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Exercises for TATA27 (PDE), spring 2021

This document will be expanded during the course. The latest version will always be available on the **course webpage**.

1 Warmup problems

Here are some exercises to get you started. Hopefully, you should be able to tackle them using only what you know from previous courses (mainly multivariable calculus – in fact, if you've taken that course here at LiU you might recognize some of the problems from there).

1.1 The chain rule. If a quantity z is described in two coordinate systems (x, y) and (u, v), then the chain rule says that (Answer.)

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x},$$
$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y},$$

or, in synonymous notation,

$$z'_x = z'_u u'_x + z'_v v'_x,$$
$$z'_y = z'_u u'_y + z'_v v'_y.$$

or, dropping the primes (as is usually done once you get past the basic level),

$$z_x = z_u u_x + z_v v_x,$$
$$z_y = z_u u_y + z_v v_y.$$

This way of writing is often convenient (and we will use it all the time), but it is a bit imprecise.

Your task here is to write down a precise formulation of the chain rule, in terms of the **functions** f, g, α, β that describe the relationships between the various quantities:

$$\begin{aligned} u &= \alpha(x, y), \\ v &= \beta(x, y), \\ z &= f(x, y) = g(u, v) = g\big(\alpha(x, y), \beta(x, y)\big). \end{aligned}$$

(We assume these functions to be differentiable, of course.) Include the points where the derivatives are supposed to be evaluated, so that your answer looks something like this:

$$\frac{\partial f}{\partial x}(a, b) = \cdots,$$
$$\frac{\partial f}{\partial y}(a, b) = \cdots.$$

Discuss some advantages and disadvantages with these two ways of writing the chain rule.

1.2 Notational weirdness. Consider the coordinate systems (x, y) and (u, v), related via the change of variables $u = x^2 - 3y$, $v = x$. (Answer.)

(a) Compute $\partial u/\partial x$ and $\partial x/\partial u$. Is it true that $(\partial u/\partial x) \cdot (\partial x/\partial u) = 1$?

(b) Compute all possible partial derivatives of the quantities involved here, i.e., compute the Jacobian matrices

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \qquad \text{and} \qquad \frac{\partial(x,y)}{\partial(u,v)} = \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}.$$

What's the relationship between these two matrices?

(c) Let $f(x,y)$ be a differentiable function. Use the chain rule to express $\partial f/\partial x$ in terms of $\partial f/\partial u$ and $\partial f/\partial v$. How come $\partial f/\partial x \neq \partial f/\partial v$ although $x = v$?

1.3 Thermodynamics. Consider a system (such as a gas in a piston) of temperature T, pressure p and volume V. Any two of these quantities determine the third, so that the system's energy E can be described as a function of (T,p), (T,V) or (p,V). Show that (Answer.)

$$\left(\frac{\partial E}{\partial T}\right)_p = \left(\frac{\partial E}{\partial T}\right)_V + \left(\frac{\partial E}{\partial V}\right)_T \left(\frac{\partial V}{\partial T}\right)_p,$$

where the subscript indicates the quantity that is held constant when computing the derivative. (For example, $(\partial E/\partial T)_p$ means that we consider E as a function of T and p, and take the partial derivative of this function $E(T,p)$ with respect to T, treating p as a constant.)

1.4 A funny triple product. Suppose that the constraint $F(x,y,z) = 0$ defines any of the three variables (x,y,z) as a differentiable function of the other two. (Answer.)

(a) What hypotheses should be fulfilled, in order for the implicit function theorem to guarantee that this is the case (locally)?

(b) Show that

$$\frac{\partial x}{\partial y} \cdot \frac{\partial y}{\partial z} \cdot \frac{\partial z}{\partial x} = -1.$$

1.5 Rotated coordinates. For a fixed angle α, consider the linear change of variables (Answer.)

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}.$$

(a) Draw a figure to illustrate how the (ξ,η) axes are situated in the (x,y) coordinate system.

(b) Show that the Laplacian is rotationally invariant, i.e.,

$$u_{\xi\xi} + u_{\eta\eta} = u_{xx} + u_{yy}.$$

1.6 Transforming partial derivatives to polar coordinates. Consider polar coordinates in $\mathbf{R}^2$, defined by $x = r\cos\varphi$ and $y = r\sin\varphi$. (Answer.)

(a) Express the partial derivatives u_r and u_φ in terms of u_x and u_y.

(b) Invert the relationship to obtain u_x and u_y in terms of u_r and u_φ.

(c) Express u_{xx}, u_{xy} and u_{yy} in terms of partial derivatives with respect to r and φ. In particular, derive the expression for the Laplacian $\Delta u = u_{xx} + u_{yy}$ in polar coordinates.

1.7 The Laplacian in polar coordinates again. Using some facts from vector calculus, we can derive the expression for Δu in polar coordinates with less effort than in problem 1.6. (Answer.)

(a) Recall that the **gradient** of a function $f(x,y)$ is defined as

$$\nabla f = \operatorname{grad} f = (f_x, f_y) = f_x\,\mathbf{e}_x + f_y\,\mathbf{e}_y,$$

where $\mathbf{e}_x = (1,0)$ and $\mathbf{e}_y = (0,1)$ are the standard basis vectors for $\mathbf{R}^2$, and that the **divergence** of a vector field

$$\mathbf{v}(x,y) = \big(X(x,y), Y(x,y)\big) = X(x,y)\,\mathbf{e}_x + Y(x,y)\,\mathbf{e}_y$$

is defined as

$$\nabla\cdot\mathbf{v} = \operatorname{div}\mathbf{v} = X_x + Y_y.$$

Deduce that the Laplacian is **the divergence of the gradient**:

$$\Delta u = u_{xx} + u_{yy} = \nabla\cdot\nabla u = \operatorname{div}(\operatorname{grad} u).$$

(b) Recall from vector calculus (or prove for yourself) that for a function $f(r,\varphi)$ expressed in polar coordinates, the gradient is

$$\nabla f = f_r\,\frac{1}{1}\,\mathbf{e}_r + f_\varphi\,\frac{1}{r}\,\mathbf{e}_\varphi,$$

where the highlighted factors 1 and r are the scale factors associated with the r and φ directions (as in the general theory for orthogonal curvilinear coordinate systems).

Also recall (or prove) that for a vector field $\mathbf{v}(R,\varphi) = R(r,\varphi)\,\mathbf{e}_r + \Phi(r,\varphi)\,\mathbf{e}_\varphi$ expressed in polar coordinates, the divergence is

$$\nabla\cdot\mathbf{v} = \frac{1}{1\cdot r}\left(\frac{\partial}{\partial r}\Big(R(r,\varphi)\cdot r\Big) + \frac{\partial}{\partial\varphi}\Big(1\cdot\Phi(r,\varphi)\Big)\right).$$

Use this to (again) derive the formula for Δu, where $u = u(r,\varphi)$.

1.8 The Laplacian in spherical coordinates. In a similar way as in problem 1.7, derive the expression for $\Delta u = \nabla \cdot \nabla u = u_{xx} + u_{yy} + u_{zz}$ when $u = u(x, y, z)$ is expressed in spherical coordinates (r, θ, φ) defined by (Answer.)

$$\begin{aligned} x &= r \sin\theta \cos\varphi, \\ x &= r \sin\theta \sin\varphi, \\ z &= r \cos\theta. \end{aligned}$$

You will need to remember that the scale factors associated with the r, θ and φ directions are 1, r and $r\sin\theta$, so that the gradient of the function $f(r, \theta, \varphi)$ is

$$\nabla f = f_r \frac{1}{1} \mathbf{e}_r + f_\theta \frac{1}{r} \mathbf{e}_\theta + f_\varphi \frac{1}{r\sin\theta} \mathbf{e}_\varphi$$

and the divergence of the vector field

$$\mathbf{v}(r, \theta, \varphi) = R(r, \theta, \varphi)\, \mathbf{e}_r + \Theta(r, \theta, \varphi)\, \mathbf{e}_\theta + \Phi(r, \theta, \varphi)\, \mathbf{e}_\varphi$$

is

$$\begin{aligned} \nabla \cdot \mathbf{v} = \frac{1}{1 \cdot r \cdot r\sin\theta} \bigg(&\frac{\partial}{\partial r}\Big(R(r, \theta, \varphi) \cdot r \cdot r\sin\theta \Big) \\ &+ \frac{\partial}{\partial \theta}\Big(1 \cdot \Theta(r, \theta, \varphi) \cdot r\sin\theta \Big) \\ &+ \frac{\partial}{\partial \varphi}\Big(1 \cdot r \cdot \Phi(r, \theta, \varphi) \Big)\bigg). \end{aligned}$$

1.9 A simple verification using the chain rule. Suppose that g is a differentiable function of one variable. Verify that $u(x, y) = g(xe^y)$ satisfies $xu_x - u_y = 0$ identically (i.e., for all x and y). (Answer.)

1.10 Solving a PDE by changing variables. (Answer.)

(a) Find all C^1-functions $u(x, y)$ that satisfy the PDE

$$xu_x - u_y = 2x^2$$

in the right half-plane $x > 0$, by rewriting the PDE in terms of the new variables $\xi = x$, $\eta = xe^y$.

(b) Find all the functions from part (a) that satisfy the additional condition $u(1, y) = e^{-y}$ for all $y \in \mathbf{R}$.

(c) More generally, find all the functions from part (a) that satisfy $u(1, y) = f(y)$ for all $y \in \mathbf{R}$, where f is some given C^1-function. (The answer will of course be expressed in terms of f.)

2 The method of characteristics

2.1 Déjà vu all over again. Pretend that you don't know about problem 1.10, (Answer.) and use the method of characteristics to determine all C^1-functions $u(x, y)$, $x > 0$, such that

$$xu_x - u_y = 2x^2, \qquad u(1, y) = f(y).$$

2.2 Mixed problems. Use the method of characteristics to find all solutions (Answer.) of class C^1 to the following equations (assuming that the given data are also of class C^1). Are the solutions globally defined? Do your calculations suggest some suitable changes of variables that could be employed for solving these equations in another way?

(a) $(1 + x^2)u_x + u_y = 0$, with $u(0, y) = f(y)$.

(b) $u_x + u_y + u = e^{x+2y}$, with $u(x, 0) = 0$.

(c) $xu_y - yu_x = u$, with $u(x, 0) = h(x)$ (for $x \geq 0$).

(d) $xu_x + yu_y + u_z = u$, with $u(x, y, 0) = f(x, y)$.

(e) $u_x + u_y = u^2$, with $u(x, 0) = f(x)$. (This equation is not linear, but it is "semilinear", meaning that it's only u that appears nonlinearly, not any of the derivatives; in other words, it's of the form

$$a(x, y)\, u_x + b(x, y)\, u_y = c(x, y, u).$$

The method of characteristics works just the same way for first-order semilinear PDEs as for linear ones, except that you will get nonlinear ODEs along the characteristic curves instead of linear ones.)

(f) $xyu_x + (1 + y^2)u_y = y$, with $u(x, 0) = f(x)$.

(g) $e^y u_x + 2xu_y = 0$, with $u(0, y) = y$. And the same, but with $u(0, y) = e^y$.

Remark. The method of characteristics also works for first-order "quasilinear" equations, where the derivatives appear linearly but with coefficients that are allowed to be u-dependent:

$$a(x, y, u)\, u_x + b(x, y, u)\, u_y = c(x, y, u).$$

The difference in this case is that one cannot first solve for characteristic curves in the xy-plane, and then solve an ODE for u along each such curve; instead there is a simultaneous system of ODEs for characteristic curves in xyu-space.

In fact, the method can also be extended to general, fully nonlinear, first-order PDEs $F(x, y, u, u_x, u_y) = 0$, but then things become quite a lot more complicated.

3 Physical origins of some PDEs

3.1 Heat conduction in a rod. Consider heat conduction in a rod on the interval $x \in [0,1]$, described by the heat equation $u_t = u_{xx}$ in the interior $0 < x < 1$. Various types of boundary conditions can be given at the endpoints. In each of the following cases, interpret what the boundary conditions mean physically. Also determine the steady-state (i.e., time-independent) solution or solutions satisfying these conditions. (Based on physical intuition, we may expect $u(x,t)$ to approach such an equilibrium state as $t \to \infty$.) (Answer.)

(a) Boundary conditions of **Dirichlet** type, where the value of the sought function is prescribed on the boundary:

$$u(0,t) = A, \qquad u(1,t) = B.$$

(b) Boundary conditions of **Neumann** type, where the value of the derivative of the sought function is prescribed on the boundary:

$$-u_x(0,t) = A, \qquad u_x(1,t) = B.$$

In particular, what do the conditions

$$u_x(0,t) = 0, \qquad u_x(1,t) = 0$$

mean physically?

Why would we expect $\int_0^1 u(x,t)\,dx$ to be independent of t if $B = -A$? Can you prove that it's true?

(A remark: For the heat equation $u_t = \alpha \Delta u$ on some domain $\Omega \subset \mathbf{R}^n$ with $n \geq 2$, the terminology "Neumann boundary conditions" means that the value of the **normal derivative** $\partial u/\partial n$, the directional derivative in the direction of the outward normal unit vector, is prescribed at each boundary point $x \in \partial\Omega$. For the one-dimensional rod, the normal derivative is $-u_x$ at the left endpoint and u_x at the right endpoint.)

(c) **Mixed** boundary conditions, where different types of conditions are prescribed on different parts of the boundary:

$$u(0,t) = A, \qquad u_x(1,t) = B.$$

3.2 More heat conduction in a rod. Consider heat condution in a rod on the interval $x \in [0,\pi]$, described by the heat equation $u_t = u_{xx}$ in the interior $0 < x < \pi$, and with Dirichlet boundary conditions $u(0,t) = u(\pi,t) = 0$. (Answer.)

(a) For a given positive integer n, determine the function $T(t)$ such that

$$u(x,t) = T(t)\sin(nx)$$

is a solution, satisfying the initial condition $u(x,0) = \sin(nx)$.

(b) Determine the solution $u(x,t)$ if the initial condition is

$$u(x,0) = 17 \sin x - 5 \sin(3x).$$

3.3 Vibrations of a string. Consider small vibrations of a string on the interval $x \in [0,\pi]$, described by the wave equation $u_{tt} = u_{xx}$ in the interior $0 < x < \pi$, and with boundary conditions $u(0,t) = u(\pi,t) = 0$ describing that the string is attached at its endpoints. (Answer.)

(a) For a given positive integer n, determine the function $T(t)$ such that

$$u(x,t) = T(t)\,\sin(nx)$$

is a solution, satisfying the initial conditions $u(x,0) = \sin(nx)$ and $u_t(x,0) = 0$.

(b) Determine the solution $u(x,t)$ if the initial conditions are

$$u(x,0) = 17 \sin x - 5 \sin(3x), \qquad u_t(x,0) = 0.$$

(The musicians in the audience may want to think a little about the musical meaning of this. How would you demonstrate it on a guitar or a bass guitar? Piano strings are *not* very accurately described by this model – why is that, and what are the implications for piano tuning?)

3.4 General solution of the one-dimensional wave equation. Use the change of variables $\xi = x + ct$, $\eta = x - ct$ to determine all functions $u(x,t)$ of class $C^2(\mathbf{R}^2)$ that satisfy the wave equation $u_{tt} = c^2 u_{xx}$. (Answer.)

3.5 The fundamental solution of the one-dimensional heat equation. Verify that (Answer.)

$$u(x,t) = \frac{1}{\sqrt{4\pi\alpha t}} \exp\left(-\frac{x^2}{4\alpha t}\right)$$

satisfies the heat equation $u_t = \alpha u_{xx}$ for $x \in \mathbf{R}$, $t > 0$. What is the limit of this function as $t \to 0^+$?

3.6 A reaction–advection problem. Let c and r be positive constants, and consider the initial value problem (Answer.)

$$\begin{aligned} u_t + c u_x &= -r u, && \text{for } x \in \mathbf{R} \text{ and } t > 0, \\ u(x,0) &= f(x), && \text{for } x \in \mathbf{R}. \end{aligned}$$

What kind of physical situation might this describe? Use the physical intuition to find a change of variables that allows you to solve the system. (Hint: Moving coordinate system. “Go with the flow!”)

4 The Laplace equation

The weak maximum principle

4.1 Multivariable calculus. (Answer.)

(a) Make sure you remember the **second partial derivative test** from multivariable calculus:

> Let Ω be an open set in $\mathbf{R}^n$, let $f\colon \Omega \to \mathbf{R}$ be a function of class C^2, and let $H(\mathbf{x})$ denote the Hessian matrix of f, i.e., the symmetric $n \times n$ matrix of second-order partial derivatives:
>
> $$H_{ij}(\mathbf{x}) = \left(f_{x_i x_j}(\mathbf{x})\right)_{i,j=1}^{n}.$$
>
> Assume that $\mathbf{a} \in \Omega$ is a stationary point for f, i.e., $\nabla f(\mathbf{a}) = \mathbf{0}$.
>
> - If $H(\mathbf{a})$ is **positive definite**, then f has a **strict local minimum** at the point $\mathbf{a}$.
> - If $H(\mathbf{a})$ is **negative definite**, then f has a **strict local maximum** at the point $\mathbf{a}$.
> - If $H(\mathbf{a})$ is **indefinite**, then f has a **saddle** at $\mathbf{a}$ (neither a local minimum nor a local maximum).

