Hans Lundmark
Department of Mathematics
Linköping University

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TATA27 Partial Differential Equations, spring 2026
Homework assignments (UPG1)

- These homework problem are meant to be handed in continually as the course progresses.

  Ideally, everything should be finished before the written exam at the beginning of June, but it's no big deal if there are still some leftovers remaining after that date. However, once the teaching begins again in August, I will **not** be very interested in grading PDE problems, so if you're not done by then, you will have to wait until the next time the course is given.

- The problems are only graded pass/fail, and if you fail a problem, you simply hand in a corrected version later.

- Discussing the problems with the teacher and with your fellow students is allowed – feel free to ask me for hints if needed! However, you must write the solution in your own words; it's not allowed to just copy someone else's solution. Of course this also includes solutions suggested by ChatGPT or other AI services; you are meant to practice your *own* thinking, so use such tools very sparingly. A good rule of thumb is to not hand in anything that you are not prepared to explain if I should question you about it.

- The solutions should preferably be handed in **on paper**; that will make the grading much easier for me. The most convenient way to hand them in (and to get them back) is in the classroom. Otherwise you can give them to me in my office (room 3A:666) at the math department in building B, if I'm there, or else in my pigeon-hole messagebox at the northern end of the same corridor (outside room 3A:685).

  Handwritten solutions are fine, and you can write them in English or in Swedish. But please **do not write in red**, since I'll be using a red pen when marking. And only write on **one side** of the paper.

1. (Lecture 2.) The goal of this problem is to use the method of characteristics to find a C^1-function $u(x, y)$ which solves the PDE $xu_x - yu_y = u^2$ under the condition that $u(1, y) = y$ for all $y \in \mathbf{R}$.

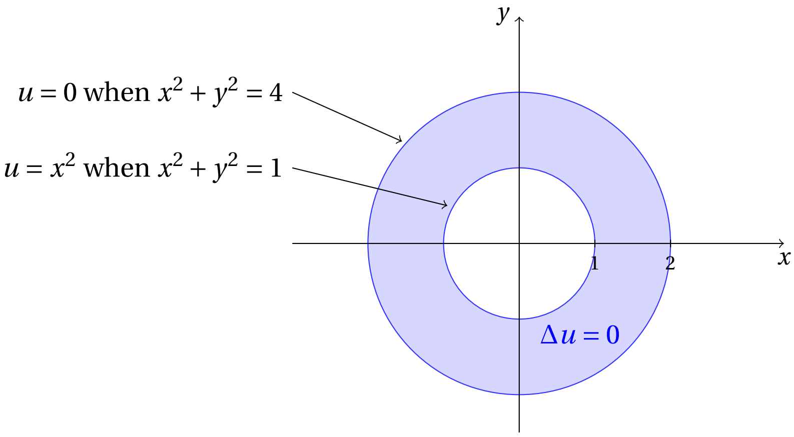
   (a) For a fixed $s \in \mathbf{R}$, compute the characteristic curve $(x(t), y(t))$ which passes through the point $(1, s)$ when $t = 0$.

   Sketch the family of characteristic curves obtained by letting the value of s vary. What region of the xy-plane does this curve family cover?

   (b) Find $z(t) = u((x(t), y(t))$ along an arbitrary characteristic curve from part (a), by solving $\dot{z} = z^2$ with $z(0) = s$. What's the maximal interval of existence for $z(t)$? What part of the curve does this time interval correspond to?

   (c) Use the information above to write the formula for the solution $u(x, y)$ and to determine the region in the xy-plane where the solution is defined. (Also draw a sketch of this region.)

2. (Lecture 4.) Solve the Dirichlet problem below! (Hint: Separation of variables in polar coordinates. But express the solution $u(x, y)$ in Cartesian coordinates.)



3. (Lecture 5.) Let $w \colon \mathbf{R}^n \to \mathbf{R}$ be a radially symmetric function (i.e. depending only on $|\mathbf{x}|$) with compact support and satisfying $\int_{\mathbf{R}^n} w(\mathbf{x})\, dV(\mathbf{x}) = 1$. Let $u \colon \mathbf{R}^n \to \mathbf{R}$ be harmonic. Show that $u * w = u$.

