Partial Differential Equations (TATA27)
Spring Semester 2019
Homework 1

Review of previous seminar

In the first seminar we studied Chapters 1 and 3. Read through these chapters to check your understanding and fill in any gaps. Then attempt the following question.

1.1 Prove that the following operators are linear operators.

(a) $\nabla = (\partial_1, \partial_2, \ldots, \partial_n)$ acting on functions $u\colon \mathbf{R}^n \to \mathbf{R}$.

(b) The divergence operator div which acts via the formula $\operatorname{div}(u) = \sum_{j=1}^n \partial_j u^j$ on functions $u = (u^1, u^2, \ldots, u^n)\colon \mathbf{R}^n \to \mathbf{R}^n$.

(c) curl acting on functions $u = (u_1, u_2, u_3)\colon \mathbf{R}^3 \to \mathbf{R}^3$ by the formula

$$\operatorname{curl}(u) = (\partial_2 u_3 - \partial_3 u_2, \partial_3 u_1 - \partial_1 u_3, \partial_1 u_2 - \partial_2 u_1).$$

(d) $\Delta := \nabla \cdot \nabla = \sum_{j=1}^n \partial_j^2$ acting on functions $u\colon \mathbf{R}^n \to \mathbf{R}$.

1.2 Classify the following equations in u as linear or non-linear (non-linear means not linear) and give the order of the equation.

(a) $u_{tt}(x,t) - u_{xx}(x,t) + xu(x,t) = 0$

(b) $u_{tt}(x,t) - u_{xx}(x,t) + x^2 = 0$

(c) $u_t(x,t) + u_{xxxx}(x,t) + \sqrt{1 + u(x,t)} = 0$

(d) $u_x(x,y) + e^y u_y(x,y) = 0$

Preparation for the next seminar

In preparation for seminar 2 read through Chapter 2 and attempt the following two problems.

1.3 Use the method of characteristics to find an explicit formula for a smooth function $u\colon \mathbf{R}^2 \to \mathbf{R}$ which solves the equation

$$u_x(x,y) + yu_y(x,y) = 0 \quad \text{for all } x, y \in \mathbf{R}$$

and satisfies the condition $u(0,y) = g(y)$ for all $y \in \mathbf{R}$ where g is a given smooth function.

1.4 Use the method of characteristics to find an explicit formula for a smooth function $u\colon \mathbf{R}^2 \to \mathbf{R}$ which solves the equation

$$(1 + x^2)u_x(x,y) + u_y(x,y) = 0 \quad \text{for all } x, y \in \mathbf{R}$$

and satisfies the condition $u(0,y) = g(y)$ for all $y \in \mathbf{R}$ where g is a given smooth function.

In-seminar group work

We will work on the following problem together in the seminar. It is best not to even read the question in advance.

1.5 Let a, b and c be real numbers and suppose that $b \neq 0$. Use the method of characteristics to find an explicit formula for a smooth function $u\colon \mathbf{R}^2 \to \mathbf{R}$ which solves the equation

$$au_x(x,t) + bu_t(x,t) + cu(x,t) = 0 \quad \text{for all } x \in \mathbf{R} \text{ and } t > 0$$

and satisfies the "initial condition" $u(x,0) = g(x)$ for all $x \in \mathbf{R}$ where g is a given smooth function.

Review exercises

Here's an addition homework exercise related to the method of characteristics that you can attempt after seminar 2.

1.6 Let $f\colon \mathbf{R}^n \times (0,\infty) \to \mathbf{R}$ and $g\colon \mathbf{R}^n \to \mathbf{R}$ be two smooth functions and $b \in \mathbf{R}^n$. Consider the equations

$$\begin{aligned} u_t(x,t) + b \cdot \nabla u(x,t) &= f(x,t) \quad \text{for } x \in \mathbf{R}^n \text{ and } t > 0, \text{ and} \\ u(x,0) &= g(x) \quad \text{for } x \in \mathbf{R}^n. \end{aligned} \tag{†}$$

Here ∇ denotes the gradient vector in the x-variables. Set $z(s) = u(x + bs, t + s)$ for fixed $x \in \mathbf{R}^n$ and $t > 0$ and derive an ODE which z satisfies. Use this ODE to find a formula for a solution u to (†). (This method simply takes the characteristic curves (X, T) to be $X(s) = x + bs$ and $T(s) = t + s$.)

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Solutions to Homework 1

1.1 (a) For $u, v\colon \mathbf{R}^n \to \mathbf{R}$ and $a, b \in \mathbf{R}$, we have

$$\begin{aligned}\nabla(au+bv) &= (\partial_1(au+bv), \partial_2(au+bv), \dots, \partial_n(au+bv)) \\ &= a(\partial_1 u, \partial_2 u, \dots, \partial_n u) + b(\partial_1 v, \partial_2 v, \dots, \partial_n v) = a\nabla u + b\nabla v,\end{aligned}$$

so ∇ is linear.

(b) For $u = (u^1, u^2, \dots, u^n)$ and $v = (v^1, v^2, \dots, v^n)$, functions from $\mathbf{R}^n$ to $\mathbf{R}^n$, and $a, b \in \mathbf{R}$, we have

$$\begin{aligned}\operatorname{div}(au+bv) &= \sum_{j=1}^{n} \partial_j(au^j + bv^j) = \sum_{j=1}^{n}(a\partial_j u^j + b\partial_j v^j) \\ &= a\left(\sum_{j=1}^{n}\partial_j u^j\right) + b\left(\sum_{j=1}^{n}\partial_j v^j\right) = a\operatorname{div} u + b\operatorname{div} v,\end{aligned}$$

so div is linear.

(c) For $u = (u^1, u^2, u^3)$ and $u = (u^1, u^2, u^3)$, functions from $\mathbf{R}^3$ to $\mathbf{R}^3$, and $a, b \in \mathbf{R}$, we have

$$\begin{aligned}&\operatorname{curl}(au+bv) \\ &= \big(\partial_2(au_3+bv_3) - \partial_3(au_2+bv_2), \partial_3(au_1+bv_1) - \partial_1(au_3+bv_3), \partial_1(au_2+bv_2) - \partial_2(au_1+bv_1)\big) \\ &= \big(a\partial_2 u_3 + b\partial_2 v_3 - a\partial_3 u_2 - b\partial_3 v_2, a\partial_3 u_1 + b\partial_3 v_1 - a\partial_1 u_3 - b\partial_1 v_3, a\partial_1 u_2 + b\partial_1 v_2 - a\partial_2 u_1 - b\partial_2 v_1\big) \\ &= a\big(\partial_2 u_3 - \partial_3 u_2, \partial_3 u_1 - \partial_1 u_3, \partial_1 u_2 - \partial_2 u_1\big) + b\big(\partial_2 v_3 - \partial_3 v_2, \partial_3 v_1 - \partial_1 v_3, \partial_1 v_2 - \partial_2 v_1\big) \\ &= a\operatorname{curl} u + b\operatorname{curl} v,\end{aligned}$$

so curl is linear.

(d) For $u, v\colon \mathbf{R}^n \to \mathbf{R}$ and $a, b \in \mathbf{R}$, we have

$$\begin{aligned}\Delta(au+bv) &= \sum_{j=1}^{n}\partial_j^2(au+bv) = \sum_{j=1}^{n}\partial_j(a\partial_j u + b\partial_j v) \\ &= \sum_{j=1}^{n}(a\partial_j^2 u + b\partial_j^2 v) = a\left(\sum_{j=1}^{n}\partial_j^2 u\right) + b\left(\sum_{j=1}^{n}\partial_j^2 v\right) \\ &= a\Delta u + b\Delta v,\end{aligned}$$

so Δ is linear.

1.2 We can see directly that the orders of the equations are (a) 2, (b) 2, (c) 4 and (d) 1. It is also easy to check that (a) and (d) are linear—which we do by check that if u and v are solutions, and a and b are constants, then $au + bv$ is a solution.

Equation (b) is also linear, although non-homogeneous. Indeed we can write the equation as $\mathcal{L}u = -x^2$ where $\mathcal{L} = \partial_t^2 - \partial_x^2$. We can check that $\mathcal{L}$ is a linear operator by considering two functions u and v and two constants α and β, and calculating

$$\mathcal{L}(\alpha u + \beta v) = (\alpha u + \beta v u + \beta v)_{tt} - (\alpha u + \beta v)_{xx} = \alpha u_{tt} - \alpha u_{xx} + \beta v_{tt} - \beta v_{xx} = \alpha\mathcal{L}(u) + \beta\mathcal{L}(v).$$

We can see that (c) is non-linear, as although the first two terms in the equation are linear, the third is not. Indeed $\sqrt{1+u} + \sqrt{1+v} = \sqrt{1+(u+v)}$ only if $4(1+u)(1+v) = 1$, so $u \mapsto u_t + u_{xxxx} + \sqrt{1+u}$ is not a linear operator.

1.3 We look for curves $t \mapsto (X(t), Y(t))$ on which a solution to

$$u_x(x,y) + yu_y(x,y) = 0 \quad \text{for all } x, y \in \mathbf{R}$$

will be constant. Thus, setting $z(t) = u(X(t), Y(t))$, we require

$$0 = z'(t) = X'(t)u_x(X(t), Y(t)) + Y'(t)u_y(X(t), Y(t))$$

so, comparing this equality with the PDE, we choose

$$X'(t) = 1 \quad \text{and}$$
$$Y'(t) = Y(t).$$

Thus, $X(t) = t + c_1$ and $Y(t) = c_2 e^t$. Let's now consider a characteristic curve which passes through an arbitrary point (x, y), say when $t = 0$. Such a curve will satisfy $x = X(0) = 0 + c_1$ and $y = Y(0) = c_2 e^0$, so $c_1 = x$ and $c_2 = y$ and we have $X(t) = t + x$ and $Y(t) = ye^t$.

To calculate the value of $u(x, y)$ we will use the fact u is constant along the characteristic curves and the fact we know $u(0, y) = g(y)$. We need to calculate for which t our characteristic curve will cross the y-axis. This happens when $0 = X(t) = t + x$, that is when $t = -x$. $Y(-x) = ye^{-x}$, so we have $u(x, t) = u(0, ye^{-x}) = g(ye^{-x})$. Thus we have calculated that the solution must be

$$u(x, y) = g(ye^{-x}).$$

We can check directly that this is indeed a solution.

1.4 We wish to find curves $t \mapsto (X(t), Y(t))$ on which a solution to

$$(1 + x^2)u_x(x, y) + u_y(x, y) = 0 \quad \text{for all } x, y \in \mathbf{R}$$

will be constant. Thus, setting $z(t) = u(X(t), Y(t))$, we require

$$0 = z'(t) = X'(t)u_x(X(t), Y(t)) + Y'(t)u_y(X(t), Y(t))$$

so, comparing this equality with the PDE, we choose

$$X'(t) = 1 + X(t)^2 \quad \text{and}$$
$$Y'(t) = 1.$$

Thus, $Y(t) = t + c_Y$ and

$$t = \int dt = \int \frac{X'(t)}{1 + X(t)^2} dt = \arctan(X(t)) + c_X$$

so $X(t) = \tan(t - c_X)$, for constants $c_X, c_Y \in \mathbf{R}$.

For a given point (x, y), we can find a characteristic curve which passes through (x, y), when say $t = 0$, by taking $Y(t) = t + y$ and $X(t) = \tan(t + \arctan(x))$ (that is, by choosing $c_Y = y$ and $c_X = -\arctan(x)$). Then, using the fact u is constant on characteristic curves and the condition $u(0, y) = g(y)$,

$$\begin{aligned} u(x, y) = u(X(0), Y(0)) &= u(X(-\arctan x), Y(-\arctan x)) \\ &= u(-\arctan x + y, 0) \\ &= g(y - \arctan x). \end{aligned}$$

1.5 For fixed (x, t), $z(s) = u(x + bs, t + s)$ so we can compute

$$z'(s) = \frac{d}{ds}u(x + bs, t + s) = b \cdot \nabla u(x + bs, t + s) + u_t(x + bs, t + s),$$

so, using the PDE, we see that

$$z'(s) = f(x + bs, t + s).$$

Integrating with respect to s from $-t$ to 0, we obtain

$$\int_{-t}^{0} f(x+bs, t+s)ds = \int_{-t}^{0} z'(s)ds = z(0) - z(-t) = u(x, t) - u(x - bt, 0) = u(x, t) - g(x - bt),$$

so

$$u(x, t) = g(x - bt) + \int_0^t f(x - bs, t - s)ds. \tag{‡}$$

1.6 We search for appropriate curves (X, T) such that the solution on the curves $s \mapsto z(s) := u(X(s), T(s))$ behaves nicely. We have

$$z'(s) = \frac{d}{ds} u(X(s), T(s)) = X'(s)\partial_1 u(X(s), T(s)) + T'(s)\partial_2 u(X(s), T(s)),$$

so it seems reasonable to set $X'(s) = a$ and $T'(s) = b$. Thus $X(s) = as + c_X$ and $T(s) = bs + c_T$ for constants $c_X, c_T \in \mathbf{R}$. We can then rewrite the PDE as

$$z'(s) + cz(s) = a\partial_1 u(X(s), T(s)) + b\partial_2 u(X(s), T(s)) + cu(X(s), T(s)) = 0.$$

This is an ODE with general solution $z(s) = Ae^{-cs}$ for any $A \in \mathbf{R}$.

Now fix (x, t). If we choose $c_X = x$ and $c_T = t$, then $X(s) = as + x$ and $T(s) = bs + t$, and when $s = 0$ the characteristic curve passes through $(X(0), T(0)) = (x, t)$ and when $s = -t/b$ the curve passes through $(X(-t/b), T(-t/b)) = (x - at/b, 0)$. When $s = -t/b$ we can use the initial condition to find the value of z:

$$z(-t/b) = u(X(-t/b), T(-t/b)) = u(x - at/b, 0) = g(x - at/b).$$

But on the other hand, using the form of the general solution to the characteristic ODE, $z(-t/b) = Ae^{ct/b}$, so $A = g(x - at/b)e^{-ct/b}$. Equally, for $s = 0$,

$$u(x, t) = z(0) = Ae^{-c0} = g(x - at/b)e^{-ct/b},$$

which gives us an expression for the solution u.

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Homework 2

Preparation for the next seminar

In preparation for Seminar 3 read through Chapter 4 and Section 5.1 and attempt the following two problems.

2.1 Let $\mathbf{R}^2_+ = \mathbf{R} \times (0, \infty)$, $C(\overline{\mathbf{R}^2_+})$ denote the set of continuous real-valued functions on $\overline{\mathbf{R}^2_+}$ and $C^1(\mathbf{R}^2_+)$ denote the set of continuously differentiable real-valued functions on $\mathbf{R}^2_+$. Consider the boundary-value problem

$$\begin{cases} u_x(x,y) + y u_y(x,y) = 0 & \text{for all } (x,y) \in \mathbf{R}^2_+, \text{ and} \\ u(x,0) = \phi(x) & \text{for all } x \in \mathbf{R}. \end{cases}$$

(a) Show that if $\phi(x) = x$ for all $x \in \mathbf{R}$, then no solution exists in $C(\overline{\mathbf{R}^2_+}) \cap C^1(\mathbf{R}^2_+)$.

(b) Show that if $\phi(x) = 1$ for all $x \in \mathbf{R}$, then there are many solutions in $C(\overline{\mathbf{R}^2_+}) \cap C^1(\mathbf{R}^2_+)$.

2.2 Fix $\ell > 0$ and consider the following boundary-value problem. Given a function $f\colon (0,\ell) \to \mathbf{R}$ we wish to find $u\colon [0,\ell] \to \mathbf{R}$ which is twice continuously differentiable such that

$$\begin{cases} u''(x) + u'(x) = f(x) & \text{for all } x \in (0,\ell), \text{ and} \\ u'(0) = u(0) = \frac{1}{2}(u'(\ell) + u(\ell)). \end{cases}$$

(a) Prove that if a solution u exists, it is not unique.

(b) Find two conditions we must place on f for a solution to exist.

Group work

We will work on the following exercise at the end of the seminar then we will discuss possible solutions together in Seminar 4.

2.3 Let Ω be a bounded open set. Prove that continuous functions $u\colon \overline{\Omega} \to \mathbf{R}$ which satisfy

$$\Delta u(\mathbf{x}) + \mathbf{x} \cdot \nabla u(\mathbf{x}) \geq 0$$

for $\mathbf{x} \in \Omega$ also satisfy the weak maximum principle:

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u.$$

Review exercises

Here's a couple of additional exercises for you to try.

2.4 Suppose $u\colon \mathbf{R}^2 \to \mathbf{R}$ is a harmonic function.

(a) For constants $a, b \in \mathbf{R}$ show that $v\colon \mathbf{R}^2 \to \mathbf{R}$ defined by

$$v(x,y) = u(x+a, y+b) \quad \text{for all } x, y \in \mathbf{R}$$

is harmonic.

(b) For a constant $\alpha \in \mathbf{R}$ show that $w\colon \mathbf{R}^2 \to \mathbf{R}$ defined by

$$w(x,y) = u(x\cos\alpha + y\sin\alpha, y\cos\alpha - x\sin\alpha) \quad \text{for all } x, y \in \mathbf{R}$$

is harmonic.

This exercise shows that Laplace's equation in the plane is invariant under rigid motions (translations and rotations).