(b) Prove it!

(c) Give examples with $n = 2$ and $\mathbf{a} = (0,0)$ to show that the test is inconclusive if $H(\mathbf{a})$ is **positive semidefinite** or **negative semidefinite**. (Or if it's both at the same time! Yes, this can happen, but only for a very particular matrix. Which one?)

4.2 Harmonic functions on an unbounded domain. Consider the unbounded domain (Answer.)

$$\Omega = \{(x, y) \in \mathbf{R}^2 : x > 0\},$$

i.e., the open right half-plane in $\mathbf{R}^2$. Then, of course, the closure

$$\overline{\Omega} = \{(x, y) \in \mathbf{R}^2 : x \geq 0\}$$

is the closed right half-plane, and the boundary $\partial\Omega$ is the y-axis.

Your task, in each part below, is to try to construct an example of a continuous function $u\colon \overline{\Omega} \to \mathbf{R}$ which is harmonic on Ω and satisfies the given condition. (If you can't find any examples of your own, at least verify that the examples in the answer key have the claimed properties.)

Hint: For parts (a)–(c), there are very simple examples. The other parts are considerably more difficult; see problem 4.3 below for a possible approach.

(a) u is unbounded both from above and from below (and consequently has no maximum or minimum).

(b) u is unbounded from above (and consequently has no maximum), and attains its minimum on the boundary $\partial\Omega$.

(c) $u = 0$ on $\partial\Omega$. (Find at least two different examples, showing that the Dirichlet problem for the Laplace equation does not have a unique solution for this domain Ω.)

(d) u is bounded from above, but has no maximum, and attains its minimum on the boundary $\partial\Omega$.

(e) u is non-constant and attains a maximum and a minimum on $\partial\Omega$.

(f) u is bounded from above and from below, but has no maximum or minimum.

4.3 Complex analysis. For those of you who know some complex analysis: (Answer.)

(a) Let u and v be the real and imaginary parts of the analytic function f:

$$f(x+iy) = u(x,y) + i\,v(x,y), \qquad \text{with } u \text{ and } v \text{ real-valued.}$$

Use the Cauchy–Riemann equations $u_x = v_y$ and $u_y = -v_x$ to prove that $\Delta u = 0$ and $\Delta v = 0$.

(b) Show that the Möbius transformation $w = f(z) = (z-1)/(z+1)$ maps the closed right half-plane $\{z \in \mathbf{C} : \operatorname{Re} z \geq 0\}$ to the closed unit disk minus a point, $\{w \in \mathbf{C} : |w| \leq 1,\ w \neq 1\}$. Calculate $u(x,y) = \operatorname{Re} f(x+iy)$ and $v(x,y) = \operatorname{Im} f(x+iy)$, and explain why they solve problems 4.2(d) and (e) above, respectively.

(c) Consider the mapping $w = f(z) = \operatorname{Log}(z+1)$, where Log denotes the principal branch of the complex logarithm, with imaginary part in the interval $(-\pi,\pi]$. What's the image of the closed right half-plane? Calculate $v(x,y) = \operatorname{Im} f(x+iy)$, and explain why it is a solution to problem 4.2(f) above.

4.4 Another weak maximum principle. Let Ω be a bounded open set in $\mathbf{R}^n$. (Answer.) Prove that any continuous function $u : \overline{\Omega} \to \mathbf{R}$ such that $\Delta u(\mathbf{x}) + \mathbf{x} \cdot \nabla u(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \Omega$ attains its maximum on $\partial\Omega$.

Poisson's formula (for a disk)

4.5 A bit of algebra. If $\mathbf{x} = (r\cos\theta, r\sin\theta)$ and $\mathbf{y} = (a\cos\varphi, a\sin\varphi)$, show that (Answer.)

$$|\mathbf{x}-\mathbf{y}|^2 = r^2 - 2ar\cos(\theta-\varphi) + a^2,$$

and deduce that the following two ways of writing Poisson's formula are the same:

$$u(\mathbf{x}) = \frac{a^2 - r^2}{2\pi} \int_0^{2\pi} \frac{h(\varphi)\,d\varphi}{r^2 - 2ar\cos(\theta-\varphi) + a^2} = \frac{a^2 - |\mathbf{x}|^2}{2\pi a} \int_{|\mathbf{y}|=a} \frac{\tilde{h}(\mathbf{y})\,d\sigma(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^2}.$$

4.6 The Laplace equation in a wedge. Let Ω be the wedge in $\mathbf{R}^2$ described in polar coordinates by $0 < r < a$ and $0 < \theta < \beta$ (where $\beta < 2\pi$). Use separation of variables $u(r,\theta) = R(r)\,\Theta(\theta)$ to find a continuous function $u\colon \overline{\Omega} \to \mathbf{R}$ which is harmonic on Ω and satisfies the boundary conditions given in polar coordinates by (Answer.)

$$\begin{aligned} u(r,0) = u(r,\beta) &= 0 && \text{for } 0 < r < a, \\ u(a,\theta) &= h(\theta) && \text{for } 0 < \theta < \beta. \end{aligned}$$

4.7 More separation of variables. Let Ω be the square in $\mathbf{R}^2$ described by $0 < x < \pi$ and $0 < y < \pi$. Use separation of variables $u(x,y) = X(x)\,Y(y)$ to find a continuous function $u\colon \overline{\Omega} \to \mathbf{R}$ which is harmonic on Ω and satisfies the boundary conditions (Answer.)

$$\begin{aligned} &u_y(x,0) = u_y(x,\pi) && \text{for } 0 < x < \pi, \\ &u(0,y) = 0, \quad u(\pi,y) = \cos^2 y && \text{for } 0 < y < \pi. \end{aligned}$$

4.8 Discontinuity. (Answer.)

(a) Show that

$$u(x,y) = \begin{cases} 17, & \text{if } x = 0 \text{ or } y = 0, \\ 43, & \text{otherwise,} \end{cases}$$

satisfies $u_{xx} + u_{yy} = 0$ at the origin, even though it's not even continuous there.

(Moral: The mere existence of the pure second partial derivatives doesn't imply much about niceness for u as a whole.)

(b) Think a moment about why most sources define a function to be **harmonic** on an open set Ω if it satisfies the Laplace equation at every point in Ω **and** belongs to the class $C^2(\Omega)$. (Or, more generally, why a "classical solution" of a PDE of order k is usually assumed to belong to the class $C^k(\Omega)$.)

(c) When deriving the weak maximum principle and the Poisson formula (with the corollary "harmonic functions are smooth"), we didn't use the full strength of the assumption $u \in C^2(\Omega)$, but we did use that $u \in C(\overline{\Omega})$, in order to be able to apply the extreme value theorem. Here's an example to show what can go wrong otherwise:

Let $f(z) = e^{-1/z^4}$ for $z \neq 0$, and let

$$u(x,y) = \begin{cases} \operatorname{Re} f(x+iy), & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$$

Show that $u_{xx} + u_{yy} = 0$ everywhere (including at the origin), but that u is not continuous at the origin.

(d) If we were to take the values on the unit circle $x^2 + y^2 = 1$ of the function u from part (c) and plug them into Poisson's formula, what harmonic function on the unit disk would we obtain?

(Hint: Use some indirect reasoning. Trying to compute the Poisson integral directly would be horrible!)

5 The Laplace equation (cont.)

The mean value property & the strong maximum principle

5.1 Bounds at the origin. Let u be a continuous solution to the problem (Answer.)

$$\begin{aligned} \Delta u &= -1, \quad \text{if } |x| < 1 \text{ and } |y| < 1, \\ u &= 0, \quad \text{if } |x| = 1 \text{ or } |y| = 1. \end{aligned}$$

By considering the function $v(x, y) = u(x, y) + \frac{1}{4}(x^2 + y^2)$, determine an interval that $u(0,0)$ must belong to.

5.2 Absolute value. Suppose u is harmonic on the bounded open set Ω and continuous on $\overline{\Omega}$. Show that its absolute value $|u|$ satisfies the weak maximum principle: (Answer.)

$$\max_{\overline{\Omega}} |u| = \max_{\partial\Omega} |u|.$$

What about the strong maximum principle?

5.3 Unbounded domain. Let Ω be a connected open set, not necessarily bounded. Show that if $u \in C^2(\Omega)$ satisfies $\Delta u \geq 0$ and attains its maximum at some point in Ω, then u is constant on Ω. (Answer.)

5.4 Local extrema. Use the mean value property to show that a harmonic function on an open set Ω cannot have any strict local extrema in Ω. (Can you say something about non-strict local extrema?) (Answer.)

5.5 Converse to the mean value property. Assume that $u \in C^2(\Omega)$ and that (Answer.)

$$u(\mathbf{x}) = \fint_{\partial B(\mathbf{x},r)} u \, dS$$

for each ball $B(\mathbf{x}, r) \subset U$. Show that u is harmonic in Ω.

6 The Laplace equation (cont.)

Dirichlet's principle

6.1 A variational problem with no solution. Consider the class of functions that are continuous on $[-1,1]$, continuously differentiable on $(-1,1)$, and satisfy the boundary conditions $f(-1) = -1$ and $f(1) = 1$. Show that the problem of minimizing the integral (Answer.)

$$\int_{-1}^{1} \left(x f'(x)\right)^2 dx$$

over all such f has no solution.

(This was the example given by Weierstrass in 1870 as a general objection to the assumption that the variational problem in Dirichlet's principle has a solution.)

6.2 A minimizing sequence that doesn't converge. (Answer.)

(a) Let Ω be the unit disk $x^2 + y^2 < 1$. Compute the Dirichlet energy integral $E(u) = \int_\Omega |\nabla u|^2 \, dx\,dy$ for the continuous and piecewise differentiable function

$$u(x,y) = u(r\cos\theta, r\sin\theta) = \begin{cases} C \ln R, & 0 \le r \le R^2, \\ C \ln(r/R), & R^2 \le r \le R, \\ 0, & R \le r \le 1, \end{cases}$$

where $C \in \mathbf{R}$ and $R \in (0,1)$ are constants.

(b) Determine sequences C_n and R_n, with corresponding functions $u_n(x,y)$ as in part (a), such that $E(u_n) \to 0$ as $n \to \infty$ (so that u_n is a minimizing sequence for the energy integral), but $u_n(0,0) \to -\infty$ as $n \to \infty$ (so that the sequence doesn't converge to a limiting function on Ω).

6.3 Infinite energy. (Answer.)

(a) Go back to the derivation of Poisson's formula using separation of variables, to recall that if $h(\theta)$ is a continuous 2π-periodic function with (not necessarily convergent) Fouries series

$$h(\theta) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos k\theta + b_k \sin k\theta\right),$$

then the unique solution of the Dirichlet problem for the Laplace equation on the unit disk with boundary values $u(\cos\theta, \sin\theta) = h(\theta)$ is given for $r < 1$ by the convergent Fourier series

$$u(r\cos\theta, r\sin\theta) = \underbrace{\frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k r^k \cos k\theta + b_k r^k \sin k\theta\right)}_{\text{call this } U(r,\theta)}.$$

(b) Also remind yourself about Parseval's identity for Fourier series (with notation as above):

$$\frac{1}{2\pi}\int_0^{2\pi} h(\theta)^2\,d\theta = \frac{a_0^2}{4} + \frac{1}{2}\sum_{k=1}^{\infty}\left(a_k^2 + b_k^2\right).$$

(c) Show that the Dirichlet energy integral in polar coordinates is

$$E(u) = \iint\limits_{x^2+y^2<1} \left(u_x^2 + u_y^2\right) dx\,dy = \iint\limits_{\substack{0<r<1 \\ 0\le\theta<2\pi}} \left(U_r^2 + r^{-2}\,U_\theta^2\right) r\,dr\,d\theta.$$

(d) Compute the Fourier series for U_r and U_θ by termwise differentiation in the series for $U(r,\theta)$, and use Parseval's identity to show that

$$E_R(u) = \iint\limits_{\substack{0<r<R \\ 0\le\theta<2\pi}} \left(U_r^2 + r^{-2}\,U_\theta^2\right) r\,dr\,d\theta = \pi\sum_{k=1}^{\infty} k\left(a_k^2 + b_k^2\right)R^{2k}$$

for $0 \le R < 1$.

(e) Deduce that the following holds for $N \ge 1$ and $0 \le R < 1$:

$$\pi\sum_{k=1}^{N} k\left(a_k^2 + b_k^2\right)R^{2k} \le E_R(u) \le \pi\sum_{k=1}^{\infty} k\left(a_k^2 + b_k^2\right),$$

where the series on the right-hand side may be convergent or divergent (to ∞). Let $R \to 1$, and then $N \to \infty$, to show that $E(u)$ equals that series (so that the energy of u is finite if and only if the series converges.)

(f) Consider the boundary values

$$h(\theta) = \sum_{m=1}^{\infty} \frac{\sin(m!\,\theta)}{m^2}.$$

Show that h is continuous, and that the corresponding solution u has infinite energy.

(This example was given by Hadamard.)

6.4 A nonlinear problem involving the Laplacian. Let Ω be the open unit ball in $\mathbf{R}^n$, and consider the problem of finding $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfying $\Delta u = u^3$ on Ω and $u = 0$ on the boundary. Show that $u = 0$ is the only solution. Hint: Consider the flow of $u\nabla u$ through the unit sphere. (Answer.)

The fundamental solution

6.5 Gradient of a radially symmetric function.

(a) Let $r(\mathbf{x}) = |\mathbf{x}| = \sqrt{x_1^2 + \cdots + x_n^2}$, for $\mathbf{x} \in \mathbf{R}^n$. By considering the level sets of the function r in the cases $n = 2$ and $n = 3$, convince yourself geometrically that it is reasonable to expect that

$$\nabla r(\mathbf{x}) = \frac{\mathbf{x}}{|\mathbf{x}|} \qquad (\mathbf{x} \neq \mathbf{0}),$$

or, written more shortly using the notation $\mathbf{e}_r(\mathbf{x})$ or $\hat{\mathbf{r}}(\mathbf{x})$ for the vector field $\frac{\mathbf{x}}{|\mathbf{x}|}$,

$$\nabla r = \mathbf{e}_r = \hat{\mathbf{r}}.$$

Verify this by direct computation of the derivatives $\partial r / \partial x_k$.

(b) Deduce using the chain rule that if $u(\mathbf{x}) = R(|\mathbf{x}|) = R(r(\mathbf{x}))$ is a radially symmetric function, then

$$\nabla u(\mathbf{x}) = R'(r(\mathbf{x}))\, \mathbf{e}_r(\mathbf{x}),$$

or, written more shortly,

$$\nabla u = R'(r)\, \mathbf{e}_r.$$

Verify that this agrees with the general expressions for the gradient in polar and spherical coordinates (see problems 1.7 and 1.8).

6.6 One-dimensional fundamental solution. (Answer.)

(a) Recall that the fundamental solution for the operator $-\Delta$ in $\mathbf{R}^n$ is a radially symmetric function $\Phi(\mathbf{x}) = R(r)$ such that $R'(r) = \frac{-1}{A_n r^{n-1}}$, where A_n is the $(n-1)$-dimensional "surface area" of the unit sphere in $\mathbf{R}^n$. So what is the fundamental solution $\Phi(x)$ in the case $n = 1$? How do you interpret the quantity A_1?

(b) Verify that the fundamental solution in $\mathbf{R}^1$ indeed satisfies $-\Phi''(x) = \delta(x)$, where δ is the Dirac delta distribution. (You can use that $H' = \delta$ in the sense of distributions, where $H(x)$ is the Heaviside step function, $H(x) = 0$ for $x < 0$ and $H(x) = 1$ for $x > 0$.)

6.7 Computing a normal derivative. Let $\Phi(\mathbf{x}) = -\frac{1}{2\pi} \ln |\mathbf{x}|$ be the fundamental solution for $-\Delta$ in $\mathbf{R}^2$. Let Ω be the open disk of radius $R > 0$ centered at the origin, and let $\mathbf{a} = (r\cos\theta, r\sin\theta)$ (with $0 \le r < R$) be an interior point. If $\Phi_{\mathbf{a}}(\mathbf{x}) = \Phi(\mathbf{x} - \mathbf{a})$ denotes the shifted fundamental solution centered at $\mathbf{a}$, compute the normal derivative $\frac{\partial \Phi_{\mathbf{a}}}{\partial n}(\mathbf{x})$ at the boundary point $\mathbf{x} = (R\cos\varphi, R\sin\varphi)$, in terms of r, θ, R and φ. (Answer.)

7 The Laplace equation (cont.)

Green's functions

7.1 Green's function in one dimension. Compute Green's function $G_a(x)$ for the operator $-\Delta = -d^2/dx^2$ on the interval $(0,1)$, and verify that the property $G_a(x) = G_x(a)$ holds. (Answer.)