   (The star denotes convolution: $(u * w)(\mathbf{x}) = \int_{\mathbf{R}^n} u(\mathbf{x} - \mathbf{y})\, w(\mathbf{y})\, dV(\mathbf{y})$. Compute this integral using spherical shells, and make use of the mean value property for harmonic functions.)

4. (Lecture 7.) In this problem we'll find two different expressions for the Green's function $G_{(a,b)}(x,y)$ for the operator $-\Delta$ on the square $\Omega = (0,\pi)^2 \subset \mathbf{R}^2$, with Dirichlet boundary conditions.

   (a) Consider Poisson's equation $-\Delta u = f$ on Ω with boundary conditions $u|_{\partial\Omega} = 0$. Since u should be zero on the lines $x = 0$, $x = \pi$, $y = 0$ and $y = \pi$, let us seek the solution in the form of a double Fourier sine series:

   $$u(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} \sin(mx) \sin(ny), \qquad (x,y) \in \Omega.$$

   Develop the right-hand side f in the same way:

   $$f(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c_{mn} \sin(mx) \sin(ny), \qquad (x,y) \in \Omega.$$

   (Write down the formulas for the coefficients c_{mn} as integrals; they will be needed soon.) In terms of these coefficients c_{mn}, how should we choose a_{mn} in order for $-\Delta u = f$ to be satisfied? Once you know this, make use of your integral formulas for c_{mn}, and you will find an expression for the solution u in terms of the right-hand side f.

   Now recall that the solution u which we seek should also be given in terms of the Green's function by

   $$u(x,y) = \iint_{\Omega} G_{(a,b)}(x,y)\, f(a,b)\, da\, db.$$

   By comparing this to the expression for u that you found above, read off an expression for $G_{(a,b)}(x,y)$ in the form of a double sine series!

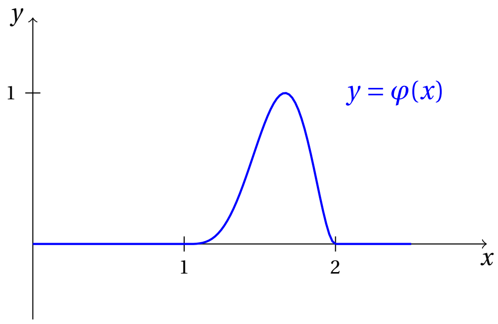
   (As a sanity check, note that your answer should satisfy the symmetry property $G_{(a,b)}(x,y) = G_{(x,y)}(a,b)$.)

   (b) There is another expression for the same Green's function, in the form of a *single* series, which can be computed numerically much more efficiently. Here we fix $(a,b) \in \Omega$ and directly look for a function of the form

   $$G_{(a,b)}(x,y) = \sum_{n=1}^{\infty} f_n(x) \sin(ny)$$

   which satisfies $-\Delta G_{(a,b)}(x,y) = \delta_{(a,b)}(x,y) = \delta_a(x)\delta_b(y)$ with Dirichlet boundary conditions. What should the functions $f_n(x)$ be in order for this plan to succeed?

5. (Lectures 9–10.) Let $\varphi\colon [0,\infty) \to \mathbf{R}$ be a (smooth) function whose graph looks something like this:



(a) Let $u(x,t)$ be the solution to the wave equation $u_{tt} = u_{xx}$ for $x > 0$ and $t > 0$, with the boundary condition $u(0,t) = 0$ and the initial conditions $u(x,0) = \varphi(x)$ and $u_t(x,0) = 0$. Draw the graphs of $u(x,1)$, $u(x,2)$ and $u(x,3)$.

(b) Let $u(x,y,z,t)$ be the solution to the wave equation $u_{tt} = u_{xx} + u_{yy} + u_{zz}$ with initial conditions $u(x,y,z,0) = \varphi(r)$ and $u_t(x,y,z,0) = 0$, where $r = \sqrt{x^2+y^2+z^2}$. Since the initial conditions are radially symmetric, the solution is symmetric too, so it has the form

$$u(x,y,z,t) = U(r,t)$$

for some function U such that $U(r,0) = \varphi(r)$. Sketch (approximately) the graphs of $U(r,1)$, $U(r,2)$ and $U(r,3)$.

(Hint: See problem 10.2.)

(To be continued!)