2.5 The Schrödinger equation is a good model for the behaviour of particles at the atomic and subatomic level. Solutions $u\colon \mathbf{R}^3 \times \mathbf{R} \to \mathbf{C}$ are complex-valued and are related to the probability that a particle can be found in a specific region. The equation which models the motion of an electron around a hydrogen nucleus has the form

$$-i\hbar \frac{\partial u}{\partial t}(\mathbf{x}, t) = \frac{\hbar^2}{2m}\Delta u(\mathbf{x}, t) + \frac{e^2}{|\mathbf{x}|}u(\mathbf{x}, t)$$

for real constants $\hbar$, m and e and all $\mathbf{x} \in \mathbf{R}^3$ and $t \in \mathbf{R}$. Assume that u and $\partial_t u$ are continuous functions, and u, $\partial_t u$ and ∇u satisfy the estimate $|u(\mathbf{x}, t)|^2 + |\partial_t u(\mathbf{x}, t)|^2 + |\nabla u(\mathbf{x}, t)|^2 \leq C(1+|\mathbf{x}|)^{-2-\varepsilon}$ for some $C, \varepsilon > 0$, so we can interchange integration and differentiation according to the formula

$$\frac{d}{dt}\int_B u(\mathbf{x}, t)d\mathbf{x} = \int_B \partial_t u(\mathbf{x}, t)d\mathbf{x}$$

where B is any ball in $\mathbf{R}^3$ (see *Strauss*, p. 420, for the result for one spatial variable, but the same rule applies in $\mathbf{R}^3$). Show that if

$$\int_{\mathbf{R}^3} |u(\mathbf{x}, t_0)|^2 d\mathbf{x} = 1$$

for some $t_0 \in \mathbf{R}$, then

$$\int_{\mathbf{R}^3} |u(\mathbf{x}, t)|^2 d\mathbf{x} = 1$$

for all $t \in \mathbf{R}$.

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Solutions to Homework 2

2.1 Using the method of characteristics, we set $s(t) = u(X(t), Y(t))$ where $(X, Y): \mathbf{R} \to \mathbf{R}^2$ are the characteristic curves. If they satisfy the equations

$$\begin{cases} X'(t) = 1 \\ Y'(t) = Y(t) \end{cases}$$

then $s'(t) = u_x(X(t), Y(t)) + Y(t)u_y(X(t), Y(t)) = 0$, so s is a constant function. Solving the ODEs above, we find $X(t) = t + c$ and $Y(t) = Ce^t$ for constants c and C. Thus any solution to the PDE is constant on the lines $y = Ce^x$ and so is of the form

$$u(x, y) = f(ye^{-x}) \tag{1}$$

for an arbitrary function $f: [0, \infty) \to \mathbf{R}$. The requirement that $u \in C^1(\mathbf{R}^2_+)$ implies we need f to be continuously differentiable on $(0, \infty)$. If u is to belong to $C(\overline{\mathbf{R}^2_+})$, then in particular

$$u(x, 0) = \lim_{y \to 0} u(x, y).$$

Substituting in the boundary condition $u(x, 0) = \phi(x)$ and (1), we find

$$\phi(x) = u(x, 0) = \lim_{y \to 0} u(x, y) = \lim_{y \to 0} f(ye^{-x}) = f(0).$$

Thus, if the PDE is to have a solution in $C(\overline{\mathbf{R}^2_+}) \cap C^1(\mathbf{R}^2_+)$, ϕ must be a constant function. Consequetly, (a) if $\phi(x) = x$ there are no such solutions, and (b) if $\phi(x) = 1$, then (1) is a solution for any continuous f which is continuously differentiable on $(0, \infty)$ and such that $f(0) = 1$.

2.2 [Olle Abrahamsson] We will show that the equation has more than one solution by solving the equation directly. We multiply the equation by the integrating factor e^x to find

$$f(x)e^x = u''(x)e^x + u'(x)e^x = \frac{d}{dx}\left(u'(x)e^x\right)$$

so

$$u'(x) = e^{-x} \int_0^x f(t)e^t dt + c_0 e^{-x}$$

for a constant c_0. Thus

$$\begin{aligned} u(x) &= \int_0^x e^{-s} \int_0^s f(t)e^t dt ds - c_0 e^{-x} + c_1 \\ &= \int_0^x f(t)(1 - e^{t-x}) dt - c_0 e^{-x} + c_1, \end{aligned}$$

where c_1 is a constant.

The condition $u'(0) = u(0)$ says that $0 + c_0 = 0 - c_0 + c_1$, so $c_1 = 2c_0$, and $u'(0) = \frac{1}{2}(u'(\ell) + u(\ell))$ says

$$\begin{aligned} c_0 &= \frac{1}{2}\left(e^{-\ell} \int_0^\ell f(t)e^t dt + c_0 e^{-\ell} + \int_0^\ell f(t)(1 - e^{t-\ell}) dt - c_0 e^{-\ell} + 2c_0\right) \\ &= \frac{1}{2} \int_0^\ell f(t) dt + c_0, \end{aligned}$$

so we require $\int_0^\ell f(t)dt = 0$, but no restriction on c_0. Thus we have that

$$u(x) = \int_0^x f(t)(1 - e^{t-x}) dt - c_0 e^{-x} + 2c_0 \tag{2}$$

is a solution for any $c_0 \in \mathbf{R}$. We can check directly that such a u will be twice continuously differentiable if and only if f is continuous. Therefore we have shown that

(b) For a solution to exist in $C^2([0,\ell])$ we require f to be continuous and $\int_0^\ell f(t)dt = 0$.

(a) If these conditions are satisfied then we have an infinite number of solutions in $C^2([0,\ell])$ given by (2) for an arbitrary $c_0 \in \mathbf{R}$.

2.3 For $\varepsilon > 0$ set $v(\mathbf{x}) = u(\mathbf{x}) + \varepsilon|\mathbf{x}|^2$. As the sum of two continuous functions, v is continuous on $\overline{\Omega}$ and so must attain a maximum somewhere in the compact set $\overline{\Omega} = \Omega \cup \partial\Omega$. We will now rule out the possibility that v attains its maximum in Ω. Suppose to the contrary that v attains this maximum $\mathbf{x} \in \Omega$. Then we know $\mathbf{x}$ is a critical point, so $\nabla v(\mathbf{x}) = 0$ and, by the second derivative test, $\Delta v(\mathbf{x}) = \sum_{j=1}^n \partial_j^2 v(\mathbf{x}) \le 0$. Therefore

$$\Delta v(\mathbf{x}) + \mathbf{x} \cdot \nabla v(\mathbf{x}) = \Delta v(\mathbf{x}) + 0 \le 0 + 0 = 0.$$

But on the other hand, we can compute

$$\Delta v(\mathbf{x}) + \mathbf{x} \cdot \nabla v(\mathbf{x}) = \Delta u(\mathbf{x}) + \mathbf{x} \cdot \nabla u(\mathbf{x}) + 2\varepsilon|\mathbf{x}|^2 + 2\varepsilon n \ge 2\varepsilon|\mathbf{x}|^2 + 2\varepsilon n > 0,$$

via the differential inequality u satisfies. These two inequalities contradict each other, so v cannot attain its maximum in Ω.

Therefore v must attain its maximum at a point $\mathbf{y} \in \partial\Omega$. Thus, for any $\mathbf{x} \in \overline{\Omega}$,

$$u(\mathbf{x}) \le v(\mathbf{x}) \le v(\mathbf{y}) = u(\mathbf{y}) + \varepsilon|\mathbf{y}|^2 \le u(\mathbf{y}) + \varepsilon C^2 \le \max_{\partial\Omega} u + \varepsilon C^2,$$

where C is the constant obtained from the fact Ω is bounded. Since the above inequality holds for any $\varepsilon > 0$, we have $u(\mathbf{x}) \le \max_{\partial\Omega} u$ for any $\mathbf{x} \in \overline{\Omega}$, so

$$\max_{\overline{\Omega}} u \le \max_{\partial\Omega} u$$

Because $\partial\Omega \subseteq \overline{\Omega}$ we have that $\max_{\partial\Omega} u \le \max_{\overline{\Omega}} u$ and combining these two inequalities we get that $\max_{\overline{\Omega}} u = \max_{\partial\Omega} u$ and the maximum of u is attained on $\partial\Omega$.

2.4 (a) We can compute

$$\partial_x v = (\partial_1 u)(x+a, y+b), \qquad \partial_x^2 v = (\partial_1^2 u)(x+a, y+b),$$
$$\partial_y v = (\partial_2 u)(x+a, y+b) \qquad \text{and} \qquad \partial_y^2 v = (\partial_2^2 u)(x+a, y+b).$$

so

$$\begin{aligned} \Delta v(x,y) &= \partial_x^2 v(x,y) + \partial_y^2 v(x,y) \\ &= (\partial_1^2 u)(x+a, y+b) + (\partial_2^2 u)(x+a, y+b) = \Delta u(x+a, y+b) = 0. \end{aligned}$$

(b) We can compute

$$\begin{aligned} \partial_x w(x,y) &= \cos\alpha(\partial_1 u)(x\cos\alpha + y\sin\alpha, y\cos\alpha - x\sin\alpha) \\ &\quad - \sin\alpha(\partial_2 u)(x\cos\alpha + y\sin\alpha, y\cos\alpha - x\sin\alpha), \quad \text{and} \\ \partial_x^2 w(x,y) &= \cos^2\alpha(\partial_1^2 u)(x\cos\alpha + y\sin\alpha, y\cos\alpha - x\sin\alpha) \\ &\quad - 2\cos\alpha\sin\alpha(\partial_2\partial_1 u)(x\cos\alpha + y\sin\alpha, y\cos\alpha - x\sin\alpha) \\ &\quad + \sin^2\alpha(\partial_2^2 u)(x\cos\alpha + y\sin\alpha, y\cos\alpha - x\sin\alpha) \end{aligned}$$

And

$$\begin{aligned} \partial_y w(x,y) &= \sin\alpha(\partial_1 u)(x\cos\alpha + y\sin\alpha, y\cos\alpha - x\sin\alpha) \\ &\quad + \cos\alpha(\partial_2 u)(x\cos\alpha + y\sin\alpha, y\cos\alpha - x\sin\alpha), \quad \text{and} \\ \partial_x^2 w(x,y) &= \sin^2\alpha(\partial_1^2 u)(x\cos\alpha + y\sin\alpha, y\cos\alpha - x\sin\alpha) \\ &\quad + 2\cos\alpha\sin\alpha(\partial_2\partial_1 u)(x\cos\alpha + y\sin\alpha, y\cos\alpha - x\sin\alpha) \\ &\quad + \cos^2\alpha(\partial_2^2 u)(x\cos\alpha + y\sin\alpha, y\cos\alpha - x\sin\alpha) \end{aligned}$$

so

$$\begin{aligned}\Delta w(x,y) &= \cos^2\alpha(\partial_1^2 u)(x\cos\alpha + y\sin\alpha, y\cos\alpha - x\sin\alpha)\\
&\quad - 2\cos\alpha\sin\alpha(\partial_2\partial_1 u)(x\cos\alpha + y\sin\alpha, y\cos\alpha - x\sin\alpha)\\
&\quad + \sin^2\alpha(\partial_2^2 u)(x\cos\alpha + y\sin\alpha, y\cos\alpha - x\sin\alpha)\\
&\quad + \sin^2\alpha(\partial_1^2 u)(x\cos\alpha + y\sin\alpha, y\cos\alpha - x\sin\alpha)\\
&\quad + 2\cos\alpha\sin\alpha(\partial_2\partial_1 u)(x\cos\alpha + y\sin\alpha, y\cos\alpha - x\sin\alpha)\\
&\quad + \cos^2\alpha(\partial_2^2 u)(x\cos\alpha + y\sin\alpha, y\cos\alpha - x\sin\alpha)\\
&= (\cos^2\alpha + \sin^2\alpha)(\partial_1^2 u)(x\cos\alpha + y\sin\alpha, y\cos\alpha - x\sin\alpha)\\
&\quad + (\cos^2\alpha + \sin^2\alpha)(\partial_2^2 u)(x\cos\alpha + y\sin\alpha, y\cos\alpha - x\sin\alpha)\\
&= (\partial_1^2 u)(x\cos\alpha + y\sin\alpha, y\cos\alpha - x\sin\alpha)\\
&\quad + (\partial_2^2 u)(x\cos\alpha + y\sin\alpha, y\cos\alpha - x\sin\alpha)\\
&= \Delta w(x\cos\alpha + y\sin\alpha, y\cos\alpha - x\sin\alpha) = 0.\end{aligned}$$

2.5 Fix $N > 0$ and let $B_N = \{\mathbf{x} \in \mathbf{R}^n \mid |\mathbf{x}| < N\}$ be a ball in $\mathbf{R}^n$ centred at the origin. Then

$$\begin{aligned}&\int_{B_N} |u(\mathbf{x},t)|^2 d\mathbf{x} - \int_{B_N} |u(\mathbf{x},t_0)|^2 d\mathbf{x} = \int_{t_0}^t \frac{d}{ds}\left(\int_{B_N} |u(\mathbf{x},s)|^2 d\mathbf{x}\right) ds\\
&= \int_{t_0}^t \int_{B_N} \frac{\partial}{\partial s}\left(|u(\mathbf{x},s)|^2\right) d\mathbf{x}ds = \int_{t_0}^t \int_{B_N} \frac{\partial}{\partial s}\left(u(\mathbf{x},s)\overline{u(\mathbf{x},s)}\right) d\mathbf{x}ds \qquad (3)\\
&= \int_{t_0}^t \int_{B_N} \left(\partial_s u(\mathbf{x},s)\overline{u}(\mathbf{x},s) + u(\mathbf{x},s)\partial_s\overline{u}(\mathbf{x},s)\right) d\mathbf{x}ds,\end{aligned}$$

where the second equality (commuting the derivative and integral) is be justified by our assumptions. From the Schrödinger's equations, we have that

$$\frac{\partial u}{\partial s}(\mathbf{x},s) = \frac{i\hbar}{2m}\Delta u(\mathbf{x},s) + \frac{ie^2}{\hbar|\mathbf{x}|}u(\mathbf{x},s)$$

and taking complex conjugates

$$\frac{\partial \overline{u}}{\partial s}(\mathbf{x},s) = -\frac{i\hbar}{2m}\Delta \overline{u}(\mathbf{x},s) - \frac{ie^2}{\hbar|\mathbf{x}|}\overline{u}(\mathbf{x},s).$$

Thus

$$(\partial_s u(\mathbf{x},s))\overline{u}(\mathbf{x},s) + u(\mathbf{x},s)(\partial_s\overline{u}(\mathbf{x},s)) = \frac{i\hbar}{2m}\left(\Delta u(\mathbf{x},s)\overline{u}(\mathbf{x},s) - u(\mathbf{x},s)\Delta\overline{u}(\mathbf{x},s)\right).$$

The divergence theorem tells us

$$\begin{aligned}&\int_{B_N} \left((\partial_s u(\mathbf{x},s))\overline{u}(\mathbf{x},s) + u(\mathbf{x},s)(\partial_s\overline{u}(\mathbf{x},s))\right) d\mathbf{x}\\
&= \int_{B_N} \frac{i\hbar}{2m}\left(\Delta u(\mathbf{x},s)\overline{u}(\mathbf{x},s) - u(\mathbf{x},s)\Delta\overline{u}(\mathbf{x},s)\right) d\mathbf{x}\\
&= -\frac{i\hbar}{2m}\int_{B_N} \left(\nabla u(\mathbf{x},s)\cdot\nabla\overline{u}(\mathbf{x},s) - \nabla u(\mathbf{x},s)\cdot\nabla\overline{u}(\mathbf{x},s)\right) d\mathbf{x}\\
&\quad + \frac{i\hbar}{2m}\int_{\partial B_N} \left(\frac{\partial u}{\partial\mathbf{n}}(\mathbf{x},s)\overline{u}(\mathbf{x},s) - u(\mathbf{x},s)\frac{\partial\overline{u}}{\partial\mathbf{n}}(\mathbf{x},s)\right) d\sigma(\mathbf{x})\\
&= \frac{i\hbar}{2m}\int_{\partial B_N} \left(\frac{\partial u}{\partial\mathbf{n}}(\mathbf{x},s)\overline{u}(\mathbf{x},s) - u(\mathbf{x},s)\frac{\partial\overline{u}}{\partial\mathbf{n}}(\mathbf{x},s)\right) d\sigma(\mathbf{x}).\end{aligned}$$

Substituting this into (3) and using our assumptions about the decay of u, we find

$$\begin{aligned}&\left|\int_{B_N} |u(\mathbf{x},t)|^2 d\mathbf{x} - \int_{B_N} |u(\mathbf{x},t_0)|^2 d\mathbf{x}\right|\\
&= \left|\frac{i\hbar}{2m}\int_{t_0}^t \int_{\partial B_N} \left(\frac{\partial u}{\partial\mathbf{n}}(\mathbf{x},s)\overline{u}(\mathbf{x},s) - u(\mathbf{x},s)\frac{\partial\overline{u}}{\partial\mathbf{n}}(\mathbf{x},s)\right) d\sigma(\mathbf{x})ds\right|\\
&\le \frac{C|t-t_0|\hbar}{2m}|\partial B_N|(1+N)^{-2-\varepsilon}\end{aligned}$$

where $|\partial B_N|$ is the area of the set ∂B_N and equals $3\alpha(3)N^2$, where $\alpha(3)$ is the volume of the unit ball in $\mathbf{R}^3$. Thus $|\partial B_N|(1+N)^{-2-\varepsilon} = 3\alpha(3)N^2(1+N)^{-2-\varepsilon} \to 0$ as $N \to \infty$, which proves

$$\int_{\mathbf{R}^3} |u(\mathbf{x},t)|^2 d\mathbf{x} = \int_{\mathbf{R}^3} |u(\mathbf{x},t_0)|^2 d\mathbf{x} = 1,$$

as required.

Partial Differential Equations (TATA27)
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Homework 3

Review of previous seminar

To finish off our work from Seminar 3 read Section 5.2.

Preparation for the next seminar

In preparation for Seminar 4 read through Section 5.3 and attempt the following problem.