7.2 Normal derivative of Green's function for the unit ball. Recall that Green's function for the operator $-\Delta$ on the unit ball in $\mathbf{R}^n$, at the interior point $\mathbf{a}$, is (Answer.)

$$G_{\mathbf{a}}(\mathbf{x}) = \Phi\big(\mathbf{x}-\mathbf{a}\big) - \Phi\big(|\mathbf{a}|\,(\mathbf{x}-\mathbf{b})\big),$$

where Φ is the fundamental solution for $-\Delta$ in $\mathbf{R}^n$ and where $\mathbf{b} = \mathbf{a}/|\mathbf{a}|^2$ is the image of $\mathbf{a}$ under inversion in the unit sphere. Compute the *Poisson kernel*, i.e., the value of

$$-\frac{\partial G_{\mathbf{a}}}{\partial n}(\mathbf{x})$$

for $|\mathbf{x}| = 1$, where the normal derivative $\partial/\partial n$ refers to the normal vector pointing out of the ball.

7.3 Green's function for a half-ball. Determine Green's function $H_{\mathbf{a}}(\mathbf{x})$ for the upper half of the unit ball in $\mathbf{R}^n$, (Answer.)

$$\big\{\mathbf{x} \in \mathbf{R}^n : |\mathbf{x}| < 1,\ x_n > 0\big\},$$

in terms of Green's function $G_{\mathbf{a}}(\mathbf{x})$ for the whole unit ball.

7.4 A reflection principle. Let Ω be the half-space $\mathbf{R}^n_+ = \{x_n > 0\}$ in $\mathbf{R}^n$. Suppose that u is harmonic on Ω and continuous on $\overline{\Omega}$ with $u = 0$ on $\partial\Omega$. Extend u to a function v on the whole space $\mathbf{R}^n$ which is odd with respect to the last coordinate x_n, as follows: (Answer.)

$$v(x_1,\dots,x_{n-1},x_n) = \begin{cases} u(x_1,\dots,x_{n-1},x_n), & x_n \ge 0,\\ -u(x_1,\dots,x_{n-1},-x_n), & x_n < 0.\end{cases}$$

(a) Obviously, v is harmonic for $x_n > 0$ and continuous on $\mathbf{R}^n$. Show that v is also harmonic for $x_n < 0$.

(b) Show that v is harmonic everywhere. Hint: Given a point $\mathbf{a}$ on the hyperplane $x_n = 0$, use the Poisson formula for a ball in $\mathbf{R}^n$ to produce a function w which is harmonic on the ball $B(\mathbf{a}, r)$ and agrees with v on the boundary. Show that w must agree with v on the ball (argue separately for the upper and lower halves).

7.5 Uniqueness for the Dirichlet problem on a half-space. Like in exercise 7.4, let $\Omega = \mathbf{R}^n_+$. Use the results from that exercise, together with Liouville's theorem, to prove that there can exist at most one **bounded** solution to the Dirichlet problem $\Delta u = f$ on Ω, $u = g$ on $\partial\Omega$. (Answer.)

7.6 Green's function for the positive quadrant. (Answer.)

(a) Determine Green's function $G_{(a,b)}(x, y)$ for $-\Delta$ in the quadrant $\{x > 0,\ y > 0\}$ in $\mathbf{R}^2$.

(b) Use it to write down a solution (the unique bounded one) to the Laplace equation $\Delta u = 0$ in the quadrant, with boundary data $u(x,0) = g(x)$ for $x > 0$ and $u(0, y) = h(y)$ for $y > 0$, where g and h are continuous and bounded.

8 The wave equation in one dimension

8.1 Initial conditions. In each part, find the solution (Answer.)

$$u(x,t) = f(x + ct) + g(x - ct)$$

of the wave equation $u_{tt} = c^2 u_{xx}$ satisfying the given initial conditions. What does the the solution look like? Try to illustrate it graphically. (The solutions in parts (c) and (d) will only satisfy the PDE in a weak sense.)

(a) $u(x,0) = e^{-x^2}$ and $u_t(x,0) = \frac{1}{1+x^2}$.

(b) $u(x,0) = 0$ and $u_t(x,0) = \cos x$.

(c) $u(x,0) = \begin{cases} 1 - |x|, & |x| < 1, \\ 0, & |x| \geq 1, \end{cases}$ and $u_t(x,0) = 0$.

(d) $u(x,0) = 0$ and $u_t(x,0) = \begin{cases} 1, & |x| < 1, \\ 0, & |x| > 1. \end{cases}$

8.2 Wave equation with source. (Answer.)

(a) Solve $u_{tt} - c^2 u_{xx} = 1$ with $u(x,0) = 0$ and $u_t(x,0) = 0$.

(b) Solve $u_{tt} - c^2 u_{xx} = \cos x$ with $u(x,0) = \sin x$ and $u_t(x,0) = 0$.

(c) Solve $u_{tt} - u_{xx} = \sin(\omega t)\sin x$ (where $\omega > 0$) with $u(x,0) = 0$ and $u_t(x,0) = 0$.

8.3 Alternative derivation. Your task here is to fill in the steps below to give another derivation of the solution to $u_{tt} - c^2 u_{xx} = f(x,t)$ with initial conditions $u(x,0) = \varphi(x)$, $u_t(x,0) = \psi(x)$, namely (Answer.)

$$u(x_0, t_0) = \frac{\varphi(x + ct) + \varphi(x - ct)}{2} + \frac{1}{2c}\int_{x_0 - ct_0}^{x_0 + ct_0} \psi(x)\,dx + \frac{1}{2c}\iint_D f(x,t)\,dx\,dt,$$

where $D = D_{(x_0,t_0)}$ is the triangle with corners at (x_0, t_0) and $(x_0 \pm c t_0, 0)$.

(a) Begin with
$$\iint_D f(x,t)\,dx\,dt = \iint_D (u_{tt} - c^2 u_{xx})\,dx\,dt,$$
and rewrite the right-hand side as a line integral over ∂D using Green's theorem from vector calculus; this line integral splits into three integrals, one over each edge of the triangle.

(b) Show that the contribution from the edge on the x-axis is
$$-\int_{x_0 - c t_0}^{x_0 + c t_0} \psi(x)\,dx.$$

(c) Show that the edge from $(x_0 + c t_0, 0)$ to (x_0, t_0) contributes
$$c\,u(x_0, t_0) - c\,\varphi(x_0 + c t_0).$$

(d) Similarly, show that the edge from (x_0, t_0) to $(x_0 - c t_0, 0)$ contributes
$$c\,u(x_0, t_0) - c\,\varphi(x_0 - c t_0).$$

(e) Add it all up!

8.4 Energy. Let $e(x,t) = \frac{1}{2}(u_t^2 + c^2 u_x^2)$ be the energy density for the wave equation $u_{tt} = c^2 u_{xx}$. (Answer.)

(a) Show by direct differentiation that $e(x,t)$ satisfies the wave equation if $u(x,t)$ does (assuming $u \in C^3$).

(b) Express $e(x,t)$ in terms of the function f and g in the general solution $u(x,t) = f(x+ct) + g(x-ct)$.

(c) Meditate upon the results.

8.5 Damped wave equation. For the *damped* wave equation $u_{tt} + r u_t = c^2 u_{xx}$, where $r > 0$, show that the energy $E(t) = \frac{1}{2}\int_{-\infty}^{\infty}(u_t^2 + c^2 u_x^2)\,dx$ is a nonincreasing function of t (assuming that all functions in sight are sufficiently nice and decay as $|x| \to \infty$). (Answer.)

9 The wave equation in one dimension (cont.)

9.1 Wave equation on a half-line with Neumann condition. Solve $u_{tt} = c^2 u_{xx}$ for $x > 0$ and $t > 0$ with initial conditions $u(x,0) = \varphi(x)$ and $u_t(x,0) = \psi(x)$ for $x > 0$ and the Neumann boundary condition $u_x(0,t) = 0$ for $t > 0$. What assumptions on φ and ψ are needed in order to obtain a classical solution (class C^2)? (Answer.)

9.2 Series solution on a finite interval. Solve $u_{tt} = c^2 u_{xx}$ for $0 < x < \pi$ and $t > 0$, with boundary conditions for $t > 0$ and initial conditions for $0 < x < \pi$ as specified. (The answer will be given in terms of a series. It might be useful to look back at exercise 3.3.) (Answer.)

(a) Dirichlet conditions $u(0,t) = u(\pi,t) = 0$, initial data $u(x,0) = x(\pi - x)$ and $u_t(x,0) = 0$.

(b) Neumann conditions $u_x(0,t) = u_x(\pi,t) = 0$, initial data $u(x,0) = \cos^2 x$ and $u_t(x,0) = 1 - \cos 3x$.

(c) Mixed boundary conditions $u_x(0,t) = u(\pi,t) = 0$, initial data $u(x,0) = 0$ and $u_t(x,0) = \pi^2 - x^2$.

9.3 Reflections on a finite interval. Use reflections and d'Alembert's formula to solve $u_{tt} = c^2 u_{xx}$ for $0 < x < \pi$ and $t > 0$, with $u(0,t) = u(\pi,t) = 0$ and $u(x,0) = \sin(nx)$ (for some integer $n > 0$), $u_t(x,0) = 0$. Does the result agree with what you would expect? (Answer.)

9.4 Odd solution. Show that if u is a (classical) solution to the wave equation (on the whole real line) with **odd** initial data $u(x,0) = \varphi(x)$ and $u_t(x,0) = \psi(x)$, then u is an odd function of x for any $t > 0$. (Answer.)

10 The wave equation in higher dimensions

10.1 Maxwell's equations. In electromagnetism, Maxwell's equations for the electric field $\mathbf{E}(\mathbf{x},t)$ and the magnetic field $\mathbf{B}(\mathbf{x},t)$ are (Answer.)

$$\nabla\cdot\mathbf{E} = \frac{\rho}{\varepsilon_0}, \quad \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t},$$
$$\nabla\cdot\mathbf{B} = 0, \quad \nabla\times\mathbf{B} = \mu_0\left(\mathbf{J} + \varepsilon_0\frac{\partial\mathbf{E}}{\partial t}\right),$$

where ε_0 and μ_0 are physical constants ("vacuum permittivity" and "vacuum permeability"), $\rho(\mathbf{x},t)$ is electric charge density per unit volume, and $\mathbf{J}(\mathbf{x},t)$ is electric current density per unit area.

Show that in empty space, where $\rho = 0$ and $\mathbf{J} = \mathbf{0}$, each component of the vector fields $\mathbf{E}$ and $\mathbf{B}$ satisfies the wave equation, with $c = (\varepsilon_0\mu_0)^{-1/2}$ (the speed of light).

10.2 Spherical waves. (Answer.)

(a) A spherical solution to the wave equation $u_{tt} = \Delta u$ is a solution which is radial with respect to the spatial variables $\mathbf{x} \in \mathbf{R}^n$, i.e., it takes the form

$$u(\mathbf{x},t) = U(|\mathbf{x}|, t)$$

for some function $U(r,t)$ (which we can think of as being defined for $r \ge 0$, or as being defined for all $r \in \mathbf{R}$ with the requirement that it's an *even* function of r).

Derive the PDE that U must satisfy in order for u to be spherical solution. (Does it look familiar?)

(b) What's the general form of a spherical solution in the case $n = 3$?

(c) For $n = 3$, use part (b) to solve $u_{tt} = \Delta u$ with initial conditions

$$u(\mathbf{x}, 0) = e^{-|\mathbf{x}|^2}, \qquad u_t(\mathbf{x}, 0) = 0.$$

(d) Similarly, for $n = 3$, solve $u_{tt} = \Delta u$ with initial conditions

$$u(\mathbf{x}, 0) = \begin{cases} 1, & |\mathbf{x}| \le 1, \\ 0, & |\mathbf{x}| > 1, \end{cases} \qquad u_t(\mathbf{x}, 0) = 0.$$

(The solution will obviously only satisfy the PDE in a weak sense.)

10.3 Time derivative of a solution is a solution. (Answer.)

(a) Show that if u is a C^3-solution of the wave equation $u_{tt} = c^2 \Delta u$, then u_t is a C^2-solution.

(b) If the initial conditions for u are $u(\mathbf{x}, 0) = 0$ and $u_t(\mathbf{x}, 0) = \psi(\mathbf{x})$, what are the initial conditions for u_t?

(c) Compare the explicit solution formulas for the wave equation in $\mathbf{R}$, $\mathbf{R}^2$ and $\mathbf{R}^3$, in terms of given initial data $u = \varphi$ and $u_t = \psi$ at time $t = 0$. These formulas have a common feature which is explained by parts (a) and (b) above – what is it?

10.4 Duhamel's principle for the inhomogeneous wave equation. (Answer.)

(a) Consider the wave equation with a source term,

$$u_{tt} - c^2 \Delta u = f(\mathbf{x}, t),$$

with zero initial conditions $u = u_t = 0$ at $t = 0$. Prove that the solution is

$$u(\mathbf{x}, t) = \int_0^t v(\mathbf{x}, t; s)\, ds,$$

where $v(\mathbf{x}, t; s)$ is the solution of the following initial value problem (starting at time s) for the usual homogeneous wave equation, with the source term f appearing in the initial conditions instead:

$$\begin{aligned} v_{tt}(\mathbf{x}, t; s) &= c^2 \Delta v(\mathbf{x}, t; s) \qquad \text{for } t > s, \\ v(\mathbf{x}, s; s) &= 0, \\ v_t(\mathbf{x}, s; s) &= f(\mathbf{x}, s). \end{aligned}$$

(b) Explain the physical intuition behind this!

(c) Using the known solution formulas for the homogeneous wave equation, write down explicitly what the resulting solution formulas for the inhomogeneous problem above look like for $n = 1$ and $n = 3$.

11 The heat equation on a bounded domain

11.1 Fourier series solutions. Solve the heat equation $u_t = u_{xx}$ on the interval $0 < x < \pi$, with the given initial and boundary conditions. (As usual, it is understood that the solutions should be continuous on the whole domain $[0,\pi] \times [0,\infty)$.) (Answer.)

(a) $u_x(0,t) = u_x(\pi,t) = 0$ for $t > 0$, $u(x,0) = x$ for $0 \le x \le \pi$.

(b) $u(0,t) = 0$ and $u(\pi,t) = 1$ for $t > 0$, $u(x,0) = \sin(x/2)$ for $0 \le x \le \pi$.

(Hint: Consider the difference between u and the equilibrium solution that you expect to see in the limit as $t \to \infty$.)

(c) $u(0,t) = 0$ and $u(\pi,t) = e^{-t}$ for $t > 0$, $u(x,0) = x/\pi$ for $0 \le x \le \pi$.

(Hint: Consider the difference between u and some (not too complicated) function that satisfies the boundary conditions. That will give you homogeneous boundary conditions, but now with an inhomogeneous heat equation instead. Solve it by expanding the sought function, and also the source term in the equation, in the same type of series that you would use for the standard heat equation, but with unknown time-dependent coefficients.)

11.2 A comparison principle. Notation: Let $\Omega \subset \mathbf{R}^n$ be open and bounded, write $\Omega_T = \Omega \times (0,T)$ for $0 < T \le \infty$, and let (Answer.)

$$\Gamma_T = \left\{ (\mathbf{x},t) \in \partial\Omega_T : \mathbf{x} \in \partial\Omega \text{ or } t = 0 \right\}$$

be the parabolic boundary of Ω_T.

(a) Suppose that f and g are functions such that $f \le g$ on Ω_∞, that u and v are continuous on $\overline{\Omega_\infty}$ and satisfy $u_t = \Delta u + f$ and $v_t = \Delta u + g$ on Ω_∞. Show that if $u \le v$ on Γ_T, then $u \le v$ on $\overline{\Omega_T}$.

(b) If v is continuous on $\overline{\Omega_\infty}$, satisfies $v_t = v_{xx} + \sin x$ for $0 < x < \pi$ and $t > 0$, and if $v(0,t) \ge 0$, $v(\pi,t) \ge 0$ and $v(x,0) \ge \sin x$, show that $v(x,t) \ge (1 - e^{-t}) \sin x$ for $0 \le x \le \pi$ and $t \ge 0$.

11.3 Backwards heat equation. Consider the heat equation $u_t = u_{xx}$ on the interval $0 < x < \pi$ for *negative* time $t < 0$, with boundary conditions $u(0,t) =$ (Answer.)

$u(\pi, t) = 0$ and "initial" (or maybe rather "final") condition $u(x, 0) = C\sin(nx)$ for some positive integer n.

(a) Show that the problem has a solution (by finding one explicitly).

(b) This type of problem is **ill-posed** (i.e., not well-posed). Why?

11.4 Another maximum principle. (Answer.)

(a) Show that if $u \geq 0$ and $u_t \leq \Delta u - cu$, where $c \geq 0$ is a constant, then

$$\max_{\overline{\Omega_T}} u = \max_{\Gamma_T} u.$$

(b) Can you find a counterexample with the condition $u \geq 0$ removed?

12 The heat equation on $\mathbf{R}^n$

12.1 Symmetries of the heat equation. If $u(x, y) = f(x, t)$ is a solution of the heat equation $u_t = u_{xx}$, show that so are the following: (Answer.)

