Svar, TATA24, 2020-08-27

1. $2\sqrt{6}$

2. $a=7$

3. $\frac{2}{3}$

4. $\frac{5}{2}, \frac{1}{2}$

5. $a=-4$

6. -10

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7. Sätt $\bar{u}_1 = \underline{e}\begin{pmatrix}1\\0\\2\\0\\1\end{pmatrix}$, $\bar{u}_2 = \underline{e}\begin{pmatrix}-2\\1\\-1\\1\\-2\end{pmatrix}$, $\bar{u}_3 = \underline{e}\begin{pmatrix}-1\\5\\-1\\2\\-3\end{pmatrix}$.

Gram-Schmidt: $\bar{f}_1 = \bar{u}_1$,

$$\bar{f}_2 = \bar{u}_2 - \frac{\bar{u}_2\cdot\bar{f}_1}{|\bar{f}_1|^2}\bar{f}_1 = \underline{e}\begin{pmatrix}-2\\1\\-1\\1\\-2\end{pmatrix} - \frac{-6}{6}\underline{e}\begin{pmatrix}1\\0\\2\\0\\1\end{pmatrix} = \underline{e}\begin{pmatrix}-1\\1\\1\\1\\-1\end{pmatrix},$$

$$\bar{f}_3 = \bar{u}_3 - \frac{\bar{u}_3\cdot\bar{f}_1}{|\bar{f}_1|^2}\bar{f}_1 - \frac{\bar{u}_3\cdot\bar{f}_2}{|\bar{f}_2|^2}\bar{f}_2 = \underline{e}\begin{pmatrix}-1\\5\\-1\\2\\-3\end{pmatrix} - \frac{-6}{6}\underline{e}\begin{pmatrix}1\\0\\2\\0\\1\end{pmatrix} - \frac{10}{5}\underline{e}\begin{pmatrix}-1\\1\\1\\1\\-1\end{pmatrix} = \underline{e}\begin{pmatrix}2\\3\\-1\\0\\0\end{pmatrix}.$$

Normering ger: <u>Svar</u>: $\underline{e}\frac{1}{\sqrt{6}}\begin{pmatrix}1\\0\\2\\0\\1\end{pmatrix}$, $\underline{e}\frac{1}{\sqrt{5}}\begin{pmatrix}-1\\1\\1\\1\\-1\end{pmatrix}$, $\underline{e}\frac{1}{\sqrt{14}}\begin{pmatrix}2\\3\\-1\\0\\0\end{pmatrix}$

8. Sätt $A = \begin{pmatrix}3&-2&2\\-2&3&-1\\-6&6&-4\end{pmatrix}$, så fås

$$\begin{pmatrix}a_n\\b_n\\c_n\end{pmatrix} = A\begin{pmatrix}a_{n-1}\\b_{n-1}\\c_{n-1}\end{pmatrix} = \ldots = A^n\begin{pmatrix}a_0\\b_0\\c_0\end{pmatrix} = A^n\begin{pmatrix}1\\1\\1\end{pmatrix}.$$

Diagonalisering av A:

$$\begin{vmatrix}3-\lambda&-2&2\\-2&3-\lambda&-1\\-6&6&-4-\lambda\end{vmatrix} \overset{\substack{k1+k2\\k2+k3}}{=} \begin{vmatrix}1-\lambda&0&2\\1-\lambda&2-\lambda&-1\\0&2-\lambda&-4-\lambda\end{vmatrix} = (1-\lambda)(2-\lambda)\begin{vmatrix}1&0&2\\1&1&-1\\0&1&-4-\lambda\end{vmatrix} = \ldots$$

$$= (1-\lambda)(2-\lambda)(-1-\lambda) = 0 \quad \text{ger} \quad \lambda = 2, 1, -1.$$

$\lambda = 2$: $\left(\begin{array}{ccc|c}1&-2&2&0\\-2&1&-1&0\\-6&6&-6&0\end{array}\right) \sim \left(\begin{array}{ccc|c}1&-2&2&0\\0&-3&3&0\\0&-6&6&0\end{array}\right)$ ger $t\begin{pmatrix}0\\1\\1\end{pmatrix}$, $t\in\mathbb{R}$.