3.1 Consider two points $\mathbf{x}, \mathbf{y} \in \mathbf{R}^2$ with polar coordinates (r, θ) and (a, ϕ), respectively. Using a geometric argument (or otherwise) show that

$$|\mathbf{x} - \mathbf{y}|^2 = r^2 - 2ar\cos(\theta - \phi) + a^2.$$

Use this fact to help you rewrite the Poisson formula

$$u(r, \theta) = \frac{(a^2 - r^2)}{2\pi} \int_0^{2\pi} \frac{h(\phi)}{a^2 - 2ar\cos(\theta - \phi) + r^2} d\phi \tag{5.6}$$

as

$$u(\mathbf{x}) = \frac{(a^2 - |\mathbf{x}|^2)}{2\pi a} \int_{|\mathbf{y}|=a} \frac{\tilde{h}(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|^2} d\sigma(\mathbf{y}). \tag{5.7}$$

Group work

We will work on the following exercise at the end of the seminar then we will discuss possible solutions together in Seminar 5.

3.2 Let $W = \{\mathbf{x} = (r, \theta) \in \mathbf{R}^2 \mid 0 < r < a \text{ and } 0 < \theta < \beta\}$ denote a wedge of length a and angle β (where (r, θ) are polar coordinates). Using the same procedure as we used to derive the Poisson formula for D derive a analogous formula for the solution u to

$$\begin{cases} \Delta u = 0 & \text{in } W, \\ u(r, 0) = u(r, \beta) = 0 & \text{for } r \in (0, a), \text{ and} \\ u(a, \theta) = h(\theta) & \text{for } \theta \in (0, \beta). \end{cases}$$

Review exercises

Heres an additional exercise for you to try.

3.3 Using the method of separation of variables find a function $u \colon \overline{S} \to \mathbf{R}$ which is harmonic on the square $S = \{(x, y) \mid 0 < x < \pi, 0 < y < \pi\}$ and which satisfies the boundary conditions

$$u_y(x, 0) = u_y(x, \pi) = 0 \quad \text{for } 0 < x < \pi,$$
$$u(0, y) = 0 \quad \text{for } 0 < y < \pi, \text{ and}$$
$$u(\pi, y) = \cos^2 y \quad \text{for } 0 < y < \pi.$$

[Hint: The coordinate system you separate variables in should be chosen based on the geometry of S.]

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Solutions to Homework 3

3.1 Consider two points $\mathbf{x}, \mathbf{y} \in \mathbf{R}^2$ with polar coordinates (r, θ) and (a, ϕ), respectively. Looking at Figure 1 we can see that the right-angled triangle with vertices $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ has hypotenuse of length $|\mathbf{x}-\mathbf{y}|$ and the other two sides are of length $r - a\cos(\phi - \theta)$ and $a\sin(\phi - \theta)$. Thus, by Pythagoras' theorem

$$\begin{aligned} |\mathbf{x}-\mathbf{y}|^2 &= (r - a\cos(\phi-\theta))^2 + (a\sin(\phi-\theta))^2 \\ &= r^2 - 2ar\cos(\phi-\theta) + a^2(\cos^2(\phi-\theta) + \sin^2(\phi-\theta)) \\ &= r^2 - 2ar\cos(\phi-\theta) + a^2. \end{aligned} \tag{1}$$

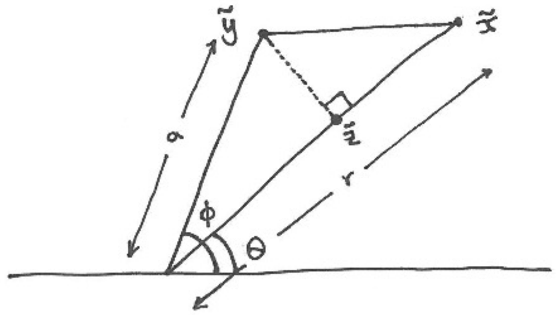

Figure 1: Two points $\mathbf{x}, \mathbf{y} \in \mathbf{R}^2$ with polar coordinates (r, θ) and (a, ϕ), respectively.

A line integral of $f\colon \gamma \to \mathbf{R}$ over a curve $\gamma \subset \mathbf{R}^2$ is defined to be

$$\int_\gamma f(\mathbf{y})d\sigma(\mathbf{y}) = \int_0^{2\pi} f(\mathbf{r}(\phi))|\mathbf{r}'(\phi)|d\phi$$

where $\mathbf{r}\colon [0, 2\pi] \to \mathbf{R}$ is a parametrisation of γ. Thus, if we take the parametrisation $\mathbf{r}(\phi) = (a\cos\phi, a\sin\phi)$ of the cicle $\{\mathbf{y} \in \mathbf{R}^2 \mid |\mathbf{y}| = a\}$, then $|\mathbf{r}'(\phi)| = a$ and

$$\int_{|\mathbf{y}|=a} \frac{\tilde{h}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^2} d\sigma(\mathbf{y}) = \int_0^{2\pi} \frac{\tilde{h}(\mathbf{r}(\phi))}{|\mathbf{x}-\mathbf{r}(\phi)|^2} a d\phi = a\int_0^{2\pi} \frac{h(\phi)}{r^2 - 2ar\cos(\phi-\theta) + a^2} d\phi,$$

where we used (1) (observing $\mathbf{r}(\phi)$ has polar coordinates (a, ϕ)). Since $|\mathbf{x}|^2 = r^2$, this implies

$$\frac{(a^2 - |\mathbf{x}|^2)}{2\pi a} \int_{|\mathbf{y}|=a} \frac{\tilde{h}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^2} d\sigma(\mathbf{y}) = \frac{(a^2 - r^2)}{2\pi} \int_0^{2\pi} \frac{h(\phi)}{a^2 - 2ar\cos(\theta-\phi) + r^2} d\phi.$$

3.2 We search for solutions of the form $u(r, \theta) = R(r)\Theta(\theta)$ via the method of separation of variables, just as we did in Section 5.3.2. Exactly as before we wish to solve the two separate ODEs

$$\Theta'' + \lambda\Theta = 0 \quad \text{and} \quad r^2R'' + rR' - \lambda R = 0,$$

for $\lambda \in \mathbf{R}$, but instead of wanting to find periodic Θ as we did for the disc, the boundary conditions $u(r, 0) = u(r, \beta) = 0$ imply we need

$$\Theta(0) = \Theta(\beta) = 0.$$

Solving the ODE for Θ with these boundary conditions gives

$$\lambda = \left(\frac{m\pi}{\beta}\right)^2 \quad \text{and} \quad \Theta(\theta) = \sin(m\pi\theta/\beta)$$

for $m = 1, 2, \ldots$. We now solve the ODE for R, which is of Euler form. We find that $R(r) = r^\alpha$, where $\alpha^2 = \lambda$. We reject negative exponents α, as they produce solutions R which are not continuous at the origin (the vertex of the wedge).[1] Thus we have a solution

$$u(r,\theta) = R(r)\Theta(\theta) = r^{m\pi/\beta}\sin(m\pi\theta/\beta)$$

for each positive integer m. In order to try to satisfy the boundary condition $u(a,\theta) = h(\theta)$ we consider linear combinations of these,

$$u(r,\theta) = \sum_{m=1}^{\infty} A_m r^{m\pi/\beta}\sin(m\pi\theta/\beta),$$

and consider the boundary value

$$h(\theta) = u(a,\theta) = \sum_{m=1}^{\infty} A_m a^{m\pi/\beta}\sin(m\pi\theta/\beta),$$

which has the form of a Fourier sine series for h, so it is natural to choose

$$A_m = \frac{2}{\beta a^{m\pi/\beta}}\int_0^\beta h(\phi)\sin(m\pi\phi/\beta)d\phi$$

and so

$$u(r,\theta) = \frac{2}{\beta}\sum_{m=1}^{\infty}\left(\frac{r}{a}\right)^{m\pi/\beta}\int_0^\beta h(\phi)\sin(m\pi\phi/\beta)\sin(m\pi\theta/\beta)d\phi.$$

3.3 We search for a solution which separates in the Cartesian coordinates x and y, that is, we search for a solution u of the form $u(x,y) = X(x)Y(y)$. In this case the equation $\Delta u(x,y) = 0$ can be rewritten as $X''(x)Y(y) + X(x)Y''(y) = 0$, and will be fullfiled if X and Y satisfy

$$X''(x) = \lambda X(x) \quad \text{and} \quad Y''(x) = -\lambda Y(x)$$

for some constant λ.

For negative λ, the general solution for Y has the form

$$Y(y) = Ae^{\sqrt{-\lambda}y} + Be^{-\sqrt{-\lambda}y}.$$

But the only choice of constants A and B which can satisfy the first two boundary conditions $Y'(0) = Y'(\pi) = 0$ is $A = B = 0$.

For non-negative λ, the general solutions for Y has the form

$$Y(y) = Ae^{i\sqrt{\lambda}y} + Be^{-i\sqrt{\lambda}y},$$

when $\lambda > 0$ or $Y(y) = A + By$ in the case $\lambda = 0$. The boundary conditions $Y'(0) = Y'(\pi) = 0$ require $A = B$ and $\lambda = n^2$ for $n = 1, 2, \ldots$ and $B = 0$ when $\lambda = 0$. Thus for each non-negative integer n,

$$Y_n(y) = A_n\cos(ny)$$

solves $Y_n''(x) = -\lambda_n Y_n(x)$ with $\lambda_n = n^2$.

Functions X_n which solve $X_n''(x) = \lambda_n X_n(x)$ and the boundary condition $X_n(0) = 0$ are $X_n(x) = \sinh(nx)$ for positive n and $X_0(x) = x$.

Since the Laplacian is a linear operator and the first three boundary conditions are homogeneous, we can take linear combinations of $X_n(x)Y_n(y)$ to constructed harmonic functions which satisfies the first three boundary conditions:

$$u(x,y) = A_0 x + \sum_{n=1}^{\infty} A_n\sinh(nx)\cos(ny).$$

[1]It is of course interesting to think about what would happen if we do not impose such continuity, but makes the solution more involved.

In order to choose A_n so that the last boundary condition is satisfied, we write

$$\cos^2(y) = \frac{\cos(2y) + 1}{2}$$

and so we require

$$\frac{\cos(2y) + 1}{2} = \cos^2(y) = u(\pi, y) = A_0\pi + \sum_{n=1}^{\infty} A_n \sinh(n\pi)\cos(ny).$$

This can be achieved by choosing $A_0 = 1/(2\pi)$, $A_2 = 1/(2\sinh(2\pi))$ and all the other A_n equal to zero. Thus, our sought-after function is

$$u(x, y) = \frac{x}{2\pi} + \frac{\sinh(2x)\cos(2y)}{2\sinh(2\pi)}.$$

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Homework 4

Preparation for the next seminar

In preparation for Seminar 5 read through Sections 5.4.1, 5.4.2 and 5.4.4, and attempt the following problem.

4.1 Green's second identity says that for two functions $u, v \in C^2(\overline{\Omega})$

$$\int_\Omega u(\mathbf{x})\Delta v(\mathbf{x}) - v(\mathbf{x})\Delta u(\mathbf{x}) d\mathbf{x} = \int_{\partial\Omega} u(\mathbf{x})\frac{\partial v}{\partial \mathbf{n}}(\mathbf{x}) - v(\mathbf{x})\frac{\partial u}{\partial \mathbf{n}}(\mathbf{x}) d\sigma(\mathbf{x}). \tag{5.10}$$

Use Green's first identity (5.8) to prove (5.10).

Group work

You should work on the following problem after Seminar 5, and then we will discuss possible solutions together in Seminar 6.

4.2 Let Ω be an open set with C^1 boundary, and let $f\colon \Omega \to \mathbf{R}$ and $h\colon \partial\Omega \to \mathbf{R}$. Use Green's first identity (5.8) to prove the uniqueness of solutions $u \in C^2(\overline{\Omega})$ to the following boundary value problems.

(a)

$$\begin{cases} \Delta u = f & \text{in } \Omega, \text{ and} \\ u = h & \text{on } \partial\Omega \end{cases}$$

(b)

$$\begin{cases} \Delta u = f & \text{in } \Omega, \text{ and} \\ \frac{\partial u}{\partial \mathbf{n}} + au = h & \text{on } \partial\Omega \end{cases}$$

where $\partial u/\partial \mathbf{n} := \mathbf{n} \cdot \nabla u$, $\mathbf{n}$ is the outward unit normal to $\partial\Omega$ and $a > 0$ is a constant.

Review exercises

Heres an additional exercise for you to try.

4.3 Consider a solution $u \in C^2(\overline{\Omega})$ to the boundary value problem

$$\begin{cases} \Delta u = f & \text{in } \Omega, \text{ and} \\ \frac{\partial u}{\partial \mathbf{n}} = h & \text{on } \partial\Omega \end{cases}$$

Observe that for any $c \in \mathbf{R}$ $u + c$ is also a solution. Could there be any more $C^2(\overline{\Omega})$ solutions?

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Solutions to Homework 4

4.1 For two functions $u, v \in C^2(\overline{\Omega})$, Green's first identity (5.8) says

$$\int_{\partial\Omega} v(\mathbf{x})\frac{\partial u}{\partial \mathbf{n}}(\mathbf{x})d\sigma(\mathbf{x}) = \int_{\Omega}(\nabla v(\mathbf{x}) \cdot \nabla u(\mathbf{x}) + v(\mathbf{x})\Delta u(\mathbf{x}))d\mathbf{x}.$$

Reversing the roles of u and v, we also have

$$\int_{\partial\Omega} u(\mathbf{x})\frac{\partial v}{\partial \mathbf{n}}(\mathbf{x})d\sigma(\mathbf{x}) = \int_{\Omega}(\nabla u(\mathbf{x}) \cdot \nabla v(\mathbf{x}) + u(\mathbf{x})\Delta v(\mathbf{x}))d\mathbf{x}.$$

Subtracting the first equality from the second, we obtain

$$\int_{\Omega} u(\mathbf{x})\Delta v(\mathbf{x}) - v(\mathbf{x})\Delta u(\mathbf{x})d\mathbf{x} = \int_{\partial\Omega} u(\mathbf{x})\frac{\partial v}{\partial \mathbf{n}}(\mathbf{x}) - v(\mathbf{x})\frac{\partial u}{\partial \mathbf{n}}(\mathbf{x})d\sigma(\mathbf{x}).$$

4.2 Let Ω be an open set with C^1 boundary, and let $f\colon \Omega \to \mathbf{R}$ and $g\colon \partial\Omega \to \mathbf{R}$. Suppose we had two solutions $u \in C^2(\Omega)$ to the following boundary value problems.

(a) Suppose we had two solutions $u_1, u_2 \in C^2(\Omega)$ to the boundary value problem

$$\begin{cases} \Delta u = f & \text{in } \Omega, \text{ and} \\ u = h & \text{on } \partial\Omega. \end{cases}$$

Then $v = u_1 - u_2$ solves

$$\begin{cases} \Delta v = 0 & \text{in } \Omega, \text{ and} \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Therefore, using (5.8),

$$\int_{\Omega} |\nabla v(\mathbf{x})|^2 d\mathbf{x} = \int_{\Omega} \nabla v(\mathbf{x}) \cdot \nabla v(\mathbf{x}) d\mathbf{x} = -\int_{\Omega} v(\mathbf{x})\Delta v(\mathbf{x})d\mathbf{x} = 0$$

which implies $\nabla v = 0$, so v must be a constant. However, since v is zero on $\partial\Omega$, it must be that $v = 0$. Therefore $u_1 = u_2$.

(b) Suppose we had two solutions $u_1, u_2 \in C^2(\Omega)$ to the boundary value problem

$$\begin{cases} \Delta u = f & \text{in } \Omega, \text{ and} \\ \frac{\partial u}{\partial \mathbf{n}} + au = h & \text{on } \partial\Omega \end{cases}$$

Then $v = u_1 - u_2$ solves

$$\begin{cases} \Delta v = 0 & \text{in } \Omega, \text{ and} \\ \frac{\partial v}{\partial \mathbf{n}} + av = 0 & \text{on } \partial\Omega \end{cases}$$

Therefore, using (5.8),

$$\begin{aligned} 0 \le \int_{\Omega} |\nabla v(\mathbf{x})|^2 d\mathbf{x} = \int_{\Omega} \nabla v(\mathbf{x}) \cdot \nabla v(\mathbf{x}) d\mathbf{x} &= -\int_{\Omega} v(\mathbf{x})\Delta v(\mathbf{x})d\mathbf{x} + \int_{\partial\Omega} v(\mathbf{x})\frac{\partial v}{\partial \mathbf{n}}(\mathbf{x})d\sigma(\mathbf{x}) \\ &= \int_{\partial\Omega} v(\mathbf{x})\frac{\partial v}{\partial \mathbf{n}}(\mathbf{x})d\sigma(\mathbf{x}) \\ &= -a\int_{\partial\Omega} |v(\mathbf{x})|^2 d\sigma(\mathbf{x}) \le 0 \end{aligned}$$

which implies $\nabla v = 0$, so v must be a constant. This means $\frac{\partial v}{\partial \mathbf{n}} = 0$ on $\partial\Omega$ and so the boundary condition then tells us that $av = 0$, which implies $v = 0$, since $a > 0$. Alternatively, we also see from the above calculation that

$$\int_{\partial\Omega} |v(\mathbf{x})|^2 d\sigma(\mathbf{x}) = 0,$$

so if v is a constant it must be zero.

In any case, we can thus conclude that $u_1 = u_2$.