(a) $u(x, y) = f(x - c, t)$

(b) $u(x, y) = f(x, t - c)$

(c) $u(x, y) = f(cx, c^2 t)$

(d) $u(x, y) = e^{-cx+c^2 t} f(x - 2ct, t)$

(e) $u(x, y) = \dfrac{1}{\sqrt{1+4ct}} \exp\left(\dfrac{-cx^2}{1+4ct}\right) f\left(\dfrac{x}{1+4ct}, \dfrac{t}{1+4ct}\right)$

(f) $u(x, y) = \dfrac{1}{\sqrt{4\pi t}} \exp\left(-\dfrac{x^2}{4t}\right) f\left(\dfrac{x}{t}, \dfrac{-1}{t}\right)$

12.2 Heat polynomials. (Answer.)

(a) Check that the function $u(x, t) = e^{xz+tz^2}$ satisfies $u_t = u_{xx}$ for every $z \in \mathbf{R}$ (or $z \in \mathbf{C}$, if you like).

(b) The *heat polynomials* $p_n(x, t)$ are defined by the power series expansion

$$e^{xz+tz^2} = \sum_{n=0}^{\infty} p_n(x, t) \frac{z^n}{n!}.$$

Compute the first few heat polynomials, for example by multiplying the power series for e^{xz} and e^{tz^2}, and check that they satisfy the heat equation.

(c) Play around with the transformations in exercise 12.1, with some heat polynomial as a starting point, to generate some more impressive-looking solutions.

12.3 Duhamel's principle for the inhomogeneous heat equation. From exercise 10.4, recall Duhamel's principle for the wave eqation $u_{tt} - c^2\Delta u = f(\mathbf{x}, t)$ with zero initial conditions $u(\mathbf{x}, t) = u_t(\mathbf{x}, 0) = 0$. What do you think the corresponding statement would be for the heat equation $u_t - D\Delta u = f(\mathbf{x}, t)$ with zero initial condition $u(\mathbf{x}, t) = 0$? Formulate a conjecture, and prove it! (Answer.)

12.4 Heaviside initial data. The goal of this exercise is to solve the heat equation $u_t = u_{xx}$ with initial values given by the Heaviside function H, (Answer.)

$$u(x,0) = H(x) = \begin{cases} 0, & x < 0, \\ 1/2, & x = 0, \\ 1, & x > 0. \end{cases}$$

(Since H is discontinuous, we obviously cannot require the solution u to be continuous on the whole closed domain $(x, t) \in \mathbf{R} \times [0, \infty)$, but let's require it to be continuous except at the origin, and to satisfy $\lim_{t\to 0^+} u(0, t) = 1/2$.)

(a) Since the initial data on the x-axis are unchanged when doing the transformation $v(x, t) = u(cx, ct^2)$, let's seek a solution of the form $u(x, t) = g(x/\sqrt{t})$ (for $t > 0$) which is also invariant under this transformation. What ODE does g have to satisfy, in order for u to solve the heat equation?

(b) Find the general solution of that ODE, and use the initial data to pick out the particular function that we want. Write down the resulting solution $u(x, t)$, and check that the initial condition really is satisfied in the sense specified above.

(c) Compute $u_x(x, t)$. Does the result look familiar?

12.5 Heat equation on a half-line. (Answer.)

(a) How would you solve the heat equation $u_t = u_{xx}$ for $x > 0$ and $t > 0$ with initial condition $u(x, 0) = \varphi(x)$ (a bounded function) for $x > 0$ and with the Dirichlet boundary condition $u(0, t) = 0$ for $t > 0$?

(b) The same question, but with the Neumann boundary condition $u_x(0, t) = 0$ for $t > 0$.

(c) Show that

$$v(x, t) = \frac{x}{t\sqrt{t}} e^{-\frac{x^2}{4t}}$$

satisfies $v_t = v_{xx}$ for $x > 0$ and $t > 0$ and that $v(x, t) \to 0$ as $x \to 0^+$ with $t > 0$ fixed and also as $t \to 0^+$ with $x > 0$ fixed. What does that say about uniqueness of the solution in part (a)? Is v bounded?

13 Classification of second-order linear PDEs

13.1 Transformation to new variables. (Answer.)

(a) Consider the general second-order linear PDE in two variables:

$$\begin{aligned}&A(x,y)\,u_{xx}+2B(x,y)\,u_{xy}+C(x,y)\,u_{yy}\\&+D(x,y)\,u_x+E(x,y)\,u_y+F(x,y)\,u+G(x,y)=0.\end{aligned}$$

Transform this equation to new variables $r=r(x,y)$, $s=s(x,y)$, i.e., write out the expressions for the coefficients in the transformed equation

$$\begin{aligned}&\widetilde{A}(r,s)\,u_{rr}+2\widetilde{B}(r,s)\,u_{rs}+\widetilde{C}(r,s)\,u_{ss}\\&+\widetilde{D}(r,s)\,u_r+\widetilde{E}(r,s)\,u_s+\widetilde{F}(r,s)\,u+\widetilde{G}(r,s)=0\end{aligned}$$

in terms of the original coefficients and of partial derivatives of $r(x,y)$ and $s(x,y)$. (It is assumed that the change of variables is of class C^2 and invertible, and that the inverse is also of class C^2.)

(b) Show that the type of the PDE (elliptic/parabolic/elliptic) at a given point does not depend on the coordinate system. That is, show that the sign of $\widetilde{A}\widetilde{C}-\widetilde{B}^2$ at the point $(r(x,y),s(x,y))$ is the same as the sign of $AC-B^2$ at the point (x,y).

(c) Suppose that the PDE is parabolic ($AC-B^2=0$, but not $A=B=C=0$) in some domain in $\mathbf{R}^2$. Explain how to find a change of variables which makes $\widetilde{B}=\widetilde{C}=0$.

13.2 Some hyperbolic PDEs. (Answer.)

(a) Show that the equation

$$y^2\,u_{yy}-2xy\,u_{xy}+2x\,u_x=0$$

is hyperbolic away from the coordinate axes, and compute the general C^2-solution (say in the first quadrant $x>0$, $y>0$) by changing to characteristic coordinates.

(b) Show that the equation

$$xy\,u_{xx}+(x^2-y^2)\,u_{xy}-xy\,u_{yy}+\frac{(y^3-3x^2y)\,u_x+(3xy^2-x^3)\,u_y}{x^2+y^2}=8xy$$

is hyperbolic away from the origin, and compute the general C^2-solution (say in the right half-plane $x>0$) by changing to characteristic coordinates.

13.3 A parabolic PDE. (Answer.)

(a) Show that the equation

$$4y^2 u_{xx} - 4y u_{xy} + u_{yy} - 2u_x = 6y$$

is parabolic, and compute the general C^2-solution by changing to characteristic coordinates (as in part (c) of exercise 13.1 above).

(b) Find the particular solution satisfying the conditions $u(x,0) = x^2$ and $u_y(x,0) = \sin x$.

13.4 A mixed-type PDE. (Answer.)

(a) Determine the type of the **Tricomi equation**[1]

$$y\, u_{xx} + u_{yy} = 0,$$

at each point $(x, y) \in \mathbf{R}^2$.

(b) In the region where the equation is hyperbolic, determine the characteristic curves, and express the equation in characteristic coordinates.

(c) In the region where the equation is elliptic, express it in terms of the new variables $(w, z) = (x, \frac{2}{3} y^{3/2})$.

(For an idea of where the inspiration for this change of variables comes from, see the remark in the answer for part (b).)

14 Generalized solutions

14.1 Distributions (on R). (Answer.)

(a) Show that $H' = \delta$ in the sense of distributions, where H is the Heaviside function: $H(x) = 0$ if $x < 0$ and $H(x) = 1$ if $x > 0$. (Opinions vary regarding what the value $H(0)$ should be, but this value is irrelevant here.)

(b) Let f be a smooth function. Show that $f\delta = f(0)\,\delta$, and derive analogous expressions for $f\delta'$ and $f\delta''$.

(c) If $u(x) = e^{-|x|}$, compute u' and u'' in the sense of distributions.

(d) Show that $(fT)' = f'T + fT'$ if f is a smooth function and T is a distribution.

(e) It can be shown that if T is a distribution such that $T' = 0$, then $T = C$ (i.e., T equals the distribution T_f associated with the constant function $f(x) = C$) for some C. Use this to find all distributions that satsify the ODE $T' - 3T = \delta$.

[1] This equation arises in the study of so-called *transonic flow*, like air flow around the wings of a plane flying close to the speed of sound, where there are regions with subsonic flow as well as regions with supersonic flow.

(f) Consider the heat kernel $S(x,t) = \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{x^2}{4t}\right)$, $x \in \mathbf{R}$, $t > 0$. For each $t > 0$, the function $S(\cdot, t)$ defines a distribution on $\mathbf{R}$. Show that $S(\cdot, t) \to \delta$ in the sense of distributions as $t \to 0^+$, i.e., that $\int_{\mathbf{R}} S(x,t)\,\varphi(x)\,dx \to \varphi(0)$ for every test function φ.

14.2 Formal adjoint. For Lu as given below, write down $L^*\varphi$, where L^* is the formal adjoint of L, defined such that $\langle Lu, \varphi\rangle = \langle u, L^*\varphi\rangle$ for test functions φ. (Answer.)

(a) $Lu = u_t - u_{xx}$.

(b) $Lu = u_{xx} + u_{yy}$.

(c) $Lu = xy\,u_{xx} + u_{xyy}$

14.3 The wave equation. Verify that $u = f(x - ct)$ is a weak solution of the wave equation $u_{tt} = c^2 u_{xx}$ if f is locally integrable. (A similar argument works for $u = g(x + ct)$, and by linearity it follows that $u = f(x - ct) + g(x + ct)$ is a weak solution.) (Answer.)

14.4 The inviscid Burgers equation. Find weak solutions of $u_t + uu_x = 0$ (for $t > 0$) with the following initial conditions $u(x,0) = u_0(x)$: (Answer.)

(a)

$$u_0(x) = \begin{cases} 1, & x \le 0, \\ 1 - x, & 0 < x < 1, \\ 0, & x \ge 1. \end{cases}$$

(b)

$$u_0(x) = \begin{cases} 0, & x < 0, \\ x - 1, & x > 0. \end{cases}$$

14.5 Inviscid Burgers with damping. Consider the equation $u_t + uu_x + au = 0$, where $a > 0$ is a constant. (Answer.)

(a) Define what we should mean by a weak solutions of this PDE.

(b) Go through the derivation of the Rankine–Hugoniot jump condition to check that the extra term au doesn't make any difference; the velocity of a shock must still be the average of the values of u to the left and to the right of the jump, just like for the equation $u_t + uu_x = 0$.

(c) In order to study the initial value problem with $u(x,0) = u_0(x)$, compute the characteristic curve starting at $(x,t,z) = (x_0, 0, u_0(x_0))$.

(d) Find a continuous weak solution with

$$\begin{cases} 0, & x \leq 0, \\ x, & 0 < x < 1, \\ 1, & x \geq 1. \end{cases}$$

(e) Find a shock wave solution with

$$\begin{cases} 1, & x < 0, \\ 0, & x > 0. \end{cases}$$

(f) Can you give a condition for $u_0(x)$ which will guarantee that no shocks are formed?

14.6 The Cole–Hopf transformation. Show that if v satisfies the heat eqation $v_t = \mu u_{xx}$, then $u = -2\mu v_x / v$ satisfies the Burgers equation $u_t + uu_x = \mu u_{xx}$. (Answer.)

15 Numerical methods

15.1 Explicit finite difference scheme. Consider the heat equation $u_t = u_{xx}$ for $0 < x < 1$ and $t > 0$, with zero boundary values $u(0, t) = u(1, t) = 0$ and initial condition $u(x, 0) = \sin(\pi x)$. (Answer.)

(a) Compute the exact solution $u(x, t)$.

(b) Recall the standard explicit finite difference scheme for the heat equation,

$$\frac{U(k, m+1) - U(k, m)}{\tau} = \frac{U(k+1, m) - 2U(k, m) + U(k-1, m)}{h^2},$$

where $U(k, m)$ is the approximation to the solution $u(x, t)$ at the grid point $(x, t) = (kh, m\tau)$. For a very crude approximation, take $h = 1/2$, so that there are just two subintervals $(0, \frac{1}{2})$ and $(\frac{1}{2}, 1)$ and one interior grid point $(1, m)$ at each time step m. (The boundary values are $U(0, m) = 0 = U(2, m)$ for all $m \geq 0$, of course.) What's the appropriate initial value $U(1, 0)$? Compute $U(1, m)$ exactly in terms of τ, for $m \geq 0$.

(c) For a slightly better approximation, take $h = 1/4$, so that there are three internal grid points $(1, m)$, $(2, m)$ and $(3, m)$ at each time step. What are the appropriate initial values $U(1, 0)$, $U(2, 0)$ and $U(3, 0)$? Compute $U(1, m)$, $U(2, m)$ and $U(3, m)$ exactly in terms of τ, for $m \geq 0$.

(d) The same as in part (c), but with $u(x, 0) = \sin(3\pi x)$.

(e) The same as in part (c), but with $u(x, 0) = \sin(4\pi x)$.

(f) The same as in part (c), but with $u(x,0) = \sin(5\pi x)$.

15.2 Crank–Nicolson scheme. Repeat exercise 15.1 with the Crank–Nicolson scheme (Answer.)

$$\frac{U(k,m+1)-U(k,m)}{\tau} = \frac{1}{2}\frac{U(k+1,m)-2U(k,m)+U(k-1,m)}{h^2} + \frac{1}{2}\frac{U(k+1,m+1)-2U(k,m+1)+U(k-1,m+1)}{h^2}.$$

15.3 Neumann condition. In the explicit finite difference scheme for the heat equation, how would you handle a Neumann-type boundary condition such as $u_x(0,t) = g(t)$? (Answer.)

15.4 Solving tridiagonal linear systems. Show how to factor a tridiagonal $n\times n$ matrix A into a product of two bidiagonal $n\times n$ matrices L and R, as follows (with $n = 5$, for example; omitted matrix entries are understood to be zero): (Answer.)

$$\underbrace{\begin{pmatrix} a_1 & b_1 & & & \\ c_1 & a_2 & b_2 & & \\ & c_2 & a_3 & b_3 & \\ & & c_3 & a_4 & b_4 \\ & & & c_4 & a_5 \end{pmatrix}}_{A} = \underbrace{\begin{pmatrix} 1 & & & & \\ l_1 & 1 & & & \\ & l_2 & 1 & & \\ & & l_3 & 1 & \\ & & & l_4 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} m_1 & r_1 & & & \\ & m_2 & r_2 & & \\ & & m_3 & r_3 & \\ & & & m_4 & r_4 \\ & & & & m_5 \end{pmatrix}}_{R}.$$

Explain how to use this to solve the tridiagonal linear system $A\mathbf{x} = \mathbf{d}$ with an amount of work proportional to n.

15.5 FEM solution of Poisson's equation in one dimension. (Answer.)

(a) Consider the ODE $-u''(x) = f(x)$ for $0 < x < 1$, with Dirichlet boundary conditions $u(0) = u(1) = 0$. Write down the equations that determine the FEM solution with regularly space mesh points at $x = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$.

(b) Compare the equations in part (a) with those that you would get if you instead used a finite-difference approximation of u''.

(c) Do the same as in part (a), but with mixed boundary conditions $u(0) = 0$ and $u'(1) = 0$.

(d) Repeat the previous parts with an irregularly spaced mesh, $x = \{0, \frac{1}{3}, \frac{1}{2}, 1\}$.

15.6 Element stiffnesses. For a triangulation of a domain in $\mathbf{R}^2$, consider a particular triangle T with vertices at nodes number a, b and c, and angles α, β and γ, respectively, at these nodes. Show that the element stiffnesses, which are defined as (Answer.)

$$K_{ij}^T = \int_T \nabla\varphi_i(x,y)\cdot\nabla\varphi_j(x,y)\,dx\,dy$$

where $\varphi_i(x,y)$ is the standard "tent" basis function at node i, are given by

$$K^T_{aa} = \tfrac{1}{2}(\cot\beta + \cot\gamma), \qquad K^T_{bc} = -\tfrac{1}{2}\cot\alpha,$$

and similarly for the other vertices (by symmetry).

16 Separation of variables in higher dimensions

16.1 Some properties of the Laplace operator. Let Ω be a (sufficiently nice) bounded domain in $\mathbf{R}^n$, and consider the eigenvalue problem $-\Delta u = \lambda u$ in Ω, with boundary values $u = 0$. (Answer.)

(a) Show that the operator $-\Delta$ is *symmetric* when acting on functions that are zero on the boundary: $\int_\Omega (-\Delta u)\, v\, dV = \int_\Omega u\,(-\Delta v)\, dV$.

(b) Show that the operator $-\Delta$ is also *positive definite*, that is, $\int_\Omega (-\Delta u)\, u\, dV > 0$ for all u that are zero on the boundary but not identically zero in Ω.

(c) Show that eigenfunctions corresponding to different eigenvalues are *orthogonal*: $\int_\Omega u_i\, u_j\, dV = 0$ if $\lambda_i \neq \lambda_j$.

(d) Show using positive definiteness that the eigenvalues λ are *positive*. (You may assume that the eigenvalues are real to begin with; this follows from symmetry, as was shown in the lecture.)