Pss ger $\lambda=1$: $t\begin{pmatrix}1\\1\\0\end{pmatrix}$, och $\lambda=-1$: $t\begin{pmatrix}1\\0\\-2\end{pmatrix}$.

forts. →

8. forts. Sätt $T = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & -2 \end{pmatrix}$, $D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, så att $A = TDT^{-1}$.

Nu fås $\begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} = A^n \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = TD^nT^{-1}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \ldots = \begin{pmatrix} 2-(-1)^n \\ -2^n+2 \\ -2^n+2(-1)^n \end{pmatrix}$.

Svar: $\begin{cases} a_n = 2-(-1)^n \\ b_n = -2^n+2 \\ c_n = -2^n+2(-1)^n \end{cases}$

9. $F: \mathbb{P}_3 \to \mathbb{P}_2$: $F(p(x)) = (3x+1)p''(x) - 3p'(x) + p(0)$.

$F(1) = (3x+1)\cdot 0 - 3\cdot 0 + 1 = 1$, $F(x) = (3x+1)\cdot 0 - 3\cdot 1 + 0 = -3$,

$F(x^2) = (3x+1)\cdot 2 - 3\cdot 2x + 0 = 2$, $F(x^3) = (3x+1)\cdot 6x - 3\cdot 3x^2 + 0 = 6x+9x^2$.

Så F:s matris i baserna $(1\ x\ x^2\ x^3)$ för $\mathbb{P}_3$ och $(1\ x\ x^2)$ för $\mathbb{P}_2$

är $\begin{pmatrix} 1 & -3 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 9 \end{pmatrix}$. Ekv. för N(F): $\begin{array}{c} \begin{array}{cccc} a_0 & a_1 & a_2 & a_3 \end{array} \\ \left(\begin{array}{cccc|c} 1 & -3 & 2 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 9 & 0 \end{array}\right) \end{array} \sim$

$\sim \left(\begin{array}{cccc|c} 1 & -3 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$ ger $\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = s\begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, och vi

har pivotelt. i kolumn 1 och 4.

Svar: Bas för N(F): $3+x$, $-2+x^2$.

Bas för V(F): 1, $6x+9x^2$.

10. F är symmetrisk, dvs $(F(\bar{u})|\bar{v}) = (\bar{u}|F(\bar{v}))$ för alla $\bar{u},\bar{v} \in \mathbb{V}$.

Låt $v \in \mathbb{V}$. Vill visa: $(\bar{v}\,|\,F(\bar{u})-\bar{u}) = 0$, $\bar{u} \in \mathbb{V}$ $\Leftrightarrow$ $F(\bar{v}) = \bar{v}$.

Eftersom $(\bar{v}\,|\,F(\bar{u})-\bar{u}) = (\bar{v}\,|\,F(\bar{u})) - (\bar{v}|\bar{u}) \underset{\substack{\uparrow \\ F \text{ symm.}}}{=} (F(\bar{v})\,|\,\bar{u}) - (\bar{v}|\bar{u}) =$

$= (F(\bar{v})-\bar{v}\,|\,\bar{u})$ fås:

$(\bar{v}\,|\,F(\bar{u})-\bar{u}) = 0$, $\bar{u} \in \mathbb{V}$ $\Leftrightarrow$ $(F(\bar{v})-\bar{v}\,|\,\bar{u}) = 0$, $\bar{u} \in \mathbb{V}$ $\Leftrightarrow$

$\Leftrightarrow$ $F(\bar{v}) - \bar{v} = \bar{0}$ $\Leftrightarrow$ $F(\bar{v}) = \bar{v}$.