4.3 Suppose we had two solutions $u_1, u_2 \in C^2(\Omega)$ to the boundary value problem

$$\begin{cases} \Delta u = f & \text{in } \Omega, \text{ and} \\ \frac{\partial u}{\partial \mathbf{n}} = h & \text{on } \partial\Omega \end{cases}$$

Then $v = u_1 - u_2$ solves

$$\begin{cases} \Delta v = 0 & \text{in } \Omega, \text{ and} \\ \frac{\partial v}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega \end{cases}$$

Therefore, using (5.8),

$$\begin{aligned} \int_\Omega |\nabla v(\mathbf{x})|^2 d\mathbf{x} = \int_\Omega \nabla v(\mathbf{x}) \cdot \nabla v(\mathbf{x}) d\mathbf{x} &= -\int_\Omega v(\mathbf{x}) \Delta v(\mathbf{x}) d\mathbf{x} + \int_{\partial\Omega} v(\mathbf{x}) \frac{\partial v}{\partial \mathbf{n}}(\mathbf{x}) d\sigma(\mathbf{x}) \\ &= \int_{\partial\Omega} v(\mathbf{x}) \frac{\partial v}{\partial \mathbf{n}}(\mathbf{x}) d\sigma(\mathbf{x}) = 0 \end{aligned}$$

which implies $\nabla v = 0$, so v must be a constant. This means $u_1 = u_2 + c$ for some constant $c \in \mathbf{R}$, so, no, there cannot be any more $C^2(\overline{\Omega})$ solutions.

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Homework 5

Preparation for the next seminar

I didn't go over all the material I planned to in the last seminar, so this weeks homework is a bit shorter than normal. First take a quick second look at Sections 5.4.2 and 5.4.4 from last time. Then read carefully through Section 5.4.3 (which was not part of last weeks homework), and attempt the following problem.

5.1 Let Ω be an open set with C^1 boundary. For $\lambda \geq 0$, define the energy of each continuously differentiable $v\colon \overline{\Omega} \to \mathbf{R}$ to be

$$E_\lambda[v] = \frac{1}{2}\int_\Omega (|\nabla v(\mathbf{x})|^2 + \lambda |v(\mathbf{x})|^2) d\mathbf{x}.$$

Show that a function $u \in C^2(\overline{\Omega})$ which satisfies $\Delta u - \lambda u = 0$ in Ω is such that

$$E_\lambda[u] \leq E_\lambda[v]$$

for all $v \in C^1(\overline{\Omega})$ such that $v(\mathbf{x}) = u(\mathbf{x})$ for all $\mathbf{x} \in \partial\Omega$.

Observe that the energy $E_\lambda[v]$ makes sense for functions in $C^1(\overline{\Omega})$, but (assuming a solution to the corresponding boundary value problem exists) a minimiser can sometimes be found in a better class. For example, if $\lambda = 0$, Lemma 5.5 tells us any solution u is smooth in Ω.

Group work

You should work on the following problem after Seminar 6, and then we will discuss possible solutions together in Seminar 7.

5.2 Let Ω be an open set with C^1 boundary and $h\colon \partial\Omega \to \mathbf{R}$ a C^1 function. Define the energy of each continuously differentiable $v\colon \Omega \to \mathbf{R}$ to be

$$E_h[v] = \frac{1}{2}\int_\Omega |\nabla v(\mathbf{x})|^2 d\mathbf{x} - \int_{\partial\Omega} h(\mathbf{x}) v(\mathbf{x}) d\sigma(\mathbf{x}).$$

Show that a function $u \in C^2(\overline{\Omega})$ which satisfies the boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \text{ and} \\ \frac{\partial u}{\partial \mathbf{n}} := \mathbf{n} \cdot \nabla u = h & \text{on } \partial\Omega \end{cases}$$

is such that

$$E_h[u] \leq E_h[v]$$

for all $v \in C^1(\overline{\Omega})$. Here $\mathbf{n}$ is the outward unit normal to $\partial\Omega$.

Here, in contrast to question 5.1, the boundary condition $\partial u / \partial \mathbf{n} = h$ is incorporated into the energy and we see that a solution u is a minimum of E_h over all $v \in C^1(\overline{\Omega})$ regardless of how v behaves at the boundary.

Partial Differential Equations (TATA27)
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Solutions to Homework 5

5.1 Suppose $u \in C^2(\overline{\Omega})$ satisfies $\Delta u - \lambda u = 0$ in Ω and $v \in C^1(\overline{\Omega})$ is such that $v(\mathbf{x}) = u(\mathbf{x})$ for all $\mathbf{x} \in \partial\Omega$. Then, setting $w = v - u$ and using Green's first identity (5.8), we see that

$$\begin{aligned}
E_\lambda[v] &= \frac{1}{2}\int_\Omega (|\nabla v(\mathbf{x})|^2 + \lambda |v(\mathbf{x})|^2) d\mathbf{x} \\
&= \frac{1}{2}\int_\Omega (|\nabla w(\mathbf{x}) + \nabla u(\mathbf{x})|^2 + \lambda |w(\mathbf{x}) + u(\mathbf{x})|^2) d\mathbf{x} \\
&= \frac{1}{2}\int_\Omega (|\nabla w(\mathbf{x})|^2 + \lambda |w(\mathbf{x})|^2) d\mathbf{x} + E_\lambda[u] + \int_\Omega (\nabla w(\mathbf{x}) \cdot \nabla u(\mathbf{x}) + \lambda w(\mathbf{x}) u(\mathbf{x})) d\mathbf{x} \\
&= \frac{1}{2}\int_\Omega (|\nabla w(\mathbf{x})|^2 + \lambda |w(\mathbf{x})|^2) d\mathbf{x} + E_\lambda[u] + \int_\Omega w(\mathbf{x})(-\Delta u(\mathbf{x}) + \lambda u(\mathbf{x})) d\mathbf{x} \\
&= \frac{1}{2}\int_\Omega (|\nabla w(\mathbf{x})|^2 + \lambda |w(\mathbf{x})|^2) d\mathbf{x} + E_\lambda[u] \\
&\geq E_\lambda[u].
\end{aligned}$$

5.2 Suppose $u \in C^2(\overline{\Omega})$ satisfies the boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \text{, and} \\ \frac{\partial u}{\partial \mathbf{n}} := \mathbf{n} \cdot \nabla u = h & \text{on } \partial\Omega \end{cases}$$

For $v \in C^1(\overline{\Omega})$ set $w = v - u$. Then, using Green's first identity (5.8),

$$\begin{aligned}
E_h[v] &= \frac{1}{2}\int_\Omega |\nabla v(\mathbf{x})|^2 d\mathbf{x} - \int_{\partial\Omega} h(\mathbf{x}) v(\mathbf{x}) d\sigma(\mathbf{x}) \\
&= \frac{1}{2}\int_\Omega |\nabla w(\mathbf{x}) + \nabla u(\mathbf{x})|^2 d\mathbf{x} - \int_{\partial\Omega} h(\mathbf{x})(w(\mathbf{x}) + u(\mathbf{x})) d\sigma(\mathbf{x}) \\
&= \frac{1}{2}\int_\Omega |\nabla w(\mathbf{x})|^2 d\mathbf{x} + E_h[u] + \int_\Omega \nabla w(\mathbf{x}) \cdot \nabla u(\mathbf{x}) d\mathbf{x} - \int_{\partial\Omega} h(\mathbf{x}) w(\mathbf{x}) d\sigma(\mathbf{x}) \\
&= \frac{1}{2}\int_\Omega |\nabla w(\mathbf{x})|^2 d\mathbf{x} + E_h[u] - \int_\Omega w(\mathbf{x}) \Delta u(\mathbf{x}) d\mathbf{x} + \int_{\partial\Omega} \left(\frac{\partial u}{\partial \mathbf{n}}(\mathbf{x}) - h(\mathbf{x})\right) w(\mathbf{x}) d\sigma(\mathbf{x}) \\
&= \frac{1}{2}\int_\Omega |\nabla w(\mathbf{x})|^2 d\mathbf{x} + E_h[u] \\
&\geq E_h[u].
\end{aligned}$$

Partial Differential Equations (TATA27)
Spring Semester 2019
Homework 6

Review of previous seminar

6.1 Consider the function $\Phi\colon \mathbf{R}^n \setminus \{\mathbf{0}\} \to \mathbf{R}$ defined by

$$\Phi(\mathbf{x}) = \begin{cases} -\frac{1}{2\alpha(2)} \ln|\mathbf{x}| & \text{if } n = 2, \\ \frac{1}{n(n-2)\alpha(n)} \frac{1}{|\mathbf{x}|^{n-2}} & \text{if } n > 2, \end{cases}$$

where $\alpha(n)$ is the volume of the unit ball in $\mathbf{R}^n$. (So, in particular, $\alpha(2) = \pi$ and $\alpha(3) = 4\pi/3$.) The aim of this exercise is to prove some properties of Φ stated in class and fill in the remaining gaps in the proof of Lemma 5.9.

(a) Prove that Φ is harmonic on $\mathbf{R}^n \setminus \{\mathbf{0}\}$.

(b) Consider the domain $B_r(\mathbf{0}) = \{\mathbf{y} \in \mathbf{R}^n \mid |\mathbf{y}| < r\}$. Then the outward unit normal at $\mathbf{x} \in \partial B_r(\mathbf{0})$ is $\mathbf{n}(\mathbf{x}) = \mathbf{x}/|\mathbf{x}|$. Prove that

$$\frac{\partial \Phi}{\partial \mathbf{n}}(\mathbf{x}) = \frac{-1}{n\alpha(n)} \frac{1}{|\mathbf{x}|^{n-1}}$$

for each $n = 1, 2, \dots$.

6.2 Prove the following lemma, which is a generalisation of Lemma 5.9 that does not assume that u is harmonic.

Lemma. *Let Ω be an open bounded set with C^1 boundary and suppose that $u \in C^2(\overline{\Omega})$ is such that $\Delta u = f$ for some $f \in C(\overline{\Omega})$. Then*

$$u(\mathbf{x}) = \int_{\partial\Omega} \left\{ \Phi(\mathbf{y}-\mathbf{x}) \left(\frac{\partial u}{\partial \mathbf{n}}\right)(\mathbf{y}) - \left(\frac{\partial \Phi}{\partial \mathbf{n}}\right)(\mathbf{y}-\mathbf{x}) u(\mathbf{y}) \right\} d\sigma(\mathbf{y}) - \int_{\Omega} f(\mathbf{y}) \Phi(\mathbf{y}-\mathbf{x}) d\mathbf{y}.$$

for each $\mathbf{x} \in \Omega$.

[Hint: Follow the proof of Lemma 5.9.]

Preparation for the next seminar

In preparation for seminar 7, read through sections 5.4.5 and 5.4.6.

Group work

Try this exercise after seminar 7. Please try to discuss your solution with others taking the course.

6.3 Use the lemma from question 6.2 to prove the following generalisation of Theorem 5.11.

Theorem. *Let $\Omega \subset \mathbf{R}^n$ be an open bounded set with C^2 boundary, and suppose $h \in C^2(\partial\Omega)$ and $f \in C(\overline{\Omega})$. If G is a Green's function for the Laplacian in Ω then the solution of the boundary value problem*

$$\begin{cases} \Delta u = f & \text{in } \Omega, \text{ and} \\ u = h & \text{on } \partial\Omega, \end{cases} \tag{$*$}$$

is given by

$$u(\mathbf{x}) = -\int_{\partial\Omega} \left(\frac{\partial G(\mathbf{x}, \cdot)}{\partial \mathbf{n}}(\mathbf{y}) \right) h(\mathbf{y}) d\sigma(\mathbf{y}) - \int_{\Omega} f(\mathbf{y}) G(\mathbf{x}, \mathbf{y}) d\mathbf{y}.$$

where $(\partial G(\mathbf{x}, \cdot)/\partial \mathbf{n})(\mathbf{y}) := \mathbf{n}(\mathbf{y}) \cdot \nabla_{\mathbf{y}} G(\mathbf{x}, \mathbf{y})$ is the normal derivative of $\mathbf{y} \mapsto G(\mathbf{x}, \mathbf{y})$.

We proved the uniqueness of solutions to $(*)$ in Section 5.2 of our notes, so when we can find a Green's function we have both the existence and uniqueness of solutions to $(*)$.

Extra problem

The following exercise is quite hard, so can be considered a bonus exercise to do it you have some spare time, but it is nevertheless a excellent way to check you have mastered the material we are studying.

6.4 The aim of this question is to prove Theorem 5.12. Let Ω be an open bounded set with C^2 boundary.

(a) In this part of the question we will prove that the Green's function for the Laplacian in Ω is unique. Suppose we have two Green's functions G_1 and G_2 for the Laplacian in Ω.

i. For each fixed $\mathbf{x} \in \Omega$, prove that $\mathbf{y} \mapsto G_1(\mathbf{x}, \mathbf{y}) - G_2(\mathbf{x}, \mathbf{y})$ has a continuous extension which belongs to $C^2(\overline{\Omega})$ and is harmonic in Ω.

ii. By considering a boundary value problem that $\mathbf{y} \mapsto G_1(\mathbf{x}, \mathbf{y}) - G_2(\mathbf{x}, \mathbf{y})$ solves, prove that $G_1 = G_2$.

(b) We now wish to prove the Green's function is symmetric, that is $G(\mathbf{x}, \mathbf{y}) = G(\mathbf{y}, \mathbf{x})$ for all $\mathbf{x}, \mathbf{y} \in \Omega$.

i. Fix $\mathbf{x}, \mathbf{y} \in \Omega$ with $\mathbf{x} \neq \mathbf{y}$ and consider the functions $\mathbf{z} \mapsto u(\mathbf{z}) := G(\mathbf{x}, \mathbf{z})$ and $\mathbf{z} \mapsto v(\mathbf{z}) := G(\mathbf{y}, \mathbf{z})$. Apply Green's second identity (5.10) to u and v in the domain $\Omega_r := \Omega \setminus \overline{(B_r(\mathbf{x}) \cup B_r(\mathbf{y}))}$ for $r > 0$ so small that $\overline{(B_r(\mathbf{x}) \cup B_r(\mathbf{y}))} \subset \Omega$ and $\overline{B_r(\mathbf{x})} \cap \overline{B_r(\mathbf{y})} = \emptyset$ to obtain that

$$\begin{aligned} 0 = &\int_{\partial B_r(\mathbf{x})} G(\mathbf{x}, \mathbf{z}) \frac{\partial G(\mathbf{y}, \cdot)}{\partial \mathbf{n}}(\mathbf{z}) - G(\mathbf{y}, \mathbf{z}) \frac{\partial G(\mathbf{x}, \cdot)}{\partial \mathbf{n}}(\mathbf{z}) d\sigma(\mathbf{z}) \\ &+ \int_{\partial B_r(\mathbf{y})} G(\mathbf{x}, \mathbf{z}) \frac{\partial G(\mathbf{y}, \cdot)}{\partial \mathbf{n}}(\mathbf{z}) - G(\mathbf{y}, \mathbf{z}) \frac{\partial G(\mathbf{x}, \cdot)}{\partial \mathbf{n}}(\mathbf{z}) d\sigma(\mathbf{z}). \end{aligned} \tag{$\dagger$}$$

ii. Using the definition of the Green's function, prove that

$$\int_{\partial B_r(\mathbf{x})} (G(\mathbf{x}, \mathbf{z}) - \Phi(\mathbf{z} - \mathbf{x})) \frac{\partial G(\mathbf{y}, \cdot)}{\partial \mathbf{n}}(\mathbf{z}) d\sigma(\mathbf{z}) \to 0$$

and

$$\int_{\partial B_r(\mathbf{x})} G(\mathbf{y}, \mathbf{z}) \left(\frac{\partial G(\mathbf{x}, \cdot)}{\partial \mathbf{n}}(\mathbf{z}) - \frac{\partial \Phi}{\partial \mathbf{n}}(\mathbf{z} - \mathbf{x}) \right) d\sigma(\mathbf{z}) \to 0$$

as $r \to 0$.

iii. Using the same ideas as in the proof of Lemma 5.9 prove that

$$\int_{\partial B_r(\mathbf{x})} \Phi(\mathbf{z} - \mathbf{x}) \frac{\partial G(\mathbf{y}, \cdot)}{\partial \mathbf{n}}(\mathbf{z}) d\sigma(\mathbf{z}) = 0$$

and

$$\int_{\partial B_r(\mathbf{x})} G(\mathbf{y}, \mathbf{z}) \frac{\partial \Phi}{\partial \mathbf{n}}(\mathbf{z} - \mathbf{x}) d\sigma(\mathbf{z}) = -G(\mathbf{y}, \mathbf{x}).$$

iv. Combine the results above to show that

$$\int_{\partial B_r(\mathbf{x})} G(\mathbf{x}, \mathbf{z}) \frac{\partial G(\mathbf{y}, \cdot)}{\partial \mathbf{n}}(\mathbf{z}) - G(\mathbf{y}, \mathbf{z}) \frac{\partial G(\mathbf{x}, \cdot)}{\partial \mathbf{n}}(\mathbf{z}) d\sigma(\mathbf{z}) \to G(\mathbf{y}, \mathbf{x}). \tag{$\ddagger$}$$

as $r \to 0$. (Observe the left-hand side of ($\ddagger$) is the first term on the right-hand side of ($\dagger$).)

v. Swap the roles of $\mathbf{x}$ and $\mathbf{y}$ in ($\ddagger$) to conclude a similar statement for the second term on the right-hand side of ($\dagger$). Combine your answer with ($\dagger$) and ($\ddagger$) to prove $G(\mathbf{x}, \mathbf{y}) = G(\mathbf{y}, \mathbf{x})$.

Partial Differential Equations (TATA27)
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Solutions to Homework 6

6.1 Consider the function $\Phi\colon \mathbf{R}^n \setminus \{\mathbf{0}\} \to \mathbf{R}$ defined by

$$\Phi(\mathbf{x}) = \begin{cases} -\frac{1}{2\alpha(2)} \ln|\mathbf{x}| & \text{if } n = 2, \\ \frac{1}{n(n-2)\alpha(n)} \frac{1}{|\mathbf{x}|^{n-2}} & \text{if } n > 2, \end{cases}$$

where $\alpha(n)$ is the volume of the unit ball in $\mathbf{R}^n$. (So, in particular, $\alpha(2) = \pi$ and $\alpha(3) = 4\pi/3$.)