(e) What changes in the questions above if we replace the Dirichlet boundary condition $u = 0$ with the Neumann boundary condition $\partial u / \partial n = 0$?

16.2 The heat equation on a square. Solve (in the form of a double Fourier series) the heat equation $u_t = \Delta u$ on the square $\Omega = (0,\pi)^2$ with boundary condition $u(x,y,t) = 0$ for $(x,y) \in \partial\Omega$ and initial condition $u(x,y,0) = x(\pi - x)\sin^2 y$. (You may save a bit of work if reuse some of the calculations from exercise 9.2.) (Answer.)

16.3 Circular sector. (Answer.)

(a) Let $0 < \beta < 2\pi$. Use separation of variables in polar coordinates (r,φ) to determine the eigenvalues and eigenfunctions of the problem $-\Delta u = \lambda u$ on the sector $r < a$, $0 < \varphi < \beta$, with $u = 0$ on the boundary.

(b) Next, read the text **The MathWorks Logo is an Eigenfunction of the Wave Equation** and see how much you are able to understand.

16.4 Eigenfunctions as minimizers. Consider the integral (Answer.)

$$I(w) = \int_0^1 w'(x)^2\, dx,$$

where

$$w \in C^2, \qquad \int_0^1 w(x)^2 \, dx = 1, \qquad w(0) = w(1) = 0.$$

What's the smallest value that $I(w)$ can take, and for which function(s) $w(x)$ does that happen?

16.5 Eigenfunctions in a ball. (Answer.)

(a) Consider the eigenvalue problem $-\Delta u = \lambda u$ for the origin-centered ball in $\mathbf{R}^3$ of radius a, with $u = 0$ on the boundary. Separate the variables in spherical coordinates, $u = R(r)\,\Theta(\theta)\,\Phi(\varphi)$. (See exercise 1.8 for the Laplacian expressed in spherical coordinates.)

(b) In the equation for $R(r)$, make the substitution $R(r) = Q(\sqrt{\lambda}\, r)/\sqrt{r}$ and show that this leads to a Bessel ODE (with what parameter?) for $Q(\rho)$.

(c) In the equation for $\Theta(\theta)$, derive the ODE that results from making the substitution $z = \cos\theta$.

17 Dispersive waves and solitons

17.1 Dispersion relations. Determine whether the following PDEs are dispersive, by investigating the dispersion relation $\omega = \omega(k)$ for harmonic waves (Answer.)

$$u(x,t) = e^{i(kx - \omega t)}.$$

For those that are, compute the phase and group velocities:

$$c_{\text{phase}} = \frac{\omega(k)}{k}, \qquad c_{\text{group}} = \omega'(k).$$

(All parameters appearing in the equations are assumed to be positive.)

(a) The heat equation, $u_t = u_{xx}$.

(b) The advection equation, $u_t + c u_x = 0$.

(c) The wave equation, $u_{tt} - c^2 u_{xx} = 0$.

(d) The linearized Korteweg–de Vries equation, $u_t + c u_x + b u_{xxx} = 0$.

(e) The linearized Boussinesq equation, $u_{tt} - c^2 u_{xx} = b^2 u_{xxtt}$.

(f) The Klein–Gordon equation, $u_{tt} - c^2 u_{xx} + m^2 u = 0$.

(g) The telegraph equation, $u_{tt} - c^2 u_{xx} + \alpha u_t + m^2 u = 0$.

17.2 Definition of "dispersive". For a PDE (linear, constant-coefficient) to be counted as dispersive, we require that the dispersion relation $\omega = \omega(k)$ satisfies $\omega''(k) \not\equiv 0$; in other words, it's not of the form $\omega = ck + d$. The case $\omega = ck$ is clearly non-dispersive, since the phase velocity and the group velocity are both equal to c, so that wave packets travel undistorted. But why do you think the case $\omega = ck + d$ with $d \neq 0$ is excluded too? (Answer.)

17.3 Interference. Given the dispersion relation $\omega = \omega(k)$, consider the superposition of two harmonic waves with equal amplitude but slightly different wave numbers $k - \delta$ and $k + \delta$, where δ is small ($0 < \delta \ll k$): (Answer.)

$$u(x,t) = \cos\Big((k-\delta)x - \omega(k-\delta)\,t\Big) + \cos\Big((k+\delta)x - \omega(k+\delta)\,t\Big).$$

Approximate $\omega(k \pm \delta)$ with $\omega(k) \pm \delta\,\omega'(k)$, and use $\cos(A-B) + \cos(A+B) = 2\cos A \cos B$ to rewrite the sum as a product. Conclusions? What does the wave $u(x,t)$ look like?

17.4 Solution involving Airy functions. Similarly to what we did with the heat equation in exercise 12.4, consider the dispersive wave equation $u_t + u_{xxx} = 0$ with initial condition $u(x,0) = H(x)$ (the Heaviside function). Since the PDE and the initial data on the x-axis are invariant under the transformation $v(x,t) = u(cx, ct^3)$, let's seek a solution of the form $u(x,t) = g(x/(3t)^{1/3})$ for $t > 0$. (Writing $3t$ instead of t here is just to make life a little simple later on.) Derive an ODE for the function g and try to find the general solution, as well as the particular solution that matches the given initial data as $t \to 0^+$. (Answer.)

17.5 Dispersion relation for water surface waves. Consider the equations from linear water wave theory, modelling surface gravity waves over a flat bottom at depth h: (Answer.)

$$\begin{aligned} \Delta\varphi &= 0, & -h < z < 0, \\ \frac{\partial\varphi}{\partial z} &= 0, & z = -h, \\ \frac{\partial^2\varphi}{\partial t^2} + g\,\frac{\partial\varphi}{\partial z} &= 0, & z = 0, \end{aligned}$$

where $\varphi(x,y,z,t)$ is the velocity potential (i.e., $\mathbf{u} = \nabla\varphi$ is the fluid velocity), and the shape of the surface wave is given by

$$z = \zeta(x,y,t) = -\frac{1}{g}\frac{\partial\varphi}{\partial t}(x,y,0,t).$$

Determine the dispersion relation $\omega = \omega(k)$ for harmonic wave solutions of the form $\varphi(x,y,z,t) = Z(z)\,e^{i(kx-\omega t)}$. What approximations do you obtain in the limiting cases $0 < k \ll h$ (long wavelenght compared to the depth, or equivalently shallow water compared to the wavelenght) and $k \to \infty$ (very short waves compared to the depth, or equivalently very deep water)?

17.6 KdV solitary wave. Show that in order for $u(x,t) = f(x-ct)$ to be a travelling wave solution to the Korteweg–de Vries equation $u_t + uu_x + u_{xxx} = 0$, the function $f(\xi)$ must satisfy the ODE (Answer.)

$$-cf'(\xi) + f(\xi)f'(\xi) + f'''(\xi) = 0.$$

Solve this ODE under the assumption that $f(\xi)$, $f'(\xi)$ and $f''(\xi)$ tend to zero as $\xi \to \pm\infty$, to find the solitary wave solution

$$u(x,t) = \frac{3c}{\cosh^2\left(\frac{\sqrt{c}}{2}(x-ct-x_0)\right)},$$

where $x_0 \in \mathbf{R}$ is an arbitrary constant (the location of the wave crest at $t=0$), and the wave velocity c must be positive.

17.7 The sine–Gordon equation. (Answer.)

(a) The sine–Gordon equation,

$$u_{tt} - u_{xx} + \sin u = 0,$$

is another nonlinear PDE which, like the KdV equation, is an *integrable system*. Find all solutions of the form $u(x,t) = f(x-ct)$ such that $u(x,t) \to 0$ as $x \to -\infty$ and $u(x,t) \to 2\pi$ as $x \to +\infty$.

(The quantity u in this equation represents an angle in applications – after all, we are taking the sine of it – so going from 0 to 2π takes you back to where you started. This type of solution, known as a *kink*, can therefore be considered as a type of solitary wave solution.

(b) Show that $u(x,t) = 4\arctan\dfrac{T(t)}{X(x)}$ satisfies the sine–Gordon equation if and only if

$$X\big((X^2+T^2)T_{tt} - 2TT_t^2\big) + T\big((X^2+T^2)X_{xx} - 2XX_x^2\big) + TX(X^2-T^2) = 0,$$

or equivalently

$$X^2\frac{T_{tt}}{T} + (XX_{xx} - 2X_x^2 + X^2) + T^2\frac{X_{xx}}{X} + (TT_{tt} - 2T_t^2 - T^2) = 0.$$

(c) Using part (b), verify that

$$u(x,t) = 4\arctan\frac{\sqrt{1-\omega^2}\cos(\omega t)}{\omega\cosh\big(\sqrt{1-\omega^2}\,x\big)}, \qquad 0<\omega<1,$$

satisfies the sine–Gordon equation.

(This type of solution is called a *breather* – why?)

Hints, comments, answers

1.1

$$\frac{\partial f}{\partial x}(a,b) = \frac{\partial g}{\partial u}\big(\alpha(a,b),\beta(a,b)\big)\,\frac{\partial \alpha}{\partial x}(a,b) + \frac{\partial g}{\partial v}\big(\alpha(a,b),\beta(a,b)\big)\,\frac{\partial \beta}{\partial x}(a,b),$$
$$\frac{\partial f}{\partial y}(a,b) = \frac{\partial g}{\partial u}\big(\alpha(a,b),\beta(a,b)\big)\,\frac{\partial \alpha}{\partial y}(a,b) + \frac{\partial g}{\partial v}\big(\alpha(a,b),\beta(a,b)\big)\,\frac{\partial \beta}{\partial y}(a,b).$$

1.2

(a) $\partial u/\partial x = 2x$ and $\partial x/\partial u = 0$, so the product is **not** equal to 1. See part (b) for an explanation.

(b) The important thing to understand here is that when we speak of (x, y) and (u, v) forming coordinate systems, we establish a *context* which is needed in order to interpret the notation correcty, namely that y is the "partner variable" of x, the quantity which is supposed to be held constant when computing $\partial u/\partial x$; this "hidden information" is not visible in the notation, since y isn't mentioned there. And likewise v is the "partner variable" of u, so when you are asked to compute $\partial x/\partial u$, it is understood that it is v that is supposed to be held constant.

So from the given formulas $u = x^2 - 3y$ and $v = x$, we find directly (treating y as a constant when computing the x-derivatives, and vice versa) that

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} 2x & -3 \\ 1 & 0 \end{pmatrix},$$

and in particular $u_x = \partial u/\partial x = 2x$.

For the other derivatives, we invert the change of variables to express x and y as functions of u and v:

$$x = v, \qquad y = \frac{v^2 - u}{3}.$$

Using these formulas, we find (treating v as a constant when computing the u-derivatives, and vice versa) that

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1/3 & 2v/3 \end{pmatrix},$$

and in particular $x_u = \partial x/\partial u = 0$.

A general fact in this situation, where we have a mapping from (u, v) to (x, y), and the inverse mapping from (x, y) back to (u, v), is that their Jacobian matrices must be each other's matrix inverses (this follows from the chain rule). In other words, their matrix product must be the identity matrix. And it's easy to verify that this indeed holds here:

$$\frac{\partial(u,v)}{\partial(x,y)}\frac{\partial(x,y)}{\partial(u,v)} = \begin{pmatrix} 2x & -3 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1/3 & 2v/3 \end{pmatrix} = \begin{pmatrix} 1 & 2x-2v \\ 0 & 1 \end{pmatrix} = [v = x] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(c) The chain rule gives

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} = 2x\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v},$$

which is obviously different from $\frac{\partial f}{\partial v}$, because of the extra term $2x\frac{\partial f}{\partial u}$. Explanation: Changing x (or v) keeping y fixed is not the same as doing it keeping u fixed.

1.3 Here it's helpful to use the idea from problem 1.1, and introduce named functions that describe some of the relations between the physical quantities. If

$$E = f(T,V) \qquad \text{and} \qquad V = g(T,p),$$

then

$$E = f\big(T, g(T,p)\big) = h(T,p),$$

so the chain rule gives

$$\frac{\partial h}{\partial T}(T,p) = \frac{\partial f}{\partial T}\big(T, g(T,p)\big) + \frac{\partial f}{\partial V}\big(T, g(T,p)\big)\frac{\partial g}{\partial T}(T,p),$$

which is exactly what it *means* when one writes

$$\left(\frac{\partial E}{\partial T}\right)_p = \left(\frac{\partial E}{\partial T}\right)_V + \left(\frac{\partial E}{\partial V}\right)_T\left(\frac{\partial V}{\partial T}\right)_p.$$

1.4

(a) Suppose that F is of class C^1, and that $(a,b,c) \in \mathbf{R}^3$ is a point that satisfies the equation $F = 0$. Then, if $F_x(a,b,c) \neq 0$, the implicit function says that there is a neighbourhood U of the point $(b,c) \in \mathbf{R}^2$ and a neighbourhood V of the point $a \in \mathbf{R}$, such that for each $(y,z) \in U$, there is exactly one $x \in V$ such that $F(x,y,z) = 0$; in other words, there is a *function* which given the input $(y,z) \in U$ outputs that value $x \in V$. Moreover, this function $x = f(x,y)$, with U as its domain of definition, is of class C^1. Similarly, if $F_y(a,b,c) \neq 0$ there is such a function $y = g(x,z)$, and if $F_z(a,b,c) \neq 0$ there is such a function $z = h(x,y)$. So we should assume that $F \in C^1$ and that all three partial derivatives F_x, F_y and F_z are nonzero at any point (a,b,c) such that $F(a,b,c) = 0$.

(b) Using the notation from part (a) above, implicit differentiation with respect to y of the identity $F(f(y,z),y,z)=0$, $(y,z)\in U$, shows that

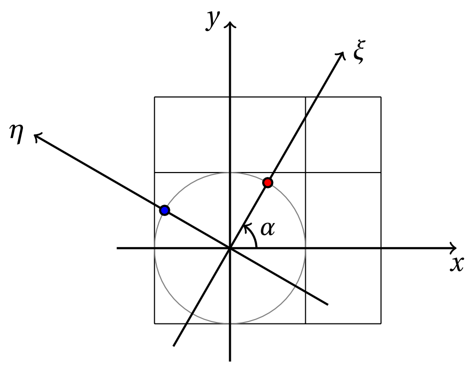
$$f_y(b,c) = -\frac{F_y(a,b,c)}{F_x(a,b,c)},$$

and (as you should be able to infer from the context) this is what is meant by the somewhat imprecise notation "$\partial x/\partial y$" in the question. Similarly for the other factors. Thus,

$$\begin{aligned}\frac{\partial x}{\partial y}\cdot\frac{\partial y}{\partial z}\cdot\frac{\partial z}{\partial x} &= f_y(b,c)\cdot g_z(a,c)\cdot h_x(a,b)\\ &= \left(-\frac{F_y(a,b,c)}{F_x(a,b,c)}\right)\left(-\frac{F_z(a,b,c)}{F_y(a,b,c)}\right)\left(-\frac{F_x(a,b,c)}{F_z(a,b,c)}\right) = -1,\end{aligned}$$

as was to be shown.

1.5 According to the given formulas, the values $(\xi,\eta)=(1,0)$ correspond to $(x,y)=(\cos\alpha,\sin\alpha)$ (the red dot in the picture), and the values $(\xi,\eta)=(0,1)$ correspond to $(x,y)=(-\sin\alpha,\cos\alpha)=\big(\cos(\alpha+\frac{\pi}{2}),\sin(\alpha+\frac{\pi}{2})\big)$ (the blue dot). So the (ξ,η) coordinate system lies rotated by the angle α with respect to the (x,y) coordinate system:



The chain rule gives $u_\xi = u_x x_\xi + u_y y_\xi = u_x\cos\alpha + u_y\sin\alpha$ and $u_\eta = u_x x_\eta + u_y y_\eta = -u_x\sin\alpha + u_y\cos\alpha$, and then

$$\begin{aligned} u_{\xi\xi} &= u_{xx}\cos^2\alpha + 2u_{xy}\sin\alpha\cos\alpha + u_{yy}\sin^2\alpha,\\ u_{\eta\eta} &= u_{xx}\sin^2\alpha - 2u_{xy}\sin\alpha\cos\alpha + u_{yy}\cos^2\alpha,\end{aligned}$$

which implies

$$u_{\xi\xi}+u_{\eta\eta} = u_{xx}\cdot 1 + u_{xy}\cdot 0 + u_{yy}\cdot 1 = u_{xx}+u_{yy},$$

since $\cos^2\alpha+\sin^2\alpha=1$.

