frac{2}{3}e^{-t}. \end{cases}$

9. $Q(\underline{e}X) = 3x_1^2 + 3x_2^2 + 7x_3^2 - x_1x_2 \qquad \left(X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right).$

Så $Q(\underline{e}X) = X^t A X$, med $A = \begin{pmatrix} 3 & -1/2 & 0 \\ -1/2 & 3 & 0 \\ 0 & 0 & 7 \end{pmatrix}$. Egenvärden:

$$\begin{vmatrix} 3-\lambda & -1/2 & 0 \\ -1/2 & 3-\lambda & 0 \\ 0 & 0 & 7-\lambda \end{vmatrix} = (7-\lambda)\left((\lambda-3)^2 - (1/2)^2\right) = (7-\lambda)(\lambda - 7/2)(\lambda - 5/2) = 0.$$

$\lambda = 7$: $\left(\begin{array}{ccc|c} -4 & -1/2 & 0 & 0 \\ -1/2 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ ger $t\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$,

$\lambda = \frac{7}{2}$: $\left(\begin{array}{ccc|c} -1/2 & -1/2 & 0 & 0 \\ -1/2 & -1/2 & 0 & 0 \\ 0 & 0 & 7/2 & 0 \end{array}\right)$ ger $t\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, $\lambda = \frac{5}{2}$: $\left(\begin{array}{ccc|c} 1/2 & -1/2 & 0 & 0 \\ -1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 9/2 & 0 \end{array}\right)$ ger $t\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

ON-basbytet $\underline{f} = \underline{e}T$, med $T = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ ger

$$Q(\underline{f}Y) = Q(\underline{e}TY) = (TY)^t A T Y = Y^t T^t A T Y = \frac{5}{2}y_1^2 + \frac{7}{2}y_2^2 + 7y_3^2.$$

Största och minsta värdet av Q på enhetssfären blir alltså:

<u>Svar:</u> 7 i $\pm\underline{e}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, och $\frac{5}{2}$ i $\pm\underline{e}\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$.

10. $F: \mathbb{R}^3 \to \mathbb{R}^3$ ges av $\frac{1}{7}\begin{pmatrix} 2 & 3 & -6 \\ 6 & 2 & 3 \\ -3 & 6 & 2 \end{pmatrix}$ i standardbasen,

a) l är egenrummet till egenvärdet -1:

$$\left(\begin{array}{ccc|c} 9 & 3 & -6 & 0 \\ 6 & 9 & 3 & 0 \\ -3 & 6 & 9 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 0 & 21 & 21 & 0 \\ 0 & 21 & 21 & 0 \\ -3 & 6 & 9 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -2 & -3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right),$$

vilket ger: <u>Svar:</u> $(x_1, x_2, x_3) = (t, -t, t)$, $t \in \mathbb{R}$.

b) Ta $\bar{u} \perp l$, t.ex. $\bar{u} = \underline{e}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. Sätt $\bar{v} = F(\bar{u}) = \underline{e}\,\frac{1}{7}\begin{pmatrix} 5 \\ 8 \\ 3 \end{pmatrix}$.

$\bar{u} \cdot \bar{v} = |\bar{u}||\bar{v}| \cos\theta$, så $\cos\theta = \dfrac{5/7 + 8/7 + 0}{\sqrt{2} \cdot \frac{1}{7}\sqrt{25+64+9}} = \dfrac{13}{14}$.

<u>Svar:</u> $\arccos \dfrac{13}{14}$.