(a) For $\mathbf{x} = (x_1, x_2, \ldots, x_n) \neq \mathbf{0}$ we have

$$\frac{\partial \Phi}{\partial x_j}(\mathbf{x}) = \frac{-x_j}{n\alpha(n)} \left(\sum_{k=1}^n x_k^2 \right)^{-n/2}$$

and

$$\frac{\partial^2 \Phi}{\partial x_j^2}(\mathbf{x}) = \frac{-1}{n\alpha(n)} \left(\sum_{k=1}^n x_k^2 \right)^{-n/2} + \frac{x_j^2}{\alpha(n)} \left(\sum_{k=1}^n x_k^2 \right)^{-(n+2)/2}.$$

Therefore

$$\Delta\Phi(\mathbf{x}) = \sum_{j=1}^n \frac{\partial^2 \Phi}{\partial x_j^2}(\mathbf{x}) = \frac{-1}{\alpha(n)} \left(\sum_{k=1}^n x_k^2 \right)^{-n/2} + \frac{\sum_{j=1}^n x_j^2}{\alpha(n)} \left(\sum_{k=1}^n x_k^2 \right)^{-(n+2)/2} = 0.$$

(b) From (a) we see that

$$\nabla\Phi(\mathbf{x}) = \frac{-1}{n\alpha(n)} \frac{\mathbf{x}}{|\mathbf{x}|^n}$$

so

$$\frac{\partial \Phi}{\partial \mathbf{n}}(\mathbf{x}) = \mathbf{n} \cdot \nabla\Phi(\mathbf{x}) = \frac{\mathbf{x}}{|\mathbf{x}|} \cdot \frac{-1}{n\alpha(n)} \frac{\mathbf{x}}{|\mathbf{x}|^n} = \frac{-1}{n\alpha(n)} \frac{1}{|\mathbf{x}|^{n-1}}.$$

6.2 We want to prove the following.

Lemma. *Let Ω be an open bounded set with C^1 boundary and suppose that $u \in C^2(\overline{\Omega})$ is such that $\Delta u = f$ for some $f \in C(\overline{\Omega})$. Then*

$$u(\mathbf{x}) = \int_{\partial\Omega} \left\{ \Phi(\mathbf{y}-\mathbf{x}) \left(\frac{\partial u}{\partial \mathbf{n}} \right)(\mathbf{y}) - \left(\frac{\partial \Phi}{\partial \mathbf{n}} \right)(\mathbf{y}-\mathbf{x}) u(\mathbf{y}) \right\} d\sigma(\mathbf{y}) - \int_\Omega f(\mathbf{y})\Phi(\mathbf{y}-\mathbf{x}) d\mathbf{y}. \quad (1)$$

for each $\mathbf{x} \in \Omega$.

Proof. We wish to apply Green's second identity (5.10) to the functions $\Phi(\cdot - \mathbf{x})$ and u in Ω. However we cannot as $\Phi(\cdot - \mathbf{x})$ is not defined at $\mathbf{x}$. We instead apply (5.10) to $\Omega_r := \Omega \setminus \overline{B_r(\mathbf{x})}$. We obtain

$$\int_{\Omega_r} f(\mathbf{y})\Phi(\mathbf{y}-\mathbf{x}) d\mathbf{y} = \int_{\partial\Omega_r} \left\{ \Phi(\mathbf{y}-\mathbf{x}) \left(\frac{\partial u}{\partial \mathbf{n}} \right)(\mathbf{y}) - \left(\frac{\partial \Phi}{\partial \mathbf{n}} \right)(\mathbf{y}-\mathbf{x}) u(\mathbf{y}) \right\} d\sigma(\mathbf{y}).$$

However, $\partial\Omega_r$ has two components, $\partial\Omega$ and $\partial B_r(\mathbf{x})$. The integral over $\partial\Omega$ appears on the right-hand side of (1), so we wish to calculate the integral over $\partial B_r(\mathbf{x})$. From (5.11) it is clear that Φ is a radial function—that is, we can write $\Phi(\mathbf{x}) = \phi(|\mathbf{x}|)$ for

$$\phi(r) = \begin{cases} -\frac{1}{2\alpha(2)} \ln|r| & \text{if } n = 2, \\ \frac{1}{n(n-2)\alpha(n)} \frac{1}{r^{n-2}} & \text{if } n > 2. \end{cases}$$

Remembering that the outward normal $\mathbf{n}$ to Ω_r is actually an inward normal to $B_r(\mathbf{x})$ on $\partial B_r(\mathbf{x})$, we have from (5.12) that

$$\frac{\partial \Phi}{\partial \mathbf{n}}(\mathbf{y}-\mathbf{x}) = \frac{1}{n\alpha(n)} \frac{1}{r^{n-1}}.$$

Thus

$$\begin{aligned}
&\int_{\partial B_r(\mathbf{x})} \left\{ \Phi(\mathbf{y}-\mathbf{x}) \left(\frac{\partial u}{\partial \mathbf{n}}\right)(\mathbf{y}) - \left(\frac{\partial \Phi}{\partial \mathbf{n}}\right)(\mathbf{y}-\mathbf{x})u(\mathbf{y}) \right\} d\sigma(\mathbf{y}) \\
=& \int_{\partial B_r(\mathbf{x})} \phi(r) \left(\frac{\partial u}{\partial \mathbf{n}}\right)(\mathbf{y}) d\sigma(\mathbf{y}) - \int_{\partial B_r(\mathbf{x})} \frac{1}{n\alpha(n)} \frac{1}{r^{n-1}} u(\mathbf{y}) d\sigma(\mathbf{y}) \\
=& \phi(r) \left(\int_{\partial B_r(\mathbf{x})} \left(\frac{\partial u}{\partial \mathbf{n}}\right)(\mathbf{y}) d\sigma(\mathbf{y}) \right) - \left(\frac{1}{n\alpha(n) r^{n-1}} \int_{\partial B_r(\mathbf{x})} u(\mathbf{y}) d\sigma(\mathbf{y}) \right).
\end{aligned}$$

Applying the divergence theorem (with $-\mathbf{n}$ being the outward unit normal), we see that

$$\left(\int_{\partial B_r(\mathbf{x})} \left(\frac{\partial u}{\partial \mathbf{n}}\right)(\mathbf{y}) d\sigma(\mathbf{y}) \right) = -\int_{B_r(\mathbf{x})} \Delta u(\mathbf{y}) d\mathbf{y} = -\int_{B_r(\mathbf{x})} f(\mathbf{y}) d\mathbf{y}.$$

Since $f \in C(\overline{\Omega})$ we know it is bounded, say by M, so

$$\phi(r) \left(\int_{\partial B_r(\mathbf{x})} \left(\frac{\partial u}{\partial \mathbf{n}}\right)(\mathbf{y}) d\sigma(\mathbf{y}) \right) = -\phi(r) \int_{B_r(\mathbf{x})} f(\mathbf{y}) d\mathbf{y} \leq \phi(r)\alpha(n) r^n M \to 0$$

as $r \to 0$.

Since $n\alpha(n)r^{n-1}$ is the surface area of $\partial B_r(\mathbf{x})$ and u is continuous,

$$\lim_{r \to 0} \left(\frac{1}{n\alpha(n) r^{n-1}} \int_{\partial B_r(\mathbf{x})} u(\mathbf{y}) d\sigma(\mathbf{y}) \right) = u(\mathbf{x}).$$

Putting these facts together, and taking the limit $r \to 0$ we see the lemma is proved. [It's worth observing that since f is bounded the integral $\int_{\Omega_r} f(\mathbf{y})\Phi(\mathbf{y}-\mathbf{x})d\mathbf{y}$ converges absolutely to $\int_\Omega f(\mathbf{y})\Phi(\mathbf{y}-\mathbf{x})d\mathbf{y}$ as $r \to 0$, since the singularity of Φ is absolutely integrable.] □

6.3 We wish to prove the following theorem.

Theorem. *Let $\Omega \subset \mathbf{R}^n$ be an open bounded set with C^2 boundary, and suppose $h \in C^2(\partial\Omega)$ and $f \in C(\overline{\Omega})$. If G is a Green's function for the Laplacian in Ω then the solution of the boundary value problem*

$$\begin{cases} \Delta u = f & \text{in } \Omega, \text{ and} \\ u = h & \text{on } \partial\Omega, \end{cases}$$

is given by

$$u(\mathbf{x}) = -\int_{\partial\Omega} \left(\frac{\partial G(\mathbf{x}, \cdot)}{\partial \mathbf{n}}(\mathbf{y}) \right) h(\mathbf{y}) d\sigma(\mathbf{y}) - \int_\Omega f(\mathbf{y}) G(\mathbf{x}, \mathbf{y}) d\mathbf{y}. \tag{2}$$

where $(\partial G(\mathbf{x}, \cdot)/\partial \mathbf{n})(\mathbf{y}) := \mathbf{n}(\mathbf{y}) \cdot \nabla_{\mathbf{y}} G(\mathbf{x}, \mathbf{y})$ is the normal derivative of $\mathbf{y} \mapsto G(\mathbf{x}, \mathbf{y})$.

Proof. Green's second identity (5.10) applied to the functions u and $\mathbf{y} \mapsto G(\mathbf{x}, \mathbf{y}) - \Phi(\mathbf{y}-\mathbf{x})$ tells us

$$-\int_\Omega (G(\mathbf{x}, \mathbf{z}) - \Phi(\mathbf{z}-\mathbf{x})) f(\mathbf{z}) d\mathbf{z} = \int_{\partial\Omega} u(\mathbf{z}) \frac{\partial (G(\mathbf{x}, \cdot) - \Phi(\cdot - \mathbf{x}))}{\partial \mathbf{n}}(\mathbf{z}) - (G(\mathbf{x}, \mathbf{z}) - \Phi(\mathbf{z}-\mathbf{x})) \frac{\partial u}{\partial \mathbf{n}}(\mathbf{z}) d\sigma(\mathbf{z}).$$

and from (1) we have

$$u(\mathbf{x}) = \int_{\partial\Omega} \left\{ \Phi(\mathbf{z}-\mathbf{x}) \left(\frac{\partial u}{\partial \mathbf{n}}\right)(\mathbf{z}) - \left(\frac{\partial \Phi}{\partial \mathbf{n}}\right)(\mathbf{z}-\mathbf{x}) u(\mathbf{z}) \right\} d\sigma(\mathbf{z}) - \int_\Omega f(\mathbf{z}) \Phi(\mathbf{z}-\mathbf{x}) d\mathbf{z}.$$

Taking the difference of the two equalities and using the fact that $G(\mathbf{x}, \mathbf{z}) = 0$ for $\mathbf{z} \in \partial\Omega$ gives (2). □

6.4 Let Ω be an open bounded set with C^2 boundary.

(a) i. Fix $\mathbf{x} \in \Omega$. Suppose we have two Green's functions G_1 and G_2 for the Laplacian in Ω. By Definition 5.10 we know that both

$$\mathbf{y} \mapsto G_1(\mathbf{x}, \mathbf{y}) - \Phi(\mathbf{y} - \mathbf{x}) \quad \text{and} \quad \mathbf{y} \mapsto G_2(\mathbf{x}, \mathbf{y}) - \Phi(\mathbf{y} - \mathbf{x})$$

have continuous extensions which belongs to $C^2(\overline{\Omega})$ and are harmonic in Ω. Here Φ is the fundamental solution defինted by (5.11). Thus the difference of these two functions

$$\mathbf{y} \mapsto (G_1(\mathbf{x}, \mathbf{y}) - \Phi(\mathbf{y} - \mathbf{x})) - (G_2(\mathbf{x}, \mathbf{y}) - \Phi(\mathbf{y} - \mathbf{x})) = G_1(\mathbf{x}, \mathbf{y}) - G_2(\mathbf{x}, \mathbf{y})$$

also has a continuous extension which belongs to $C^2(\overline{\Omega})$ and is harmonic in Ω.

ii. By Definition 5.10 we know that $G_1(\mathbf{x}, \mathbf{y}) = G_2(\mathbf{x}, \mathbf{y}) = 0$ for $\mathbf{y} \in \partial\Omega$. Thus $\mathbf{y} \mapsto G_1(\mathbf{x}, \mathbf{y}) - G_2(\mathbf{x}, \mathbf{y})$ solves (5.2) with $f = 0$ and $g = 0$. Since we know from Section 5.2, there is at most one continuous solution to (5.2) and the zero function is also a solution, we can conclude that $G_1(\mathbf{x}, \mathbf{y}) - G_2(\mathbf{x}, \mathbf{y}) = 0$. Thus there is at most one Green's function for a given Ω.

(b) i. Fix $\mathbf{x}, \mathbf{y} \in \Omega$ with $\mathbf{x} \neq \mathbf{y}$ and consider the functions $\mathbf{z} \mapsto u(\mathbf{z}) := G(\mathbf{x}, \mathbf{z})$ and $\mathbf{z} \mapsto v(\mathbf{z}) := G(\mathbf{y}, \mathbf{z})$. We apply Green's second identity (5.10) to u and v in the domain $\Omega_r := \Omega \setminus (B_r(\mathbf{x}) \cup B_r(\mathbf{y}))$ for $r > 0$ so small that $(B_r(\mathbf{x}) \cup B_r(\mathbf{y})) \subset \Omega$ and $\overline{B_r(\mathbf{x})} \cap \overline{B_r(\mathbf{y})} = \emptyset$. We obtain

$$\begin{aligned}
0 &= \int_{\partial\Omega_r} G(\mathbf{x}, \mathbf{z}) \frac{\partial G(\mathbf{y}, \cdot)}{\partial \mathbf{n}}(\mathbf{z}) - G(\mathbf{y}, \mathbf{z}) \frac{\partial G(\mathbf{x}, \cdot)}{\partial \mathbf{n}}(\mathbf{z}) d\sigma(\mathbf{z}) \\
&= \int_{\partial\Omega} G(\mathbf{x}, \mathbf{z}) \frac{\partial G(\mathbf{y}, \cdot)}{\partial \mathbf{n}}(\mathbf{z}) - G(\mathbf{y}, \mathbf{z}) \frac{\partial G(\mathbf{x}, \cdot)}{\partial \mathbf{n}}(\mathbf{z}) d\sigma(\mathbf{z}) \\
&\quad + \int_{\partial B_r(\mathbf{x})^c} G(\mathbf{x}, \mathbf{z}) \frac{\partial G(\mathbf{y}, \cdot)}{\partial \mathbf{n}}(\mathbf{z}) - G(\mathbf{y}, \mathbf{z}) \frac{\partial G(\mathbf{x}, \cdot)}{\partial \mathbf{n}}(\mathbf{z}) d\sigma(\mathbf{z}) \\
&\quad + \int_{\partial B_r(\mathbf{y})^c} G(\mathbf{x}, \mathbf{z}) \frac{\partial G(\mathbf{y}, \cdot)}{\partial \mathbf{n}}(\mathbf{z}) - G(\mathbf{y}, \mathbf{z}) \frac{\partial G(\mathbf{x}, \cdot)}{\partial \mathbf{n}}(\mathbf{z}) d\sigma(\mathbf{z}) \\
&= -\int_{\partial B_r(\mathbf{x})} G(\mathbf{x}, \mathbf{z}) \frac{\partial G(\mathbf{y}, \cdot)}{\partial \mathbf{n}}(\mathbf{z}) - G(\mathbf{y}, \mathbf{z}) \frac{\partial G(\mathbf{x}, \cdot)}{\partial \mathbf{n}}(\mathbf{z}) d\sigma(\mathbf{z}) \\
&\quad - \int_{\partial B_r(\mathbf{y})} G(\mathbf{x}, \mathbf{z}) \frac{\partial G(\mathbf{y}, \cdot)}{\partial \mathbf{n}}(\mathbf{z}) - G(\mathbf{y}, \mathbf{z}) \frac{\partial G(\mathbf{x}, \cdot)}{\partial \mathbf{n}}(\mathbf{z}) d\sigma(\mathbf{z}).
\end{aligned}$$

since $G(\mathbf{x}, \mathbf{z}) = G(\mathbf{y}, \mathbf{z}) = 0$ for $z \in \partial\Omega$.

ii. Since we know that $\mathbf{z} \mapsto G(\mathbf{x}, \mathbf{z}) - \Phi(\mathbf{z} - \mathbf{x})$ has a $C^2(\overline{\Omega})$ extension, we know that $\mathbf{z} \mapsto G(\mathbf{x}, \mathbf{z}) - \Phi(\mathbf{z} - \mathbf{x})$ is bounded near $\mathbf{x}$. We also know that $\partial G(\mathbf{y}, \cdot)/\partial \mathbf{n}$ is bounded near $\mathbf{x}$, since $\mathbf{x} \neq \mathbf{y}$. This means that the integrand in

$$\int_{\partial B_r(\mathbf{x})} (G(\mathbf{x}, \mathbf{z}) - \Phi(\mathbf{z} - \mathbf{x})) \frac{\partial G(\mathbf{y}, \cdot)}{\partial \mathbf{n}}(\mathbf{z}) d\sigma(\mathbf{z})$$

is bounded. This fact together with the fact that the measure of $\partial B_r(\mathbf{x})$ tends to zero as $r \to 0$, means the integral above tends to zero as $r \to 0$. By the same logic

$$\int_{\partial B_r(\mathbf{x})} G(\mathbf{y}, \mathbf{z}) \frac{\partial (G(\mathbf{x}, \cdot) - \Phi(\cdot - \mathbf{x}))}{\partial \mathbf{n}}(\mathbf{z}) d\sigma(\mathbf{z}) \to 0$$

as $r \to 0$.