1.6

(a) The chain rule gives $u_r = u_x x_r + u_y y_r = u_x \cos\varphi + u_y \sin\varphi$ and $u_\varphi = u_x x_\varphi + u_y y_\varphi = u_x(-r\sin\varphi) + u_y\, r\cos\varphi$, which can be written with a rotation matrix as

$$\begin{pmatrix} u_r \\ u_\varphi/r \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} u_x \\ u_y \end{pmatrix}.$$

(b) Inverting the rotation matrix gives

$$\begin{pmatrix} u_x \\ u_y \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} u_r \\ u_\varphi/r \end{pmatrix},$$

so that $u_x = \cos\varphi\, u_r - \frac{\sin\varphi}{r} u_\varphi$ and $u_y = \sin\varphi\, u_r + \frac{\cos\varphi}{r} u_\varphi$. Written in terms of differential operators, this takes the form

$$\frac{\partial}{\partial x} = \cos\varphi \frac{\partial}{\partial r} - \frac{\sin\varphi}{r}\frac{\partial}{\partial\varphi},$$
$$\frac{\partial}{\partial y} = \sin\varphi \frac{\partial}{\partial r} + \frac{\cos\varphi}{r}\frac{\partial}{\partial\varphi}.$$

Remark. One may notice that the relationship of the differential operators $\frac{\partial}{\partial r}$ and $\frac{1}{r}\frac{\partial}{\partial\varphi}$ to the operators $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ is the same as the relationship of the unit vector fields $\mathbf{e}_r$ and $\mathbf{e}_\varphi$ (sometimes denoted by $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\varphi}}$) to the constant unit vector fields $\mathbf{e}_x$ and $\mathbf{e}_y$ (or $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$, if you prefer). This is because these differential operators represent *directional derivatives* in the directions of the respective vector fields. The reason for the factor $1/r$ before $\partial/\partial\varphi$ here is that a change of $d\varphi$ in the value of the coordinate φ makes the point with polar coordinates (r,φ) move a distance of $r\,d\varphi$ (rather than just $d\varphi$) in the xy-plane, so that the derivative $\partial/\partial\varphi$, which measures sensitivity to changes in the value of θ, gives a value which is r times greater than the physically relevant directional derivative, which measures sensitivity to movements of the point by a certain distance in the angular direction.

(c) Using part (b), we find

$$\begin{aligned} u_{xx} &= \left(\tfrac{\partial}{\partial x}\right)^2 u = \left(\cos\varphi \tfrac{\partial}{\partial r} - \tfrac{\sin\varphi}{r}\tfrac{\partial}{\partial\varphi}\right)^2 u \\ &= \left(\cos\varphi \tfrac{\partial}{\partial r} - \tfrac{\sin\varphi}{r}\tfrac{\partial}{\partial\varphi}\right)\left(\cos\varphi \tfrac{\partial u}{\partial r} - \tfrac{\sin\varphi}{r}\tfrac{\partial u}{\partial\varphi}\right) \\ &= \cos^2\varphi \tfrac{\partial^2 u}{\partial r^2} - \tfrac{2\sin\varphi\cos\varphi}{r}\tfrac{\partial^2 u}{\partial r\partial\varphi} + \tfrac{\sin^2\varphi}{r^2}\tfrac{\partial^2 u}{\partial\varphi^2} + \tfrac{\sin^2\varphi}{r}\tfrac{\partial u}{\partial r} + \tfrac{2\sin\varphi\cos\varphi}{r^2}\tfrac{\partial u}{\partial\varphi} \\ &= \cos^2\varphi\, u_{rr} - \tfrac{2\sin\varphi\cos\varphi}{r} u_{r\varphi} + \tfrac{\sin^2\varphi}{r^2} u_{\varphi\varphi} + \tfrac{\sin^2\varphi}{r} u_r + \tfrac{2\sin\varphi\cos\varphi}{r^2} u_\varphi, \end{aligned}$$

and similarly

$$\begin{aligned} u_{yy} &= \sin^2\varphi\, u_{rr} + \tfrac{2\sin\varphi\cos\varphi}{r} u_{r\varphi} + \tfrac{\cos^2\varphi}{r^2} u_{\varphi\varphi} + \tfrac{\cos^2\varphi}{r} u_r - \tfrac{2\sin\varphi\cos\varphi}{r^2} u_\varphi, \\ u_{xy} &= \sin\varphi\cos\varphi\, u_{rr} + \tfrac{\cos^2\varphi-\sin^2\varphi}{r} u_{r\varphi} - \tfrac{\sin\varphi\cos\varphi}{r^2} u_{\varphi\varphi} \\ &\quad - \tfrac{\sin\varphi\cos\varphi}{r} u_r - \tfrac{\cos^2\varphi-\sin^2\varphi}{r^2} u_\varphi. \end{aligned}$$

In particular, since $\cos^2\varphi + \sin^2\varphi = 1$, the Laplacian becomes

$$\Delta u = u_{xx} + u_{yy} = u_{rr} + \tfrac{1}{r^2} u_{\varphi\varphi} + \tfrac{1}{r} u_r.$$

1.7

(a) Easy, it's just $X = u_x$ and $Y = u_y$ in the formula for the divergence.

(b) We have $\nabla u = (u_r / \boxed{1})\,\mathbf{e}_r + (u_\varphi / \boxed{r})\,\mathbf{e}_\varphi$ to begin with, so we take $R = u_r$ and $\Phi = u_\varphi / r$ in the formula for the divergence to obtain

$$\nabla \cdot \nabla u = \frac{1}{\boxed{1} \cdot \boxed{r}} \left(\frac{\partial}{\partial r}\left(u_r \cdot \boxed{r}\right) + \frac{\partial}{\partial \varphi}\left(\boxed{1} \cdot u_\varphi / r\right) \right) = \frac{\frac{\partial}{\partial r}(r \cdot u_r)}{r} + \frac{u_{\varphi\varphi}}{r^2} = u_{rr} + \tfrac{1}{r^2} u_{\varphi\varphi} + \tfrac{1}{r} u_r.$$

(Derivations of the formulas for gradient and divergence in orthogonal coordinates can be found in most vector calculus textbook. The proof for the divergence is based on computing the flow of the vector field through a "curvilinear box" using the divergence theorem, and then letting the side lengths of the box tend to zero.)

1.8 With $R = u_r / \boxed{1}$, $\Theta = u_\theta / \boxed{r}$ and $\Phi = u_\varphi / \boxed{r \sin\theta}$ in the formula for the divergence, we find

$$\begin{aligned}
\nabla \cdot \nabla u &= \frac{1}{\boxed{1} \cdot \boxed{r} \cdot \boxed{r \sin\theta}} \left(\frac{\partial}{\partial r}\left(u_r \cdot \boxed{r} \cdot \boxed{r \sin\theta}\right) + \frac{\partial}{\partial \theta}\left(\boxed{1} \cdot \frac{u_\theta}{r} \cdot \boxed{r \sin\theta}\right) + \frac{\partial}{\partial \varphi}\left(\boxed{1} \cdot \boxed{r} \cdot \frac{u_\varphi}{r \sin\theta}\right) \right) \\
&= \frac{\frac{\partial}{\partial r}(r^2 \cdot u_r)}{r^2} + \frac{\frac{\partial}{\partial \theta}(\sin\theta \cdot u_\theta)}{r^2 \sin\theta} + \frac{u_{\varphi\varphi}}{r^2 \sin^2\theta} \\
&= u_{rr} + \tfrac{1}{r^2} u_{\theta\theta} + \tfrac{1}{r^2 \sin^2\theta} u_{\varphi\varphi} + \tfrac{2}{r} u_r + \tfrac{\cos\theta}{r^2 \sin\theta} u_\theta.
\end{aligned}$$

1.9 The chain rule gives $u_x(x, y) = g'(xe^y)\, e^y$ and $u_y(x, y) = g'(xe^y)\, xe^y$, so

$$x u_x - u_y = x \cdot g'(xe^y)\, e^y - g'(xe^y)\, xe^y = 0,$$

since the two terms cancel.

Remark. It's very important here to write the derivative of g simply as g', and not as g'_x or g'_y or something like that. The function g is, in itself, a function of just *one* variable, say something like $g(t) = \sin t$. The two-variable function $u(x, y)$ is formed by composing this one-variable function $g(t)$ with the two-variable function $t = T(x, y) = xe^y$, and what the chain rule says is that $\frac{\partial u}{\partial x} = \frac{dg}{dt}\frac{\partial T}{\partial x}$. For example, if $u(x, y) = \sin(xe^y)$, then $u_x(x, y) = \cos(xe^y)\, e^y$, where $g(t) = \sin t$ and $g'(t) = \cos t$ (the ordinary one-variable derivative).

1.10

(a) Since u is assumed to be of of class C^1, it is differentiable, so we can use the chain rule to get $u_x = u_\xi \xi_x + u_\eta \eta_x = u_\xi + e^y u_\eta$ and $u_y = u_\xi \xi_y + u_\eta \eta_y = xe^y u_\xi$.

So we find

$$\begin{aligned}
&x u_x - u_y = 2x^2 \\
\iff\quad &x(u_\xi + e^y u_\eta) - x e^y u_\xi = 2x^2 \\
\iff\quad &x u_\xi = 2x^2 \\
\iff\quad &u_\xi = 2x \qquad \text{(we can cancel } x\text{, since } x > 0) \\
\iff\quad &u_\xi = 2\xi \\
\iff\quad &u = \xi^2 + g(\eta),
\end{aligned}$$

where $g(\eta)$ is a completely arbitrary function of η as far as the last step is concerned. But in order to make $u(x, y)$ a C^1-function of two variables, we need to require that $g(\eta)$ be a C^1-function of one variable. Since $x > 0$, we also have $\eta = xe^y > 0$, so only the values $g(\eta)$ for $\eta > 0$ are relevant here. Going back to the original variables, we thus obtain the general solution

$$u(x, y) = x^2 + g(xe^y), \quad x > 0,$$

where g is an arbitrary C^1-function of one (positive) variable.

(**Remark.** Note that since the PDE is *linear*, the solution has the structure "one particular solution to the PDE", namely $u_{\text{part}}(x, y) = x^2$, plus "the general solution to the homogeneous PDE $x u_x - u_y = 0$", namely $u_{\text{hom}}(x, y) = g(xe^y)$, $g \in C^1$; cf. problem 1.9.)

(b) We need to determine what the function $g(\eta)$ must be (for all $\eta > 0$) in order for the given condition $u(1, y) = e^{-y}$ to be satisfied identically (i.e., for all $y \in \mathbf{R}$). Plugging $x = 1$ into the general solution $u(x, y) = x^2 + g(xe^y)$ from part (a), we see that the condition becomes

$$u(1, y) = 1^2 + g(1 \cdot e^y) = e^{-y} \quad \iff \quad g(e^y) = \frac{1}{e^y} - 1,$$

which means that we must have

$$g(\eta) = \frac{1}{\eta} - 1, \qquad \eta > 0.$$

Plugging this function g back into the general solution $u(x, y) = x^2 + g(xe^y)$ from part (a), we find the answer

$$u(x, y) = x^2 + \frac{1}{xe^y} - 1, \qquad x > 0.$$

(c) Now the condition is $u(1, y) = 1 + g(e^y) = f(y)$, so $g(e^y) = f(y) - 1$, and hence

$$g(\eta) = f(\ln \eta) - 1, \qquad \eta > 0.$$

Plugging this function g back into the general solution $u(x, y) = x^2 + g(xe^y)$ from part (a), we find the answer

$$u(x, y) = x^2 + f\big(\ln(xe^y)\big) - 1 = x^2 + f(y + \ln x) - 1, \qquad x > 0.$$

2.1 The characteristic curve $\big(x(t), y(t)\big)$ starting at the point $(x, y) = (1, s)$ (where $u = f(s)$ is known) is determined by the ODEs

$$\dot{x} = x, \qquad x(0) = 1,$$
$$\dot{y} = -1, \qquad y(0) = s,$$

with the solution

$$x(t) = e^t, \qquad y(t) = s - t.$$

The PDE $xu_x - u_y = 2x^2$ now turns into an ODE along the characteristic curve:

$$\tfrac{d}{dt} u\big(x(t), y(t)\big) = 2(e^t)^2, \qquad u\big(x(0), y(0)\big) = f(s),$$

with the solution

$$u\big(x(t), y(t)\big) = e^{2t} + f(s) - 1,$$

i.e.,

$$u(e^t, s - t) = e^{2t} + f(s) - 1.$$

From $(x, y) = (e^t, s - t)$ we get $(t, s) = (\ln x, y + \ln x)$, and thus the answer is

$$u(x, y) = x^2 + f(y + \ln x) - 1, \qquad x > 0.$$

2.2

(a) The characteristic curve $(x(t), y(t))$ starting at $(x, y) = (0, s)$ is given by the ODEs $\dot{x} = 1 + x^2$ and $\dot{y} = 1$, with $x(0) = 0$ and $y(0) = s$. We can solve them separately, to obtain $x(t) = \tan t$ (for $|t| < \pi/2$) and $y(t) = t + s$. Along that characteristic, $z(t) = u(x(t), y(t))$ satisfies $\dot{z} = 0$ and $z(0) = f(s)$, so $z(t) = f(s)$. The values of t and s corresponding to a given point (x, y) are $t = \arctan x$ and $s = y - \arctan x$, so the value of u at that point is given by $f(s) = f(y - \arctan x)$.

Answer: $u(x, y) = f(y - \arctan x)$.

(b) Answer: $u(x, y) = \frac{1}{4} e^x (e^{2y} - e^{-2y})$.

(c) Answer, in terms of polar coordinates: $u(r\cos\varphi, r\sin\varphi) = f(r)\, e^{\varphi}$ (not globally defined).

(d) Answer: $u(x, y, z) = f(xe^{-z}, ye^{-z})\, e^z$.

(e) Answer: $u(x, y) = \dfrac{f(x - y)}{1 - y\, f(x - y)}$ (not globally defined).

(f) Answer: $u(x, y) = f\big(x/\sqrt{1 + y^2}\big) + \frac{1}{2}\ln(1 + y^2)$.

(g) Answer in the first case: $u(x, y) = \ln(e^y - x^2)$, for $y > \ln(x^2)$ (not globally defined).

Answer in the second case: $u(x, y) = g_1(e^y - x^2)$ for $x \geq 0$ and $u(x, y) = g_2(e^y - x^2)$ for $x < 0$, where g_1 and g_2 are any C^1-functions on $\mathbf{R}$ such that $g_1(s) = g_2(s) = s$ for $s \geq 0$.

In this problem, the characteristic curves in general have the form $e^y - x^2 = C$, but only the ones with $C > 0$ pass through the y-axis, so the given initial values $u(0, y)$ only propagate into the region $e^y - x^2 > 0$, i.e., the region above the curve $y = 2\ln|x|$. In the first case, the solution u tends to $-\infty$ as (x, y) approaches that curve, and therefore cannot be extended past that singularity. But in the second case, u tends to zero, so we can extend the solution to the whole plane as we like, as long as it's of class C^1 and constant on each characteristic. (Note that for each $C \leq 0$, there are actually two characteristics $x = \pm\sqrt{e^y - C}$, one in the right half-plane $x > 0$ and one in the left half-plane $x < 0$.)

3.1

(a) The boundary conditions obviously mean that the temperature is prescribed at the ends of the rod. One can imagine the rod being connected to "infinite heat reservoirs" of temperature A and B, respectively. Steady-state solution: $u(x, t) = A + (B - A)x$, for $0 \leq x \leq 1$.

(b) The heat equation is a conservation law $u_t + \partial_x(-u_x) = 0$, where the term $J = -u_x$ describes the heat flow (Fick's law). A positive/negative value of J means that heat is flowing in the positive/negative x direction. So the boundary conditions, with the signs chosen in that particular way, mean that the flow *into* the rod is prescribed to be A at $x = 0$ and B at $x = 1$. (Of course, negative values mean that the flow is actually going out of the rod.)

In particular, the case $A = B = 0$ means that the ends of the rod are *insulated* (no heat flows into or out of the rod). In this case, the total amount of heat energy in the rod should be conserved over time, and the same thing if $B = -A$, so that heat leaves the rod at one endpoint at the same rate as it enters the rod at the other endpoint. Indeed,

$$\frac{d}{dt}\int_0^1 u(x,t)\,dx = \int_0^1 u_t(x,t)\,dx = \int_0^1 u_{xx}(x,t)\,dx = [u_x(x,t)]_{x=0}^1 = B - (-A),$$

so the integral is time-independent if (and only if) $A + B = 0$. And in this case, the steady-state solution is $u(x, t) = Bx + C$ for $0 \leq x \leq 1$, where the value of C can be determined from the constant value of $\int_0^1 u(x,t)\,dx$. If $A + B \neq 0$, no steady-state solution exists (the total amount of heat energy in the rod tends to ∞ or $-\infty$ as $t \to \infty$).

(c) The temperature at the left endpoint is prescribed to be A, and the heat flow into the rod at the right endpoint is prescribed to be B. Steady-state solution: $u(x, t) = A + Bx$ for $0 \leq x \leq 1$.

3.2

(a) $T(t) = e^{-n^2 t}$.

(b) $u(x,t) = 17e^{-t}\sin x - 5e^{-9t}\sin(3x)$.

3.3

(a) $T(t) = \cos(nt)$.

(b) $u(x,t) = 17\cos t \sin x - 5\cos(3t)\sin(3x)$.

3.4 In the new variables, the PDE becomes $u_{\xi\eta} = 0$. The general solution is $u(x,t) = f(x+ct)+g(x-ct)$, where $f \in C^2(\mathbf{R})$ and $g \in C^2(\mathbf{R})$, so it's a superposition of two travelling waves with speed c, one moving to the left and the other to the right.