iii. Since Φ is a radial function, we can write $\Phi(\mathbf{z} - \mathbf{x}) = \phi(r)$ when $\mathbf{z} \in \partial B_r(\mathbf{x})$. This fact and the divergence theorem give

$$\begin{aligned}
&\int_{\partial B_r(\mathbf{x})} \Phi(\mathbf{z} - \mathbf{x}) \frac{\partial G(\mathbf{y}, \cdot)}{\partial \mathbf{n}}(\mathbf{z}) d\sigma(\mathbf{z}) \\
&= \phi(r) \int_{\partial B_r(\mathbf{x})} \frac{\partial G(\mathbf{y}, \cdot)}{\partial \mathbf{n}}(\mathbf{z}) d\sigma(\mathbf{z}) \\
&= \phi(r) \int_{\partial B_r(\mathbf{x})} \Delta(G(\mathbf{y}, \cdot))(\mathbf{z}) d\sigma(\mathbf{z}) = 0.
\end{aligned}$$

By (5.12) (homework question 6.1) we can compute

$$\begin{aligned}
&\int_{\partial B_r(\mathbf{x})} G(\mathbf{y},\mathbf{z})\frac{\partial \Phi}{\partial \mathbf{n}}(\mathbf{z}-\mathbf{x})d\sigma(\mathbf{z}) \\
&= \int_{\partial B_r(\mathbf{x})} G(\mathbf{y},\mathbf{z})\frac{-1}{n\alpha(n)}\frac{1}{|\mathbf{z}-\mathbf{x}|^{n-1}}d\sigma(\mathbf{z}) \\
&= \frac{-1}{n\alpha(n)}\frac{1}{r^{n-1}}\int_{\partial B_r(\mathbf{x})} G(\mathbf{y},\mathbf{z})d\sigma(\mathbf{z}) \\
&= -G(\mathbf{y},\mathbf{x})
\end{aligned}$$

by the mean value property for harmonic functions (Theorem 5.6).

iv. We can compute

$$\begin{aligned}
&\int_{\partial B_r(\mathbf{x})} G(\mathbf{x},\mathbf{z})\frac{\partial G(\mathbf{y},\cdot)}{\partial \mathbf{n}}(\mathbf{z}) - G(\mathbf{y},\mathbf{z})\frac{\partial G(\mathbf{x},\cdot)}{\partial \mathbf{n}}(\mathbf{z})d\sigma(\mathbf{z}) \\
&= \int_{\partial B_r(\mathbf{x})} (G(\mathbf{x},\mathbf{z})-\Phi(\mathbf{z}-\mathbf{x}))\frac{\partial G(\mathbf{y},\cdot)}{\partial \mathbf{n}}(\mathbf{z})d\sigma(\mathbf{z}) + \int_{\partial B_r(\mathbf{x})} \Phi(\mathbf{z}-\mathbf{x})\frac{\partial G(\mathbf{y},\cdot)}{\partial \mathbf{n}}(\mathbf{z})d\sigma(\mathbf{z}) \\
&\quad - \int_{\partial B_r(\mathbf{x})} G(\mathbf{y},\mathbf{z})\left(\frac{\partial G(\mathbf{x},\cdot)}{\partial \mathbf{n}}(\mathbf{z}) - \frac{\partial \Phi}{\partial \mathbf{n}}(\mathbf{z}-\mathbf{x})\right)d\sigma(\mathbf{z}) - \int_{\partial B_r(\mathbf{x})} G(\mathbf{y},\mathbf{z})\frac{\partial \Phi}{\partial \mathbf{n}}(\mathbf{z}-\mathbf{x})d\sigma(\mathbf{z}) \\
&\to 0+0-0+G(\mathbf{y},\mathbf{x})
\end{aligned}$$

as $r \to 0$ by our work above.

v. The statement we obtain is

$$\int_{\partial B_r(\mathbf{y})} G(\mathbf{y},\mathbf{z})\frac{\partial G(\mathbf{x},\cdot)}{\partial \mathbf{n}}(\mathbf{z}) - G(\mathbf{x},\mathbf{z})\frac{\partial G(\mathbf{y},\cdot)}{\partial \mathbf{n}}(\mathbf{z})d\sigma(\mathbf{z}) \to G(\mathbf{x},\mathbf{y}).$$

which is the negative of the second term on the right-hand side of (†). Thus, in the limit $r \to 0$, (†) becomes $0 = G(\mathbf{y},\mathbf{x}) - G(\mathbf{x},\mathbf{y})$.

Partial Differential Equations (TATA27)
Spring Semester 2019
Homework 7

Review of previous seminar

In seminars 8 and 9 we covered section 6.1, 6.2 and 6.3. After checking through the notes to see that you are somewhat familiar with the material, try the following exercises.

7.1 Solve (6.1) with:

(a) $g(x) = e^x$ and $h(x) = \sin x$;

(b) $g(x) = \log(1 + x^2)$ and $h(x) = 4 + x$.

7.2 Suppose both g and h are odd functions and u is the solution of (6.1). Show that $u(\cdot, t)$ is also odd for each $t > 0$.

7.3 By factorising the operator as we did in Section 6.1, solve the following initial value problems.

(a)

$$\begin{cases} \partial_{tt}u(x,t) - 3\partial_{xt}u(x,t) - 4\partial_{xx}u(x,t) = 0 & \text{for } x \in \mathbf{R} \text{ and } t > 0, \\ u(x,0) = x^2 \quad \text{and} \quad \partial_t u(x,0) = e^x & \text{for } x \in \mathbf{R}. \end{cases}$$

(b)

$$\begin{cases} \partial_{tt}u(x,t) + \partial_{xt}u(x,t) - 20\partial_{xx}u(x,t) = 0 & \text{for } x \in \mathbf{R} \text{ and } t > 0, \\ u(x,0) = x^2 \quad \text{and} \quad \partial_t u(x,0) = e^x & \text{for } x \in \mathbf{R}. \end{cases}$$

7.4 For a smooth solution u of the wave equation $\partial_{tt}u(x,t) - \partial_{xx}u(x,t) = 0$ (with $\rho = T = c = 1$, $x \in \mathbf{R}$ and $t > 0$), the energy density is defined to be

$$e(x,t) = \frac{1}{2}((\partial_t u(x,t))^2 + (\partial_x u(x,t))^2)$$

and the momentum density

$$p(x,t) = \partial_t u(x,t)\partial_x u(x,t).$$

(a) Show that $\partial e/\partial t = \partial p/\partial x$ and $\partial p/\partial t = \partial e/\partial x$.

(b) Show that e and p also satsify the wave equation.

7.5 Suppose that u is a solution of the wave equation $\partial_{tt}u(x,t) - c^2\partial_{xx}u(x,t) = 0$ ($x \in \mathbf{R}$, $t > 0$).

(a) Show that for a fixed $y \in \mathbf{R}$, v defined by $v(x,t) = u(x - y, t)$ is also a solution of the wave equation.

(b) Show that for a fixed $a \in \mathbf{R}$, w defined by $w(x,t) = u(ax, at)$ is also a solution of the wave equation.

Group work

Now try this exercise. Please try to discuss your solution with others taking the course.

7.6 Consider a solution u to the *damped string equation*

$$\partial_{tt}u(x,t) - c^2\partial_{xx}u(x,t) + r\partial_t u(x,t) = 0 \quad (x \in \mathbf{R},\ t > 0)$$

for $c^2 = T/\rho$ and $T, \rho, r > 0$. Define the energy by the same formula we used in class:

$$E[u](t) = \frac{1}{2}\int_{-\infty}^{\infty} \rho(\partial_t u(x,t))^2 + T(\partial_x u(x,t))^2 dx.$$

Assuming u and its derivatives are sufficiently smooth and decay as $x \to \pm\infty$, show that the energy $E[u]$ is a non-increasing function.

Partial Differential Equations (TATA27)
Spring Semester 2019
Solutions to Homework 7

7.1 Recall that the solution to (6.1) is given by (6.3):

$$u(x,t) = \frac{1}{2}\left(g(x+ct) + g(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} h(y)dy.$$

(a) If we take $g(x) = e^x$ and $h(x) = \sin x$ then

$$u(x,t) = \frac{1}{2}\left(e^{x+ct} + e^{x-ct}\right) - \frac{1}{2c}\left(\cos(x+ct) - \cos(x-ct)\right).$$

(b) If we take $g(x) = \log(1+x^2)$ and $h(x) = 4 + x$ then

$$u(x,t) = \frac{1}{2}\left(\log(x+ct) + \log(x-ct)\right) + \frac{4ct + xct}{c}.$$

7.2 Using (6.3), we have

$$u(-x,t) = \frac{1}{2}\left(g(-x+ct) + g(-x-ct)\right) + \frac{1}{2c}\int_{-x-ct}^{-x+ct} h(y)dy$$

But if g and h are odd functions then

$$\begin{aligned} u(-x,t) &= -\frac{1}{2}\left(g(x-ct) + g(x+ct)\right) - \frac{1}{2c}\int_{-x-ct}^{-x+ct} h(-y)dy \\ &= -\frac{1}{2}\left(g(x+ct) + g(x-ct)\right) + \frac{1}{2c}\int_{x+ct}^{x-ct} h(z)dz \\ &= -\frac{1}{2}\left(g(x+ct) + g(x-ct)\right) - \frac{1}{2c}\int_{x-ct}^{x+ct} h(z)dz \\ &= -u(x,t), \end{aligned}$$

so $u(\cdot,t)$ is an odd function for each t.

7.3 (a) We can factorise the differential equation as $(\partial_t + \partial_x)(\partial_t - 4\partial_x)u(x,t) = 0$ and so can then view it as the system

$$\begin{cases} (\partial_t + \partial_x)v(x,t) = 0; \\ (\partial_t - 4\partial_x)u(x,t) = v(x,t). \end{cases}$$

Using the method of characteristics, we can see that the general solution to the first equation is $v(x,t) = f(x-t)$ where f is any differentiable function.

To solve the second equation we first solve the homogeneous equation associated to it. Again, the method of characteristics shows the general solution to the homogeneous equation $(\partial_t - 4\partial_x)u_h(x,t) = 0$ is $u_h(x,t) = g(x+4t)$ where g is any differentiable function. It is easy to find a particular solution to the second equation by starting with the anzats $u_p(x,t) = V(x-t)$. Substituting this into the equation gives

$$-V'(x-t) - 4V'(x-t) = f(x-t),$$

so a particular solution is $u_p(x,t) = V(x-t)$ where $V(y) = -(1/5)\int f(y)dy$. We add u_p and u_h together to obtain the general solution: $u(x,t) = V(x-t) + g(x+4t)$.

Now we can make use of the initial conditions to choose g and V. We see that we require

$$\begin{cases} u(x,0) = V(x) + g(x) = x^2; \\ \partial_t u(x,0) = -V'(x) + 4g'(x) = e^x. \end{cases}$$

It is convenient to differentiate the first equation, so we obtain two conditions for V' and g':

$$\begin{cases} V'(x) + g'(x) = 2x; \\ -V'(x) + 4g'(x) = e^x. \end{cases}$$

This is easy to solve and tells us that $V'(x) = (8x - e^x)/5$ and $g'(x) = (2x + e^x)/5$. This means that $V(x,t) = (4x^2 - e^x)/5 + C_1$ and $g(x) = (x^2 + e^x)/5 + C_1$ for some constants C_1 and C_2, but the intial condition $u(x,0) = x^2$ tells us that $C_1 = -C_2$, so

$$u(x,t) = \frac{4(x-t)^2 - e^{x-t} + (x+4t)^2 + e^{x+4t}}{5}$$

(b) We can factorise the differential equation as $(\partial_t - 4\partial_x)(\partial_t + 5\partial_x)u(x,t) = 0$ and so can then view it as the system

$$\begin{cases} (\partial_t - 4\partial_x)v(x,t) = 0; \\ (\partial_t + 5\partial_x)u(x,t) = v(x,t). \end{cases}$$

Using the method of characteristics just as before, we can see that the general solution to the first equation is $v(x,t) = f(x+4t)$ where f is any differentiable function.

To solve the second equation we first again solve the homogeneous equation associated to it. The method of characteristics shows the general solution to the homogeneous equation $(\partial_t + 5\partial_x)u_h(x,t) = 0$ is $u_h(x,t) = g(x-5t)$ where g is any differentiable function. A particular solution to the second equation is found by starting with the anzats $u_p(x,t) = V(x+4t)$. Substituting this into the equation gives

$$4V'(x+4t) + 5V'(x+4t) = f(x+4t),$$

so a particular solution is $u_p(x,t) = V(x+4t)$ where $V(y) = (1/9)\int f(y)dy$. We add u_p and u_h together to obtain the general solution: $u(x,t) = V(x+4t) + g(x-5t)$.

Now we can make use of the initial conditions to choose g and V. We see that we require

$$\begin{cases} u(x,0) = V(x) + g(x) = x^2; \\ \partial_t u(x,0) = 4V'(x) - 5g'(x) = e^x. \end{cases}$$

It is convenient to differentiate the first equation, so we obtain two conditions for V' and g':

$$\begin{cases} V'(x) + g'(x) = 2x; \\ 4V'(x) - 5g'(x) = e^x. \end{cases}$$

This tells us that $g'(x) = (8x - e^x)/9$ and $V'(x) = (10x + e^x)/9$. This means that $g(x,t) = (4x^2 - e^x)/9 + C_1$ and $V(x) = (5x^2 + e^x)/9 + C_2$ for some constants C_1 and C_2, but the intial condition $u(x,0) = x^2$ tells us that $C_1 = -C_2$, so

$$u(x,t) = \frac{4(x-5t)^2 - e^{x-5t} + 5(x+4t)^2 + e^{x+4t}}{9}$$

7.4 For a solution u of the wave equation $\partial_{tt}u - \partial_{xx}u = 0$ (with $\rho = T = c = 1$), the energy density is defined to be

$$e(x,t) = \frac{1}{2}((\partial_t u(x,t))^2 + (\partial_x u(x,t))^2)$$

and the momentum density

$$p(x,t) = \partial_t u(x,t)\partial_x u(x,t).$$

(a) We compute

$$\begin{aligned} \frac{\partial e}{\partial t}(x,t) &= \partial_t u(x,t)\partial_{tt}u(x,t) + \partial_x u(x,t)\partial_{xt}u(x,t) \\ \frac{\partial e}{\partial x}(x,t) &= \partial_t u(x,t)\partial_{tx}u(x,t) + \partial_x u(x,t)\partial_{xx}u(x,t) \\ \frac{\partial p}{\partial t}(x,t) &= \partial_{tt}u(x,t)\partial_x u(x,t) + \partial_t u(x,t)\partial_{xt}u(x,t) \\ \frac{\partial p}{\partial x}(x,t) &= \partial_{tx}u(x,t)\partial_x u(x,t) + \partial_t u(x,t)\partial_{xx}u(x,t) \end{aligned}$$

So clearly if $\partial_{tt}u - \partial_{xx}u = 0$ then $\partial e/\partial t = \partial p/\partial x$ and $\partial p/\partial t = \partial e/\partial x$.

(b) Furthermore, using the above results, if $\partial_{tt}u - \partial_{xx}u = 0$ we have

$$\begin{aligned}\frac{\partial^2 e}{\partial t^2}(x,t) &= \frac{\partial^2 p}{\partial t \partial x}(x,t) = \frac{\partial^2 p}{\partial x \partial t}(x,t) = \frac{\partial^2 e}{\partial x^2}(x,t) \quad \text{and} \\ \frac{\partial^2 p}{\partial t^2}(x,t) &= \frac{\partial^2 e}{\partial t \partial x}(x,t) = \frac{\partial^2 e}{\partial x \partial t}(x,t) = \frac{\partial^2 p}{\partial x^2}(x,t).\end{aligned}$$

7.5 Suppose that u is a solution of the wave equation, so $\partial_2^2 u - c^2\partial_1^2 u = 0$.

(a) For a fixed $y \in \mathbf{R}$, $\partial_x v(x,t) = \partial_1 u(x-y,t)$, $\partial_{xx}v(x,t) = \partial_1^2 u(x-y,t)$, $\partial_t v(x,t) = \partial_2 u(x-y,t)$ and $\partial_{tt}v(x,t) = \partial_2^2 u(x-y,t)$, Thus

$$\partial_{tt}v(x,t) - c^2\partial_{xx}v(x,t) = \partial_2^2 u(x-y,t) - c^2\partial_1^2 u(x-y,t) = (\partial_2^2 u - c^2\partial_1^2 u)(x-y,t) = 0.$$

(b) For a fixed $a \in \mathbf{R}$, $\partial_x w(x,t) = a\partial_1 u(ax,at)$, $\partial_{xx}w(x,t) = a^2\partial_1^2 u(ax,at)$, $\partial_t w(x,t) = a\partial_2 u(ax,at)$ and $\partial_{tt}w(x,t) = a^2\partial_2^2 u(ax,at)$, Thus

$$\partial_{tt}w(x,t) - c^2\partial_{xx}w(x,t) = a^2\partial_2^2 u(ax,at) - c^2a^2\partial_1^2 u(ax,at) = a^2(\partial_2^2 u - c^2\partial_1^2 u)(ax,at) = 0.$$

7.6 Consider a solution u to the *damped string equation*

$$\partial_{tt}u(x,t) - c^2\partial_{xx}u(x,t) + r\partial_t u(x,t) = 0$$

for $c^2 = T/\rho$ and $T, \rho, r > 0$. Define the energy by the same formula we used in class:

$$E[u](t) = \frac{1}{2}\int_{-\infty}^{\infty} \rho(\partial_t u(x,t))^2 + T(\partial_x u(x,t))^2 dx.$$

We have

$$\begin{aligned}\frac{d}{dt}\left(\frac{1}{2}\int_{-\infty}^{\infty} \rho(\partial_t u(x,t))^2 dx\right) &= \frac{1}{2}\int_{-\infty}^{\infty} \rho\partial_t u(x,t)\partial_{tt}u(x,t)dx \\ &= \frac{1}{2}\int_{-\infty}^{\infty} \partial_t u(x,t)(T\partial_{xx}u(x,t) - r\rho\partial_t u(x,t))dx \\ &= -\frac{1}{2}\int_{-\infty}^{\infty} T\partial_{tx}u(x,t)\partial_x u(x,t)dx - \frac{1}{2}\int_{-\infty}^{\infty} r\rho(\partial_t u(x,t))^2 dx \\ &= -\frac{d}{dt}\left(\frac{1}{2}\int_{-\infty}^{\infty} T(\partial_x u(x,t))^2 dx\right) - \frac{1}{2}\int_{-\infty}^{\infty} r\rho(\partial_t u(x,t))^2 dx.\end{aligned}$$

Therefore

$$E[u]'(t) = \frac{d}{dt}\left(\frac{1}{2}\int_{-\infty}^{\infty} \rho(\partial_t u(x,t))^2 + T(\partial_x u(x,t))^2 dx\right) = -\frac{1}{2}\int_{-\infty}^{\infty} r\rho(\partial_t u(x,t))^2 dx \le 0,$$

hence the energy $E[u]$ is a non-increasing function.

Partial Differential Equations (TATA27)
Spring Semester 2019
Homework 8

Review of previous seminars

In seminars 10 and 11 we covered section 6.5. After checking through the notes to see that you are familiar with the material, try the following exercises.

8.1 Use (6.14) to find a solution to the wave equation (6.10) in three dimensions with initial conditions given by:

(a) $\phi(x, y, z) = 0$ and $\psi(x, y, z) = 1$ for all $(x, y, z) \in \mathbf{R}^3$; and

(b) $\phi(x, y, z) = 0$ and $\psi(x, y, z) = y$ for all $(x, y, z) \in \mathbf{R}^3$.

8.2 Use (6.16) to find a solution to the wave equation (6.15) in two dimensions with initial conditions given by $\phi(x, y) = 0$ and $\psi(x, y) = A$ for all $(x, y) \in \mathbf{R}^2$ and some constant $A \in \mathbf{R}$.

8.3 Given a function $u \colon \mathbf{R}^3 \to \mathbf{R}$ we can define a new function $\overline{u} \colon \mathbf{R}^3 \to \mathbf{R}$ via the formula

$$\overline{u}(\mathbf{x}) = \frac{1}{4\pi r^2} \int_{|\mathbf{y}|=r} u(\mathbf{y}) d\sigma(\mathbf{y})$$

where $r = |\mathbf{x}|$. (The function $\overline{u}$ is said to be *radial* because $\overline{u}(\mathbf{x})$ depends only on $|\mathbf{x}|$.) Prove that $\Delta\overline{u} = \overline{\Delta u}$. [Hint: Here it is easier to compute Δ using spherical polar coordinates.]

8.4 Show that formula (6.8) can be rewritten as

$$v(x, t) = \frac{\partial}{\partial t}\left(\frac{1}{2c}\int_{ct-x}^{ct+x} g_{\text{odd}}(y)dy\right) + \frac{1}{2c}\int_{ct-x}^{ct+x} h_{\text{odd}}(y)dy$$

when $0 \le x \le ct$.

Group work

8.5 A solution $u \colon \mathbf{R}^3 \times [0, \infty) \to \mathbf{R}$ to the wave equation (6.10) in three dimensions is called spherical if $u(\mathbf{x}, t) = u_0(|\mathbf{x}|, t)$ for some function $u_0 \colon \mathbf{R} \times [0, \infty) \to \mathbf{R}$ (that is, if it is radial in its spatial variables). This question investigates what form spherical solutions to the wave equation must take.

(a) By arguing similarly to how we proved d'Alembert's formula (6.3) show that an arbitrary solution to the wave equation

$$\partial_t^2 v(r, t) - \partial_r^2 v(r, t) = 0 \quad \text{for } r \in \mathbf{R} \text{ and } t > 0$$

in one dimension has the form

$$v(r, t) = f(x - t) + g(x + t)$$

for some functions $f \colon \mathbf{R} \to \mathbf{R}$ and $g \colon \mathbf{R} \to \mathbf{R}$.

(b) By making use of (6.12), show that spherical solutions of the wave equation (6.10) have the form

$$u(\mathbf{x}, t) = \frac{f(|\mathbf{x}| - t) + g(|\mathbf{x}| + t)}{|\mathbf{x}|}. \qquad (*)$$

(c) Comment on the smoothness of $(*)$ in relation to the smoothness of f and g. Is there any difference between the one and three dimensional cases?

Partial Differential Equations (TATA27)
Spring Semester 2019
Solutions 8

8.1 We recall that in lectures we claimed

$$u(\mathbf{x},t) = \frac{\partial}{\partial t}\left(\frac{1}{4\pi t}\int_{|\mathbf{y}-\mathbf{x}|=t}\phi(\mathbf{y})d\sigma(\mathbf{y})\right) + \frac{1}{4\pi t}\int_{|\mathbf{y}-\mathbf{x}|=t}\psi(\mathbf{y})d\sigma(\mathbf{y}), \tag{6.14}$$

is a solution to (6.10).

(a) We compute that if $\phi(x,y,z) = 0$ and $\psi(x,y,z) = 1$ for all $(x,y,z) \in \mathbf{R}^3$, then according to (6.14)

$$\begin{aligned} u(\mathbf{x},t) &= \frac{\partial}{\partial t}\left(\frac{1}{4\pi t}\int_{|\mathbf{y}-\mathbf{x}|=t} 0\, d\sigma(\mathbf{y})\right) + \frac{1}{4\pi t}\int_{|\mathbf{y}-\mathbf{x}|=t} 1\, d\sigma(\mathbf{y}) \\ &= 0 + \frac{4\pi t^2}{4\pi t} = t. \end{aligned}$$

(b) Equally, we compute that if $\phi(\mathbf{x}) = 0$ and $\psi(\mathbf{x}) = y$ for all $\mathbf{x} = (x,y,z) \in \mathbf{R}^3$, then according to (6.14)

$$u(\mathbf{x},t) = \frac{\partial}{\partial t}\left(\frac{1}{4\pi t}\int_{|\mathbf{y}-\mathbf{x}|=t} 0\, d\sigma(\mathbf{y})\right) + \frac{1}{4\pi t}\int_{|\mathbf{y}-\mathbf{x}|=t} v\, d\sigma(\mathbf{y}) = frac14\pi t\int_{|\mathbf{y}-\mathbf{x}|=t} v\, d\sigma(\mathbf{y})$$

where we write $\mathbf{y} = (u,v,w) \in \mathbf{R}^3$. To simplify the integral above we observe that $\mathbf{y} \mapsto v$ is harmonic, so Theorem 5.6 gives that

$$\frac{1}{4\pi t}\int_{|\mathbf{y}-\mathbf{x}|=t} v\, d\sigma(\mathbf{y}) = t\left(\frac{1}{4\pi t^2}\int_{|\mathbf{y}-\mathbf{x}|=t} v\, d\sigma(\mathbf{y})\right) = ty,$$

so $u(\mathbf{x},t) = ty$ for $\mathbf{x} = (x,y,z) \in \mathbf{R}^3$.

8.2 The formula

$$\begin{aligned} u(x,y,t) = &\frac{\partial}{\partial t}\left(\frac{1}{2\pi}\int_{a^2+b^2\le t^2}\frac{\phi(a+x,b+y)}{\sqrt{t^2-a^2-b^2}}dadb\right) \\ &+ \frac{1}{2\pi}\int_{a^2+b^2\le t^2}\frac{\psi(a+x,b+y)}{\sqrt{t^2-a^2-b^2}}dadb \end{aligned} \tag{6.16}$$

with $\phi(x,y) = 0$ and $\psi(x,y) = A$ for all $(x,y) \in \mathbf{R}^2$ reads

$$\begin{aligned} u(x,y,t) &= \frac{1}{2\pi}\int_{a^2+b^2\le t^2}\frac{A}{\sqrt{t^2-a^2-b^2}}dadb = \frac{t^2}{2\pi t}\int_{a^2+b^2\le 1}\frac{A}{\sqrt{1-a^2-b^2}}dadb \\ &= \frac{1}{2\pi}\int_0^1\int_0^{2\pi}\frac{A}{\sqrt{1-r^2}}rd\theta dr = t\int_0^1\frac{Ar}{\sqrt{1-r^2}}dr = -At\sqrt{1-r^2}\Big|_0^1 = At \end{aligned}$$

8.3 We apply the change of variables $\mathbf{y} = r\mathbf{z}$ to obtain

$$\overline{u}(\mathbf{x}) = \frac{1}{4\pi r^2}\int_{|\mathbf{y}|=r} u(\mathbf{y})d\sigma(\mathbf{y}) = \frac{1}{4\pi}\int_{|\mathbf{z}|=1} u(r\mathbf{z})d\sigma(\mathbf{z})$$

where $r = |\mathbf{x}|$. Therefore, recalling that the Laplacian in spherical coordinates is

$$\Delta = \Delta_{(r,\theta,\phi)} = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2},$$

we see that

$$\Delta\overline{u}(\mathbf{x}) = \frac{1}{4\pi}\int_{|\mathbf{z}|=1}\Delta_{(r,\theta,\phi)}u(r\mathbf{z})d\sigma(\mathbf{z}) = \frac{1}{4\pi}\int_{|\mathbf{z}|=1}\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right)u(r\mathbf{z})d\sigma(\mathbf{z}). \tag{1}$$

Writing (s,φ,ϑ) as the spherical coordinates of $\mathbf{z}$, we have

$$\left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}\right)u(r\mathbf{z})=\left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}\right)u(rs,\varphi,\vartheta)=s^2\left(\partial_1^2 u(rs,\varphi,\vartheta)+\frac{2}{rs}\partial_1 u(rs,\varphi,\vartheta)\right).$$

Since we are integrating on the unit sphere, $s=1$, and so

$$\begin{aligned}&\frac{1}{4\pi}\int_{|\mathbf{z}|=1}\left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}\right)u(r\mathbf{z})d\sigma(\mathbf{z})\\&=\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\left(\partial_1^2 u(rs,\varphi,\vartheta)+\frac{2}{rs}\partial_1 u(rs,\varphi,\vartheta)\right)\sin\vartheta d\varphi d\vartheta\\&=\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\left(\frac{\partial^2 u(r,\phi,\theta)}{\partial r^2}+\frac{2}{r}\frac{\partial u(r,\phi,\theta)}{\partial r}\right)\sin\theta d\phi d\theta\\&=\frac{1}{4\pi r^2}\int_{|\mathbf{y}|=r}\left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}\right)u(\mathbf{y})d\sigma(\mathbf{y}).\end{aligned}\tag{2}$$

Moreover

$$\begin{aligned}&\frac{1}{4\pi r^2}\int_{|\mathbf{y}|=r}\left(\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)u(\mathbf{y})d\sigma(\mathbf{y})\\&=\frac{1}{4\pi r^2}\int_0^{2\pi}\int_0^{\pi}r^2\sin\theta\left(\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)u(r,\phi,\theta)\,d\phi d\theta\\&=\frac{1}{4\pi r^2}\int_0^{2\pi}\int_0^{\pi}\left(\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}+\frac{\partial^2}{\partial\phi^2}\right)u(r,\phi,\theta)\,d\phi d\theta=0,\end{aligned}$$

since $u(r,\phi,\theta)$ is 2π-periodic in ϕ and $\sin 0=\sin\pi=0$. Combining this with (1) and (2) we find that $\Delta\overline{u}=\overline{\Delta u}$.

8.4 Since g_{odd} and h_{odd} are odd, we can write (6.8) as

$$\begin{aligned}v(x,t)&=\frac{1}{2}\left(g_{\text{odd}}(x+ct)+g_{\text{odd}}(x-ct)\right)+\frac{1}{2c}\int_{x-ct}^{x+ct}h_{\text{odd}}(y)dy\\&=\frac{1}{2}\left(g_{\text{odd}}(ct+x)-g_{\text{odd}}(ct-x)\right)+\frac{1}{2c}\left(\int_{x-ct}^{ct-x}h_{\text{odd}}(y)dy+\int_{ct-x}^{ct+x}h_{\text{odd}}(y)dy\right)\\&=\frac{\partial}{\partial t}\left(\frac{1}{2c}\int_{ct-x}^{ct+x}g_{\text{odd}}(y)dy\right)+\frac{1}{2c}\int_{ct-x}^{ct+x}h_{\text{odd}}(y)dy\end{aligned}$$

when $0\le x\le ct$.

8.5 (a) We can rewrite the wave equation

$$\partial_t^2 v(r,t)-\partial_r^2 v(r,t)=0\quad\text{for } r\in\mathbf{R}\text{ and } t>0\tag{3}$$

as the system

$$\begin{cases}\partial_t u(x,t)+\partial_x u(x,t)=0\\ \partial_t v(x,t)-\partial_x v(x,t)=u(x,t)\end{cases}$$

Via the method of characteristics, we see that a general solution to the first equation is $u(r,t)=h(r-t)$. We observe that $v(r,t)=g(r-t)$ is a solution to the second equation with $u(r,t)=h(r-t)$ provided $-2g'(r-t)=h(r-t)$. Since h was arbitrary, this says nothing more that g is differentiable. To find a general solution to the second equation we must add an arbitrary solution to the homogeneous equation $\partial_t v(x,t)-\partial_x v(x,t)=0$, which again via the method of characteristics can be seen to be $v(r,t)=f(r-t)$. Thus a general solution to (3) is

$$v(r,t)=f(r-t)+g(r+t)\tag{4}$$

for arbitrary differentiable functions $f\colon\mathbf{R}\to\mathbf{R}$ and $g\colon\mathbf{R}\to\mathbf{R}$.

(b) We can write a radial solution u as $u(\mathbf{x},t) = u_0(|\mathbf{x}|,t)$ for some $u_0\colon [0,\infty)^2 \to \mathbf{R}$. Furthermore $u_0(r,t) = \overline{u}_\mathbf{0}(r,t)$ for all $r,t > 0$, where $\overline{u}_\mathbf{0}$ is the spherical mean of u about the origin. Therefore $u_0 = \overline{u}_\mathbf{0}$ satisfies (6.12) (with $\mathbf{x} = \mathbf{0}$). By the same argument as in the notes,

$$v(r,t) := r u_0(r,t) \tag{5}$$

satisfies the wave equation (6.13) and $v(0,t) = 0$. Therefore v has the form (4) for some f and g. In order to ensure $v(0,t) = 0$ we choose f and g so that $r \mapsto v(r,t)$ is odd. One way to do this is by choosing f and g so that $f(-x) = -g(x)$ for all $x \in \mathbf{R}$. By (5) we have

$$u(\mathbf{x},t) = u_0(|\mathbf{x}|,t) = \frac{f(|\mathbf{x}|-t) + g(|\mathbf{x}|+t)}{|\mathbf{x}|}. \tag{$*$}$$

(c) At first sight it appears that $(*)$ may develop a singularity at the origin, as we divide by $|\mathbf{x}|$. However the requirement that $v(0,t) = 0$ above has the potential to mitigates this problem.

It is instructive to test a few sensible examples of f and g to see if it is possible to create a singularity at the origin. Remember we must make sure $r \mapsto f(r-t) + g(r+t)$ is odd! You should find your attempts to create a singularity are always thwarted.

Try, for example, $g(s) = s^m$ for $s \geq 0$ and $m = 2, 3$.

This is a hard problem to answer rigorously, but is a nice exercise to play with to investigate what the truth might be.

Section 2.4.1 of Evans *Partial Differetial Equations* makes concrete statements about the regularity of solutions to the wave equation. See Theorems 1 and 2 there. They state that if the initial data is sufficiently smooth, then the u will be correspondingly smooth. In broad terms there is no difference between one and three dimensions.

Partial Differential Equations (TATA27)
Spring Semester 2019
Homework 9

Review of previous seminars

In Seminar 12 we sudied Sections 7.1 and 7.2. Questions 9.1 and 9.2 are directly related to these sections. Even question 9.5 is good preparation for the next seminar, although all the remaining questions are most closely connected to Section 7.3.

9.1 Consider the initial boundary value problem

$$\begin{cases} \partial_{tt}v(x,t) - \partial_{xx}v(x,t) = 0 & \text{for } x \in (0,\ell) \text{ and } t \in (0,T], \\ v(x,0) = g(x) \quad \text{and} \quad \partial_t v(x,0) = h(x) & \text{for } x \in [0,\ell], \text{ and} \\ v(0,t) = 0 \quad \text{and} \quad v(\ell,t) = 0 & \text{for } t \in (0,T]. \end{cases} \tag{1}$$

If v solved the heat equation instead of the wave equation, then v would satify a weak maximum principle:

> The maximum value of v over the set $[0,\ell] \times [0,T]$ is attained on the set $D = ([0,\ell] \times \{0\}) \cup (\{0\} \times [0,T)) \cup (\{\ell\} \times [0,T))$.

(a) Draw a picture of the set D.

(b) Find a specific choice of functions g and h, and $T > 0$, together with a solution v to (1), which prove that such a weak maximum principle for the wave equation is false.

9.2 (a) Write down Theorem 7.1 with the words 'maximum value' replaced by 'minimum value'. Now prove this reformulation. You may use Theorem 7.1 in its original form to help you do this.

(b) Prove Theorem 7.3 using only Theorem 7.1 and 9.2(a).

(c) Use Theorem 7.1 and 9.2(a) to prove the following stability result: If u_1 and u_2 both solve (7.1) with the initial conditions ϕ_1 and ϕ_2 and boundary conditions g_1 and g_2, respectively, then

$$\max_{\mathbf{x}\in\overline{\Omega}, t\in[0,T]} |u_2(\mathbf{x},t) - u_1(\mathbf{x},t)| \leq \max_{\mathbf{x}\in\overline{\Omega}} |\phi_2(\mathbf{x}) - \phi_1(\mathbf{x})| + \sup_{\mathbf{x}\in\partial\Omega, t\in(0,T]} |g_2(\mathbf{x},t) - g_1(\mathbf{x},t)|.$$

9.3 Recall that the heat kernel $S\colon \mathbf{R} \times (0,\infty) \to \mathbf{R}$ is defined by the formula

$$S(x,t) = \frac{1}{2\sqrt{\pi t}} e^{-x^2/4t}.$$

If $I = \int_{-\infty}^{\infty} S(x,t)dx$ we can write

$$I^2 = \left(\int_{-\infty}^{\infty} S(x,t)dx\right)\left(\int_{-\infty}^{\infty} S(y,t)dy\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S(x,t)S(y,t)dxdy$$

Evaluate the repeated integral on the right by changing to polar coordinates. What is the value of $\int_0^{\infty} S(x,t)dx$ and why?