3.5 The verification is just computation. Regarding the limit, letting $t \to 0^+$ with x fixed, we find that $u(x,t)$ tends to 0 if $x \neq 0$ and to ∞ if $x = 0$. (And it does it in such a way that $\int_{-\infty}^{\infty} u(x,t) = 1$ for all $t > 0$, so the limit in the sense of distributions is the Dirac delta $\delta(x)$, for those of you who are familiar with that already. So this solution describes how an "infinite concentration of heat energy at a point" would spread out over time, in an infinitely long rod.)

3.6 A more detailed hint: $(\xi,\tau) = (x-ct, t)$. Answer: $u(x,t) = f(x-ct)\,e^{-rt}$.

4.1

(a) —

(b) Proof sketch: For a given vector $\mathbf{h}$, let $g(t) = f(\mathbf{a}+t\mathbf{h})$. Maclaurin expansion of g with second-order remainder term on Lagrange's form gives

$$g(1) = g(0) + g'(0) + \tfrac{1}{2}g''(\theta),$$

for some $\theta \in (0,1)$. By the chain rule, this is the same as

$$f(\mathbf{a}+\mathbf{h}) = f(\mathbf{a}) + \underbrace{\nabla f(\mathbf{a})}_{=\mathbf{0}} \cdot \mathbf{h} + \tfrac{1}{2}\mathbf{h}^T H(\mathbf{a}+\theta\mathbf{h})\,\mathbf{h}.$$

Since $f \in C^2$ by assumption, all entries of the Hessian matrix $H(\mathbf{x})$ are continuous functions, so if $H(\mathbf{a})$ is positive definite, then so is $H(\mathbf{a}+\theta\mathbf{h})$ if $|\mathbf{h}|$ is sufficiently small. And then, for all nonzero $\mathbf{h}$ of sufficiently small length, we have

$$f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) = \tfrac{1}{2}\mathbf{h}^T H(\mathbf{a}+\theta\mathbf{h})\,\mathbf{h} > 0,$$

i.e., f has a strict local minimum at $\mathbf{a}$. The proof for the negative definite case is extremely similar. And the proof for the indefinite case is rather similar – can you see how to do it?

(c) $f(x,y) = 0$, $f(x,y) = \pm x^2$, $f(x,y) = x^2 \pm y^4$, $f(x,y) = x^4 \pm y^4$, etc.

4.2 For example (where it is understood that $x \geq 0$ and $y \in \mathbf{R}$ in all cases):

(a) $u(x, y) = y$.

(b) $u(x, y) = x$.

(c) $u(x, y) = kx$ works, for any $k \in \mathbf{R}$. Or $u(x, y) = xy$.

(d) $u(x, y) = \dfrac{x^2 + y^2 - 1}{(x+1)^2 + y^2}$.

(e) $u(x, y) = \dfrac{y}{(x+1)^2 + y^2}$.

(f) $u(x, y) = \arctan \dfrac{y}{x+1}$.

4.3

(a) Straightforward, using the equality of mixed derivatives $u_{xy} = u_{yx}$ and $v_{xy} = v_{yx}$. (Analytic functions are infinitely differentiable, so u and v are smooth.)

(b) One can use that the points $z = \pm 1$, which are symmetric with respect to the imaginary axis, are mapped to $w = 0$ and $w = \infty$, which are symmetric with respect to the unit circle.

We have $f(x + iy) = \frac{x-1+iy}{x+1+iy} = \frac{(x-1+iy)(x+1-iy)}{(x+1)^2+y^2} = \frac{x^2+y^2-1+2yi}{(x+1)^2+y^2}$, so

$$u(x, y) = \operatorname{Re} f(x + iy) = \frac{x^2 + y^2 - 1}{(x+1)^2 + y^2} \qquad (\text{for } x \geq 0,\ y \in \mathbf{R})$$

has the half-open interval $[-1, 1)$ as its range, and

$$v(x, y) = \operatorname{Im} f(x + iy) = \frac{2y}{(x+1)^2 + y^2} \qquad (\text{for } x \geq 0,\ y \in \mathbf{R})$$

has the closed interval $[-1, 1]$ as its range.

(c) $v(x, y) = \operatorname{Im} \operatorname{Log}(x+1+iy) = \arctan \frac{y}{x+1}$ (for $x \geq 0$, $y \in \mathbf{R}$) has the open interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ as its range.

4.4 Hint: For $\varepsilon > 0$, let $v(\mathbf{x}) = u(\mathbf{x}) + \varepsilon |\mathbf{x}|^2$ and show that the maximum of v on $\overline{\Omega}$ must be attained on the boundary $\partial\Omega$, and not on Ω. (Assume that the maximum is attained at the point $\mathbf{a} \in \Omega$, so that $\nabla v(\mathbf{a}) = \mathbf{0}$ and $\Delta v(\mathbf{a}) \leq 0$, and consequently $\Delta v(\mathbf{x}) + \mathbf{x} \cdot \nabla v(\mathbf{x}) \leq 0$ for $\mathbf{x} = \mathbf{a}$. On the other hand, show that the hypotheses imply that $\Delta v(\mathbf{x}) + \mathbf{x} \cdot \nabla v(\mathbf{x}) > 0$ for all $\mathbf{x} \in \Omega$, a contradiction.) Then continue exactly as in the proof of the weak maximum principle for (sub)harmonic functions.

4.5 Hint: For example, use that $|\mathbf{x}-\mathbf{y}|^2 = (\mathbf{x}-\mathbf{y})\cdot(\mathbf{x}-\mathbf{y}) = \mathbf{x}\cdot\mathbf{x}-2\mathbf{x}\cdot\mathbf{y}+\mathbf{y}\cdot\mathbf{y}$. What's the angle between $\mathbf{x}$ and $\mathbf{y}$?

4.6 We get $\Theta''(\theta) = -\lambda\Theta(\theta)$ and $r^2R''(r)+rR'(r)-\lambda R(r) = 0$, where the boundary conditions $\Theta(0) = \Theta(\beta) = 0$ imply that $\lambda = (m\pi/\beta)^2$ for $m = 1,2,3,\ldots$, and that $\Theta(\theta)$ equals a constant times $\sin(m\pi\theta/\beta)$. Then $R(r) = Ar^{m\pi/\beta} + Br^{-m\pi/\beta}$, where we must take $B = 0$ since we need $R(r)$ to be nice at $r = 0$. Our solution u will be a linear combination of such separated solutions:

$$u(r,\theta) = \sum_{m=1}^{\infty} A_m r^{m\pi/\beta} \sin(m\pi\theta/\beta),$$

where we want to have $u(a,\theta) = h(\theta)$, i.e.,

$$\sum_{m=1}^{\infty} A_m a^{m\pi/\beta} \sin(m\pi\theta/\beta) = h(\theta), \qquad 0<\theta<\beta.$$

Multiply both sides by $\sin(n\pi\theta/\beta)$ and integrate over $\theta \in [0,\beta]$ to get

$$A_n a^{n\pi/\beta} \frac{\beta}{2} = \int_0^{\beta} h(\theta)\sin(n\pi\theta/\beta).$$

So the answer is

$$u(r,\theta) = \frac{2}{\beta}\sum_{m=1}^{\infty}\left(\int_0^{\beta} h(\varphi)\sin(m\pi\varphi/\beta)\,d\varphi\right)\left(\frac{r}{a}\right)^{m\pi/\beta}\sin(m\pi\theta/\beta).$$

4.7 Separated solutions satisfying the conditions along the top, bottom and left edges:

$$u_0(x,y) = x\cdot 1,$$
$$u_n(x,y) = \sinh(nx)\cdot\cos(ny), \quad \text{for integers } n \ge 1.$$

The condition at the right edge is

$$u(\pi,y) = \cos^2 y = \frac{1}{2}+\frac{1}{2}\cos(2y) = \frac{1}{2\pi}u_0(\pi,y)+\frac{1}{2\sinh(2\pi)}u_2(\pi,y),$$

so the answer is

$$u(x,y) = \frac{1}{2\pi}u_0(x,y)+\frac{1}{2\sinh(2\pi)}u_2(x,y) = \frac{x}{2\pi}+\frac{\sinh(2x)\cos(2y)}{2\sinh(2\pi)}.$$

4.8

(a) This shouldn't be too hard – after all, the functions $x \mapsto u(x,0)$ and $y \mapsto u(0,y)$ are constant!

(b) —

(c) The function u is harmonic away from the origin, by exercise 4.3. Regarding what happens at the origin, consider

$$g(x) = u(x,0) = \begin{cases} e^{-1/x^4}, & x \neq 0, \\ 0, & x = 0, \end{cases}$$

which is a typical example of a one-variable function which is smooth but not analytic at the origin (its derivatives of all orders can be shown to exist at the origin, and they are all zero, so that the Maclaurin expansion is identically zero and therefore does not converge to the function). In particular we have $g''(0) = 0$, which is the same thing as $u_{xx}(0,0) = 0$. And along the y-axis, it's just the same, so $u_{yy}(0,0) = 0$ too. Thus, $\Delta u(0,0) = 0$, but u is discontinuous at the origin, since $u(t,t) = \operatorname{Re} e^{-1/(t+it)^4} = e^{1/(4t^4)} \to \infty$ as $t \to 0$. (The function $f(z)$ has an essential singularity at $z = 0$.)

(d) Because of the identity $z\overline{z} = |z|^2$, we have $z = 1/\overline{z}$ when $|z| = 1$, and thus also

$$f(z) = f(1/\overline{z}) = e^{-\overline{z}^4} = \overline{e^{-z^4}} \qquad \text{when } |z| = 1.$$

The real part isn't affected by complex conjugation, so

$$u(x,y) = \operatorname{Re} f(x+iy) = \operatorname{Re} \overline{e^{-(x+iy)^4}} = \operatorname{Re} e^{-(x+iy)^4} \qquad \text{when } x^2 + y^2 = 1.$$

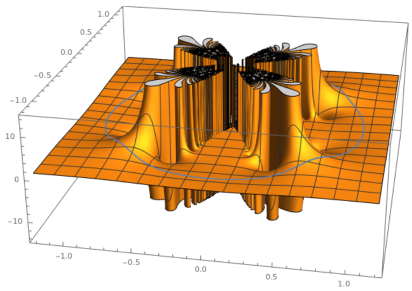
Thus, letting

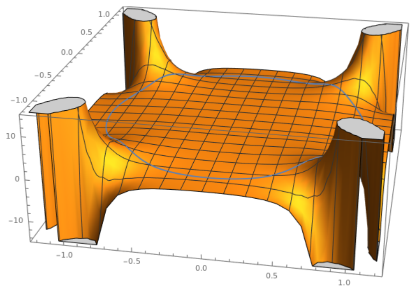
$$v(x,y) = \operatorname{Re} e^{-(x+iy)^4}$$

we obtain a function which agrees with u on the unit circle, and which is harmonic on the unit disk (and in fact everywhere).

Uniqueness of the Dirichlet problem for the Laplace equation shows that this function is what Poisson's formula would produce if we supplied it with the values of $u(x,y)$ for $x^2 + y^2 = 1$.

Below are the graphs of u (left) and v (right):





5.1 The function v is continuous and satisfies

$$\begin{aligned} \Delta v &= 0, && \text{if } |x| < 1 \text{ and } |y| < 1, \\ v &= \tfrac{1}{4}(x^2 + y^2), && \text{if } |x| = 1 \text{ or } |y| = 1. \end{aligned}$$

Moreover, $u(0,0) = v(0,0)$, and by the maximum principle (for the harmonic function v) this value lies between the minimum and the maximum of v on the boundary, which is easily seen to be $v(1,0) = \frac{1}{4}$ and $v(1,1) = \frac{1}{2}$, respectively. Answer: $u(0,0) \in [\frac{1}{4}, \frac{1}{2}]$.

5.2 Sketch of proof: The harmonic function u assumes its maximum A on the boundary, and the harmonic function $-u$ assumes its maximum B on the boundary too. And the maximum of $|u|$ is the largest of these two values, so it must be assumed on the boundary as well.

And the strong maximum principle holds too, since if the maximum of $|u|$ is attained at some interior point, then either u or $-u$ attains its maximum there, and must therefore be constant (assuming that Ω is connected), which means that $|u|$ is constant too.

5.3 The proof is the same as in the bounded case: Let M be the maximum value, and write Ω as the disjoint union of the sets

$$\Omega_1 = \{\mathbf{x} \in \Omega : u(\mathbf{x}) = M\} \qquad \text{and} \qquad \Omega_2 = \{\mathbf{x} \in \Omega : u(\mathbf{x}) < M\}.$$

The mean value property of subharmonic function (the value at a point is less than or equal to the average over spheres centered at that point) can be used to show that Ω_1 is open, and the continuity of u implies that Ω_2 is open. The set Ω_1 is nonempty, since u was assumed to assume its maximum in Ω, and since Ω is connected this means that Ω_2 must be empty.

5.4 Assume that u has a strict local maximum at the point $\mathbf{a} \in \Omega$. This means that there is a ball centered at that point, such that $u(\mathbf{x}) < u(\mathbf{a})$ for all $\mathbf{x}$ in that ball except $\mathbf{x} = \mathbf{a}$. But then the average of u over a sphere inside the ball would be less than $u(\mathbf{a})$, contradicting the mean value property. Similarly if u has a strict local minimum.

If u has a non-strict local extremum at $\mathbf{a}$, the mean value property implies that $u(\mathbf{x})$ must be constant (equal to $u(\mathbf{a})$) in some ball centered at $\mathbf{a}$, but it's not completely obvious how to extend this to the whole domain Ω. The theorem "harmonic functions are of class C^∞" has a stronger version saying "harmonic functions are real analytic" (i.e., they agree with their Taylor series), and this implies an identity theorem saying that two harmonic functions agreeing on a open subset of a connected open set Ω have to agree on all of Ω. So if u is constant on a ball in Ω, it follows that u is constant on Ω. Conclusion: a non-constant harmonic function cannot have any local extrema (strict or not).

5.5 Suppose, in order to derive a contradiction, that u is not harmonic. Then there is some point in Ω where Δu is nonzero, say $\Delta u(\mathbf{a}) > 0$. The assumption that $u \in C^2(\Omega)$ implies that Δu is continuous, so we must have $\Delta u > 0$ in some ball centered at $\mathbf{a}$. So for all sufficiently small $r > 0$ we have (using the divergence theorem)

$$0 < \int_{B(\mathbf{a},r)} \Delta u \, dV = \int_{\partial B(\mathbf{a},r)} \nabla u \cdot \mathbf{n} \, dS.$$

Dividing by the area of the sphere, to turn the integral into a mean value integral, and using the same calculation as in the proof of the mean value property, we obtain

$$0 < \fint_{\partial B(\mathbf{a},r)} \nabla u \cdot \mathbf{n} \, dS = \frac{d}{dr} \fint_{\partial B(\mathbf{a},r)} u \, dS.$$

But because of the assumption on u in this exercise, the right-hand side here equals $\frac{d}{dr} u(\mathbf{a}) = 0$, and this is the desired contradiction.

6.1 Hint: Show that the integral can take arbitrarily small positive values, but not the value zero. (Consider, for example, the function $f(x) = \arctan(x/\varepsilon) / \arctan(1/\varepsilon)$ for $\varepsilon > 0$.)

More details: The integral is clearly nonnegative, and it's zero if and only if the integrand is identically zero, which can't happen since then f would have to be constant and at the same time satisfy the boundary conditions $f(\pm 1) = \pm 1$. With f as in the hint, we have $f'(x) = \frac{1}{\arctan(1/\varepsilon)} \cdot \frac{\varepsilon}{\varepsilon^2 + x^2}$, and the integral is

$$\begin{aligned}
\int_{-1}^{1} \left(x f'(x) \right)^2 dx &= \frac{1}{\arctan^2(1/\varepsilon)} \int_{-1}^{1} \frac{x^2 \varepsilon^2 \, dx}{(\varepsilon^2 + x^2)^2} \\
&< \frac{1}{\arctan^2(1/\varepsilon)} \int_{-1}^{1} \frac{\varepsilon^2 \, dx}{\varepsilon^2 + x^2} \qquad \text{(since } 0 \le x^2 < x^2 + \varepsilon^2\text{)} \\
&= \frac{1}{\arctan^2(1/\varepsilon)} \Big[\varepsilon \arctan(x/\varepsilon) \Big]_{-1}^{1} \\
&= \frac{2\varepsilon}{\arctan(1/\varepsilon)},
\end{aligned}$$

which tends to zero as $\varepsilon \to 0^+$.

6.2

(a) Hint: Write the energy integral in polar coordinates (don't forget the Jacobian determinant r), and use that the gradient ∇u equals $(C/r)\mathbf{e}_r$ in the middle region $R^2 < r < R$ (so that $|\nabla u|^2 = C^2/r^2$ there) and is zero in the inner and outer regions.

Answer: $E(u) = -2\pi C^2 \ln R$.

(b) For example, $R_n = e^{-n}$ and $C_n = n^{-2/3}$ will work. (Then $u_n(0,0) = -n^{1/3}$ and $E(u_n) = 2\pi n^{-1/3}$.)

6.3

(a) —

(b) —

(c) This follows (for example) from the expression for the gradient in polar coordinates; see exercise 1.7.