9.4 (a) Show that the heat equation is a linear equation.

(b) Show that if u is a solution to the heat equation $\partial_t u(x,t) - \partial_{xx}u(x,t) = 0$ then $v(x,t) = u(\sqrt{\alpha}x, \alpha t)$ is also a solution to the heat equation for any fixed $\alpha > 0$.

Group work

9.5 The aim of this question is to solve the initial value problem (7.2) with initial data

$$\phi(x) = \begin{cases} 1 & \text{if } x > 0; \\ \frac{1}{2} & \text{if } x = 0; \\ 0 & \text{if } x < 0. \end{cases}$$

(a) Since the initial data ϕ is invariant under the transformation in (9.4b), we look for a solution which would also be unchanged by this transformation. Namely we look for a solution of the form

$$u(x,t) = g(x/(2\sqrt{t}))$$

for some $g\colon \mathbf{R} \to \mathbf{R}$. (We have inserted a 2 here just to make the following calculation neater[1].) Show that g solves the ordinary differential equation

$$g''(p) + 2pg'(p) = 0.$$

(b) Find the general formula for solutions to the ordinary differential equation above, then use the initial data ϕ to find the particular solution we are looking for.

(c) Observe that $\partial_x u(x,t)$ is equal to $S(x,t)$, the heat kernel. Can you justify this in any way?

[1] If you don't like it, you are welcome to repeat the question with $u(x,t) = g(x/\sqrt{t})$.

Partial Differential Equations (TATA27)
Spring Semester 2019
Solutions to Homework 9

9.1 (a) See Figure 1.

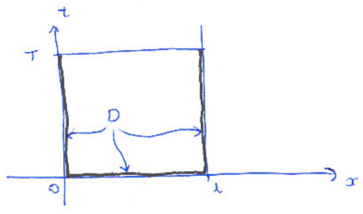

Figure 1: Here is a picture of the set D.

(b) The motivation behind why the wave equation does not satisfy a maximum principle is that wave profiles travelling in opposite directions may collide with one another and thus be larger in value that the initial data — in physics this is called *constructive interference.* We want to take this idea and apply it to construct a specific example.

We want our solution to be two waves which travel towards each other. Let $\phi \in C^\infty(\mathbf{R})$ be a function such that $|\phi(x)| \le 1$ and

$$\phi(x) = \begin{cases} 0 & \text{for } x \in (-\infty, 0) \cup (\ell/2, \infty), \\ 1 & \text{for } x \in (\ell/6, 2\ell/6). \end{cases}$$

Then set $\phi_+(x) = \phi(x)$ and $\phi_-(x) = \phi(x - \ell/2)$. Clearly $u(x,t) = \phi_+(x-t) + \phi_-(x+t)$ solves the heat equation, so to check whether or not it solves (1) we must check the initial and boundary conditions. We have $u(x,0) = \phi_+(x) + \phi_-(x)$ and $\partial_t u(x,t) = -\phi'_+(x) + \phi'_-(x)$, so we choose

$$g(x) = \phi_+(x) + \phi_-(x) \quad \text{and} \quad h(x) = -\phi'_+(x) + \phi'_-(x). \tag{‡}$$

We also have $u(0,t) = \phi_+(-t) + \phi_-(t)$, which is zero provided $t - \ell/2 \le 0 \iff t \le \ell/2$. Finally, $u(\ell,t) = \phi_+(\ell - t) + \phi_-(\ell + t)$, which is zero provided $\ell - t \ge \ell/2 \iff t \le \ell/2$. Therefore, u solves (1) with g and h as in (‡) and $T = \ell/2$.

9.2 (a) **Theorem.** Suppose $\Omega \subset \mathbf{R}^n$ is an open bounded connected set and $T > 0$. Let $u\colon \overline{\Omega} \times [0,T] \to \mathbf{R}$ be a continuous function which is also a solution to the heat equation $\partial_t u(\mathbf{x},t) - \Delta u(\mathbf{x},t) = 0$ for $(\mathbf{x},t) \in \Omega \times (0,T]$. Then the minimum value of u is attained at a point $(\mathbf{x},t) \in \overline{\Omega} \times [0,T]$ such that either $t = 0$ or $\mathbf{x} \in \partial\Omega$.

Proof. Observe that if u satisfies the hypothesis of Theorem 7.1, then so does $-u$. Since the maximum value of $-u$ is the minimum value of u we can apply Theorem 7.1 to $-u$ and conclude that u attains its minimum value at a point $(\mathbf{x},t) \in \overline{\Omega} \times [0,T]$ such that either $t = 0$ or $\mathbf{x} \in \partial\Omega$.

(b) Suppose we have two solutions u_1 and u_2 to (7.1). Then $v = u_2 - u_1$ solves

$$\begin{cases} \partial_t v(\mathbf{x},t) - \Delta v(\mathbf{x},t) = 0 & \text{for } \mathbf{x} \in \Omega \text{ and } t \in (0,T]; \\ u(\mathbf{x},0) = 0 & \text{for } \mathbf{x} \in \overline{\Omega}; \text{ and} \\ u(\mathbf{y},t) = 0 & \text{for } \mathbf{y} \in \partial\Omega \text{ and } t \in (0,T]. \end{cases}$$

Theorem 7.1 and 9.2(a) say that

$$\max_{(\mathbf{x},t) \in \overline{\Omega} \times [0,T]} |v(x,t)| = \max_D |v(x,t)|$$

where $D = (\overline{\Omega} \times \{0\}) \cup (\overline{\Omega} \times (0,T])$. But clearly $\max_D |v(x,t)| = 0$, so $v \equiv 0$ and so $u_1 \equiv u_2$.

(c) Now the difference $u_2 - u_1 = v$ solves

$$\begin{cases} \partial_t v(\mathbf{x},t) - \Delta v(\mathbf{x},t) = 0 & \text{for } \mathbf{x} \in \Omega \text{ and } t \in (0,T]; \\ u(\mathbf{x},0) = \phi_2(\mathbf{x}) - \phi_1(\mathbf{x}) & \text{for } \mathbf{x} \in \overline{\Omega}; \text{ and} \\ u(\mathbf{y},t) = g_2(\mathbf{y},t) - g_1(\mathbf{y},t) & \text{for } \mathbf{y} \in \partial\Omega \text{ and } t \in (0,T]. \end{cases}$$

and again Theorem 7.1 and 9.2(a) say

$$\max_{\mathbf{x}\in\overline{\Omega}, t\in[0,T]} |u_2(\mathbf{x},t) - u_1(\mathbf{x},t)| = \max_{(\mathbf{x},t)\in\overline{\Omega}\times[0,T]} |v(x,t)| = \max_D |v(x,t)|$$

But

$$\begin{aligned} \max_D |v(x,t)| &\le \max_{\mathbf{x}\in\overline{\Omega}} |v(\mathbf{x},t)| + \sup_{\mathbf{x}\in\partial\Omega, t\in(0,T]} |v(\mathbf{x},t)| \\ &= \max_{\mathbf{x}\in\overline{\Omega}} |\phi_2(\mathbf{x}) - \phi_1(\mathbf{x})| + \sup_{\mathbf{x}\in\partial\Omega, t\in(0,T]} |g_2(\mathbf{x},t) - g_1(\mathbf{x},t)|. \end{aligned}$$

Use Theorem 7.1 and 9.2(a) to prove the following stability result: If u_1 and u_2 both solve (7.1) with the initial conditions ϕ_1 and ϕ_2 and boundary conditions g_1 and g_2, respectively, then

$$\max_{\mathbf{x}\in\overline{\Omega}, t\in[0,T]} |u_2(\mathbf{x},t) - u_1(\mathbf{x},t)| \le \max_{\mathbf{x}\in\overline{\Omega}} |\phi_2(\mathbf{x}) - \phi_1(\mathbf{x})| + \sup_{\mathbf{x}\in\partial\Omega, t\in(0,T]} |g_2(\mathbf{x},t) - g_1(\mathbf{x},t)|.$$

Putting these estimates together gives us the required stability result.

9.3 We have

$$\begin{aligned} I^2 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi t}} e^{-x^2/4t} \frac{1}{2\sqrt{\pi t}} e^{-y^2/4t} dxdy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{4\pi t} e^{-(x^2+y^2)/4t} dxdy \\ &= \int_0^{2\pi}\int_0^{\infty} \frac{r}{4\pi t} e^{-r^2/4t} drd\theta \\ &= \int_0^{\infty} \frac{r}{2t} e^{-r^2/4t} dr = -e^{-r^2/4t}\Big|_0^{\infty} = 1. \end{aligned}$$

Therefore $I = 1$ and, since $x \mapsto S(x,t)$ is even $\int_0^{\infty} S(x,t)dx = 1/2$.

9.4 (a) Recall the definitions from Section 1.2. We need to check that the operator $\partial_t - \Delta$ is a linear operator. Take two functions u and v and two constants α and β. Then,

$$\begin{aligned} (\partial_t - \Delta)(\alpha u + \beta v) &= (\partial_t - \sum_{j=1}^n \partial_{x_j x_j})(\alpha u + \beta v) \\ &= \partial_t(\alpha u + \beta v) - (\sum_{j=1}^n \partial_{x_j x_j})(\alpha u + \beta v) \\ &= \partial_t(\alpha u + \beta v) - \sum_{j=1}^n \partial_{x_j x_j}(\alpha u + \beta v) \\ &= (\alpha\partial_t u + \beta\partial_t v) - \sum_{j=1}^n \partial_{x_j}(\alpha\partial_{x_j} u + \beta\partial_{x_j} v) \\ &= (\alpha\partial_t u + \beta\partial_t v) - \sum_{j=1}^n (\alpha\partial_{x_j x_j} u + \beta\partial_{x_j x_j} v) \\ &= \alpha\left(\partial_t u - \sum_{j=1}^n \partial_{x_j x_j} u\right) + \beta\left(\partial_t v - \sum_{j=1}^n \partial_{x_j x_j} v\right) \end{aligned}$$

Thus $\mathcal{L}(\alpha u + \beta v) = \alpha\mathcal{L}u + \beta\mathcal{L}v$ where $\mathcal{L} = (\alpha u + \beta v)$, so the operator $\mathcal{L}$ is linear and hence the heat equation is linear.

(b) We know that $\partial_2 u - \partial_{11} u = 0$. By the chain rule

$$\begin{aligned}\partial_t u(\sqrt{\alpha}x, \alpha t) &= \alpha(\partial_2 u)(\sqrt{\alpha}x, \alpha t),\\ \partial_x u(\sqrt{\alpha}x, \alpha t) &= \sqrt{\alpha}(\partial_1 u)(\sqrt{\alpha}x, \alpha t) \quad \text{and}\\ \partial_{xx} u(\sqrt{\alpha}x, \alpha t) &= \alpha(\partial_{11} u)(\sqrt{\alpha}x, \alpha t).\end{aligned}$$

Thus

$$\begin{aligned}\partial_t u(\sqrt{\alpha}x, \alpha t) - \partial_{xx} u(\sqrt{\alpha}x, \alpha t) &= \alpha(\partial_2 u)(\sqrt{\alpha}x, \alpha t) - \alpha(\partial_{11} u)(\sqrt{\alpha}x, \alpha t)\\ &= \alpha((\partial_2 u)(\sqrt{\alpha}x, \alpha t) - (\partial_{11} u)(\sqrt{\alpha}x, \alpha t))\\ &= \alpha((\partial_2 u) - (\partial_{11} u))(\sqrt{\alpha}x, \alpha t) = 0.\end{aligned}$$

9.5 (a) By applying the chain rule, we see that

$$\begin{aligned}\partial_t u(x,t) &= -\frac{x}{4t^{3/2}} g'(x/(2\sqrt{t}))\\ \partial_x u(x,t) &= \frac{1}{2t^{1/2}} g'(x/(2\sqrt{t})) \quad \text{and}\\ \partial_{xx} u(x,t) &= \frac{1}{4t} g''(x/(2\sqrt{t})).\end{aligned}$$

Therefore,

$$0 = \partial_t u(x,t) - \partial_t u(x,t) = -\frac{x}{4t^{3/2}} g'(x/(2\sqrt{t})) - \frac{1}{4t} g''(x/(2\sqrt{t})),$$

and hence

$$0 = 2\frac{x}{2\sqrt{t}} g'(x/(2\sqrt{t})) + g''(x/(2\sqrt{t}))$$

so

$$0 = 2pg'(p) + g''(p).$$

(b) Set $h = g'$, then we can solve

$$h'(p) + 2ph(p) = 0$$

by multiplying by the integrating factor e^{p^2}:

$$0 = e^{p^2} h'(p) + 2pe^{p^2} h(p) = \frac{d}{dp}\left(e^{p^2} h(p)\right).$$

Hence $e^{p^2} h(p) = A$ and $h(p) = Ae^{-p^2}$. It follows that

$$g(p) = \int_0^p Ae^{-q^2}\, dq + B$$

and hence

$$u(x,t) = \int_0^{x/(2\sqrt{t})} Ae^{-q^2}\, dq + B$$

The initial condition tells us that

$$\lim_{t\to 0} u(x,t) = \phi(x) = \begin{cases} 1 & \text{if } x \geq 0;\\ 0 & \text{if } x < 0.\end{cases}$$

But

$$\lim_{t\to 0} u(x,t) = \begin{cases} \int_0^{\infty} Ae^{-q^2}\, dq + B & \text{if } x > 0;\\ \int_0^{-\infty} Ae^{-q^2}\, dq + B & \text{if } x < 0.\end{cases} = \begin{cases} \frac{A\sqrt{\pi}}{2} + B & \text{if } x > 0;\\ -\frac{A\sqrt{\pi}}{2} + B & \text{if } x < 0.\end{cases}$$

Solving the two equations $(\sqrt{\pi}/2)A + B = 1$ and $-(\sqrt{\pi}/2)A + B = 0$ gives $A = 1/\sqrt{\pi}$ and $B = 1/2$, so

$$u(x,t) = \frac{1}{\sqrt{\pi}} \int_0^{x/(2\sqrt{t})} e^{-q^2}\, dq + \frac{1}{2}.$$

(c) Observe that, by the First Fundamental Theorem of Calculus and the chain rule,

$$\partial_x u(x,t) = \frac{1}{2\sqrt{\pi t}} e^{-x^2/4t} = S(x,t).$$

We know that the heat kernel solves the heat equation. Moreover, looking at the graph of $x \mapsto S(x,t)$ for smaller and smaller t we might guess that $x \mapsto S(x,t)$ tends towards a Dirac delta distribution as $t \to 0$. This means that it appears that S solves (7.2) with initial data being the Dirac delta distribution. While our initial data ϕ is not differentiable in the usual sense, we can differentiate it in the sense of distributions and its derivative is the Dirac delta distribution. Thus it makes sense that $\partial_x u(x,t) = S(x,t)$ since they both appear to solve the same initial value problems for the heat equations.

Partial Differential Equations (TATA27)
Spring Semester 2019
Homework 10

10.1 (a) Suppose that $\phi\colon \mathbf{R} \to \mathbf{R}$ is a bounded odd function. Show that if u is given by (7.3) (so is a solution to (7.2)) then $u(\cdot, t)$ is also odd for each $t > 0$.

(b) Now suppose that ϕ is bounded and even. Prove that $u(\cdot, t)$ given by (7.3) is also even for each $t > 0$.

10.2 (a) Use the ideas of reflections from Section 6.4.1 and 10.1(a) to solve the following boundary and initial value problem on the half line:

$$\begin{cases} \partial_t u(x,t) - \partial_{xx} u(x,t) = 0 & \text{for } x \in (0,\infty) \text{ and } t > 0, \\ u(x,0) = \phi(x) & \text{for } x \in (0,\infty), \text{ and} \\ u(0,t) = 0 & \text{for } t > 0. \end{cases} \tag{1}$$

(b) Further develop these ideas, just as we did in Section 6.4.2, to solve (7.4) via an alternative method to the separation of variables we used in Section 7.5.

(c) Make use of 10.1(b) to help you solve a similar problem to (1):

$$\begin{cases} \partial_t u(x,t) - \partial_{xx} u(x,t) = 0 & \text{for } x \in (0,\infty) \text{ and } t > 0, \\ u(x,0) = \phi(x) & \text{for } x \in (0,\infty), \text{ and} \\ \partial_x u(0,t) = 0 & \text{for } t > 0. \end{cases}$$

Here we replaced the Dirichlet boundary condition $u(0,t) = 0$ with $\partial_x u(0,t) = 0$, which is called a *Neumann condition*.

10.3 In Section 8.1 we estimated the error between derivatives and finite differences in terms of the mesh size δx for a $C^4(\mathbf{R})$ function.

(a) If u is merely a $C^3(\mathbf{R})$ function, what is the error between its first derivative and its centred difference?

(b) If u is merely a $C^2(\mathbf{R})$ function, what is the error between its first derivative and its centred difference?

(c) If u is merely a $C^3(\mathbf{R})$ function, what is the error between its second derivative and its centred second difference?

10.4 Suppose $u \in C^5(\mathbf{R})$. Can you approximate the first derivative $u'(x)$ using a similar method we used in lectures with an error of $O((\delta x)^4)$? [Hint: Make use of the function u evaluated at the points $x + k(\delta x)$ for $k = -2, -1, 1, 2$.]