(d) Termwise differentiation gives

$$U_r(r,\theta) = \sum_{k=1}^{\infty} \Big(k a_k r^{k-1} \cos k\theta + k b_k r^{k-1} \sin k\theta \Big)$$

and

$$U_\theta(r,\theta) = \sum_{k=1}^{\infty} \Big(-k a_k r^{k} \sin k\theta + k b_k r^{k} \cos k\theta \Big),$$

so that (by Parseval's identity)

$$\frac{1}{2\pi} \int_0^{2\pi} U_r(r,\theta)^2 \, d\theta = \frac{0^2}{4} + \frac{1}{2} \sum_{k=1}^{\infty} \big((k a_k r^{k-1})^2 + (k b_k r^{k-1})^2 \big)$$

and

$$\frac{1}{2\pi} \int_0^{2\pi} U_\theta(r,\theta)^2 \, d\theta = \frac{0^2}{4} + \frac{1}{2} \sum_{k=1}^{\infty} \big((-k a_k r^{k})^2 + (k b_k r^{k})^2 \big).$$

So the energy integral over the disk with radius R becomes

$$\begin{aligned}
E_R(u) &= \iint\limits_{\substack{0<r<R \\ 0\le\theta<2\pi}} \Big(U_r^2 + r^{-2} U_\theta^2 \Big) r \, dr \, d\theta \\
&= \int_0^R 2\pi \left(\frac{1}{2} \sum_{k=1}^{\infty} k^2 r^{2k-2} \big(a_k^2 + b_k^2\big) + \frac{r^{-2}}{2} \sum_{k=1}^{\infty} k^2 r^{2k} \big(a_k^2 + b_k^2\big) \right) r \, dr \\
&= 2\pi \sum_{k=1}^{\infty} \left(k^2 \big(a_k^2 + b_k^2\big) \int_0^R r^{2k-1} \, dr \right) \\
&= 2\pi \sum_{k=1}^{\infty} \left(k^2 \big(a_k^2 + b_k^2\big) \left[\frac{r^{2k}}{2k} \right]_0^R \right) \\
&= \pi \sum_{k=1}^{\infty} k \big(a_k^2 + b_k^2\big) R^{2k},
\end{aligned}$$

as claimed.

(e) The double inequality should be obvious from the result in part (d). Since it holds for all $R \in [0,1)$, and the inequalities are non-strict, it holds also in the limit as $R \to 1$ on the left-hand side and in the middle:

$$\pi \sum_{k=1}^{N} k \big(a_k^2 + b_k^2\big) \le E(u) \le \pi \sum_{k=1}^{\infty} k \big(a_k^2 + b_k^2\big).$$

And since *this* double inequality holds for all $N \geq 1$, it holds also in the limit as $N \to \infty$ on the left-hand side:

$$\pi \sum_{k=1}^{\infty} k\big(a_k^2 + b_k^2\big) \leq E(u) \leq \pi \sum_{k=1}^{\infty} k\big(a_k^2 + b_k^2\big).$$

And here the outer expressions are the same, so in fact the inequalities are equalities.

(f) The series for $h(\theta)$ is majorized by the convergent series $\sum_{m=1}^{\infty} \frac{1}{m^2}$, so it converges uniformly by the Weierstrass M-test. And a uniformly convergent sum of continuous functions is continuous.

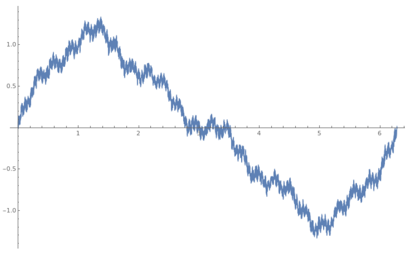
Note that this series is a very "sparse" Fourier series, the only nonzero coefficients being $b_{m!} = 1/m^2$ for $m \geq 1$:

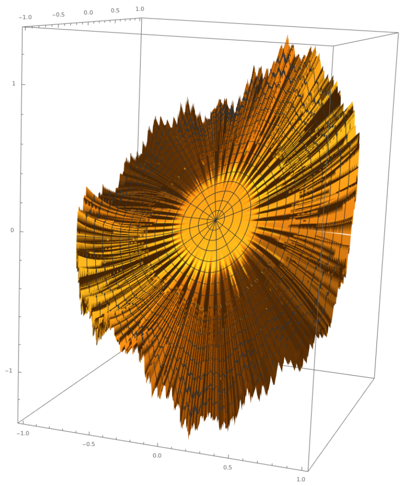
$$\begin{aligned} h(\theta) = &\frac{1}{1^2}\sin(\theta) \\ &+ \frac{1}{2^2}\sin(2\theta) + 0\sin(3\theta) + 0\sin(4\theta) + 0\sin(5\theta) \\ &+ \frac{1}{3^2}\sin(6\theta) + 0\sin(7\theta) + 0\sin(8\theta) + \cdots + 0\sin(23\theta) \\ &+ \frac{1}{4^2}\sin(24\theta) + 0\sin(25\theta) + 0\sin(26\theta) + \cdots \end{aligned}$$

The series derived in part (d) for the energy $E(u)$ of the corresponding solution u thus becomes

$$\begin{aligned} &\pi \sum_{k=1}^{\infty} k\big(a_k^2 + b_k^2\big) = \pi \sum_{k=1}^{\infty} k\big(0^2 + b_k^2\big) \\ &= \pi\big(1 \cdot b_1^2 + 2 \cdot b_2^2 + 0 + 0 + 0 + 6 \cdot b_6^2 + 0 + \cdots + 0 + 24 \cdot b_{24}^2 + \cdots\big) \\ &= \pi\left(1 \cdot \left(\frac{1}{1^2}\right)^2 + 2 \cdot \left(\frac{1}{2^2}\right)^2 + 6 \cdot \left(\frac{1}{3^2}\right)^2 + 24 \cdot \left(\frac{1}{4^2}\right)^2 + \cdots\right) \\ &= \pi \sum_{m=1}^{\infty} \frac{m!}{m^4}, \end{aligned}$$

which is clearly divergent, since the terms don't even tend to zero. And thus that solution has infinite energy. The pictures below show approximations to the graphs of $h(\theta)$ and $u(x,y)$ obtained from the partial sums $\sum_{m=1}^{20}$:





6.4 Compute the flow, as suggested:

$$0 = \left[u = 0 \text{ on } \partial\Omega\right] = \int_{\partial\Omega} u \nabla u \cdot \mathbf{n}\, dS = \int_{\Omega} \nabla \cdot (u \nabla u)\, dV = \int_{\Omega} \left(\nabla u \cdot \nabla u + u \Delta u\right) dV$$
$$= \left[\Delta u = u^3\right] = \int_{\Omega} |\nabla u|^2\, dV + \int_{\Omega} u^4\, dV.$$

Since both integrals on the right-hand side are nonnegative, they have to be zero, and this is only possible if $u = 0$.

6.6

(a) $\Phi(x) = -\frac{1}{2}|x|$. The unit sphere in $\mathbf{R}^1$ is the set $\{\pm 1\}$, and $A_1 = 2$ simply means that this set contains two points.

(b) $\Phi'(x) = -\frac{1}{2}\operatorname{sgn} x = \frac{1}{2} - H(x)$, so $\Phi''(x) = -\delta(x)$.

6.7 From

$$\Phi_{\mathbf{a}}(x,y) = -\frac{1}{2\pi} \ln \sqrt{(x - r\cos\theta)^2 + (y - r\sin\theta)^2}$$
$$= -\frac{1}{4\pi} \ln\left((x - r\cos\theta)^2 + (y - r\sin\theta)^2\right)$$

we obtain

$$\nabla \Phi_{\mathbf{a}}(x,y) = -\frac{1}{4\pi} \cdot \frac{1}{(x - r\cos\theta)^2 + (y - r\sin\theta)^2} \begin{pmatrix} 2(x - r\cos\theta) \\ 2(y - r\sin\theta) \end{pmatrix}$$

and hence

$$\frac{\partial \Phi_{\mathbf{a}}}{\partial n}(R\cos\varphi, R\sin\varphi) = \nabla\Phi_{\mathbf{a}}(R\cos\varphi, R\sin\varphi)\cdot\mathbf{n}$$

$$= -\frac{1}{2\pi}\cdot\frac{\begin{pmatrix}R\cos\varphi - r\cos\theta\\ R\sin\varphi - r\sin\theta\end{pmatrix}\cdot\begin{pmatrix}\cos\varphi\\ \sin\varphi\end{pmatrix}}{(R\cos\varphi - r\cos\theta)^2 + (R\sin\varphi - r\sin\theta)^2}$$

$$= -\frac{1}{2\pi}\cdot\frac{R - r\cos(\varphi-\theta)}{R^2 - 2Rr\cos(\varphi-\theta) + r^2}.$$

7.1 From exercice 6.6, the fundamental solution for $-\Delta$ on R^1 is $\Phi(x) = -\frac{1}{2}|x|$, and Green's function at the point $a \in (0,1)$ is $G_a(x) = \Phi(x-a) + w(x)$, where the harmonic function $w(x) = Cx + D$ is chosen such that $G_a(0) = G_a(1) = 0$. This gives

$$G_a(x) = -\tfrac{1}{2}|x-a| + (\tfrac{1}{2}-a)x + \tfrac{1}{2}a = \begin{cases}(1-a)x, & 0\le x\le a\le 1,\\ (1-x)a, & 0\le a\le x\le 1.\end{cases}$$

(Remark: The symmetry lets us extend $G_a(x)$ to be definied on the closed square $0\le a\le 1$, $0\le x\le 1$, instead of just for $0<a<1$, $0\le x\le 1$.)

7.2 From

$$\nabla\Phi(\mathbf{x}) = \frac{-1}{A_n|\mathbf{x}|^{n-1}}\frac{\mathbf{x}}{|\mathbf{x}|} = \frac{-\mathbf{x}}{A_n|\mathbf{x}|^n},$$

where A_n is the area of the unit sphere in $\mathbf{R}^n$, the chain rule gives

$$\nabla G_{\mathbf{a}}(\mathbf{x}) = \frac{-(\mathbf{x}-\mathbf{a})}{A_n|\mathbf{x}-\mathbf{a}|^n} - |\mathbf{a}|\frac{-|\mathbf{a}|(\mathbf{x}-\mathbf{b})}{A_n\big||\mathbf{a}|(\mathbf{x}-\mathbf{b})\big|^n}.$$

Now remember that the vectors $\mathbf{x}-\mathbf{a}$ and $|\mathbf{a}|(\mathbf{x}-\mathbf{b})$ have the same length when $|\mathbf{x}|=1$; this is the property used in the construction of $G_{\mathbf{a}}$ to ensure that $G_{\mathbf{a}}(\mathbf{x}) = 0$ when $|\mathbf{x}|=1$. Then the two terms in the expression above have the same denominators, and we can write

$$\nabla G_{\mathbf{a}}(\mathbf{x}) = -\frac{(\mathbf{x}-\mathbf{a}) - |\mathbf{a}|^2(\mathbf{x}-\mathbf{b})}{A_n|\mathbf{x}-\mathbf{a}|^n} = -\frac{(1-|\mathbf{a}|^2)\mathbf{x}}{A_n|\mathbf{x}-\mathbf{a}|^n}\quad\text{when } |\mathbf{x}|=1.$$

(Here we used that $\mathbf{a} - |\mathbf{a}|^2\mathbf{b} = \mathbf{0}$ by the definition of $\mathbf{b}$.) On the unit sphere, the normal vector is simply $\mathbf{n} = \mathbf{x}$, and we also have $\mathbf{x}\cdot\mathbf{x} = |\mathbf{x}|^2 = 1$ there, so that

$$-\frac{\partial G_{\mathbf{a}}}{\partial n}(\mathbf{x}) = -\nabla G_{\mathbf{a}}(\mathbf{x})\cdot\mathbf{n} = -\nabla G_{\mathbf{a}}(\mathbf{x})\cdot\mathbf{x} = \frac{(1-|\mathbf{a}|^2)\mathbf{x}\cdot\mathbf{x}}{A_n|\mathbf{x}-\mathbf{a}|^n} = \frac{1-|\mathbf{a}|^2}{A_n|\mathbf{x}-\mathbf{a}|^n}\quad\text{when } |\mathbf{x}|=1.$$

7.3 Answer: $H_{\mathbf{a}} = G_{\mathbf{a}} - G_{\mathbf{b}}$, where $\mathbf{b} = (a_1,\dots,a_{n-1},-a_n)$ and $a_n>0$.

Recall that $G_{\mathbf{a}}(\mathbf{x}) = \Phi(\mathbf{x}-\mathbf{a}) - \Phi(|\mathbf{a}|(\mathbf{x}-\mathbf{a}^*))$, where $\mathbf{a}^*$ is the inversion of $\mathbf{a}$ with respect to the unit sphere. Since $G_{\mathbf{b}}$ has its singularities in the lower half-space, our function $H_{\mathbf{a}}$ equals the fundamental solution $\Phi_{\mathbf{a}}$ plus something which is harmonic in the upper half-ball. On the unit sphere, both $G_{\mathbf{a}}$ and $G_{\mathbf{b}}$ are zero, hence so is $H_{\mathbf{a}}$. And if $\mathbf{x}$ is a point with $x_n = 0$, then the distances from $\mathbf{x}$ to $\mathbf{a}$ and to $\mathbf{b}$ will be equal, and likewise the distances from $\mathbf{x}$ to $\mathbf{a}^*$ and to $\mathbf{b}^*$ will be equal, and this makes $G_{\mathbf{a}}(\mathbf{x}) = G_{\mathbf{b}}(\mathbf{x})$ since Φ is radially symmetric, and hence $H_{\mathbf{a}}(\mathbf{x}) = 0$ when $x_n = 0$. Thus $H_{\mathbf{a}}(\mathbf{x})$ is zero on the boundary of the upper half-ball, as required.

7.4

(a) For $x_n < 0$ a simple calculation gives

$$\Delta v(x_1, \ldots, x_{n-1}, x_n) = -\Delta u(x_1, \ldots, x_{n-1}, -x_n) = 0.$$

(The derivative $\partial^2/\partial x_n^2$ produces a factor $(-1)^2$ from the chain rule.)

(b) For symmetry reasons, w is zero for all points in the ball with $x_n = 0$, so it agrees with v at those points too. Thus, w is the unique solution of the Dirichlet problem where $\Delta w = 0$ inside the upper half-ball and $w = v$ on its boundary, and therefore it agrees with v on the closed upper half-ball. Similarly for the closed lower half-ball. So $v = w$ on the whole ball, and since w is harmonic there, so is v.

7.5 Suppose u_1 and u_2 are solutions. Then $u = u_1 - u_2$ is a bounded function which is harmonic on Ω and zero on $\partial\Omega$. As in problem 7.4, extend u to a function v which is bounded and harmonic on the whole space. By Liouville's theorem, v is constant, and therefore so is u. And this constant value is of course zero, since $u = 0$ on $\partial\Omega$. Hence, $u_1 = u_2$.

7.6

(a) The fundamental solution centered at (a, b) is

$$\begin{aligned}\Phi_{(a,b)}(x, y) &= \Phi(x-a, y-b) \\ &= \frac{-1}{2\pi} \ln \sqrt{(x-a)^2 + (y-b)^2} = \frac{-1}{4\pi} \ln\big((x-a)^2 + (y-b)^2\big),\end{aligned}$$

and using reflection arguments one finds that the following works:

$$G_{(a,b)} = \Phi_{(a,b)} - \Phi_{(-a,b)} - \Phi_{(a,-b)} + \Phi_{(-a,-b)}.$$

Indeed, it's clear that $G_{(a,b)}$ equals $\Phi_{(a,b)}$ plus a function which is harmonic in the first quadrant (since the other three terms have their singularities in other quadrants), and it's zero on the boundary since

$$G_{(a,b)}(x,0) = \underbrace{\Phi_{(a,b)}(x,0) - \Phi_{(a,-b)}(x,0)}_{=0} + \underbrace{\Phi_{(-a,-b)}(x,0) - \Phi_{(-a,b)}(x,0)}_{=0}$$

and

$$G_{(a,b)}(0,y) = \underbrace{\Phi_{(a,b)}(0,y) - \Phi_{(-a,b)}(0,y)}_{=0} + \underbrace{\Phi_{(-a,-b)}(0,y) - \Phi_{(a,-b)}(0,y)}_{=0}.$$

(For example, the terms in the first of these four pairs cancel out since the point $(x,0)$ is equally far from the points (a,b) and $(a,-b)$, and Φ is radially symmetric. Similarly for the other pairs.)

(b) On the positive x-axis we have $-\partial/\partial n = \partial/\partial y$, so

$$
\begin{aligned}
-\frac{\partial G_{(a,b)}}{\partial n}(x,0) = \frac{\partial G_{(a,b)}}{\partial y}(x,0) &= \frac{-1}{2\pi}\frac{0-b}{(x-a)^2+(0-b)^2} \\
&\quad - \frac{-1}{2\pi}\frac{0-b}{(x+a)^2+(0-b)^2} \\
&\quad - \frac{-1}{2\pi}\frac{0+b}{(x-a)^2+(0+b)^2} \\
&\quad + \frac{-1}{2\pi}\frac{0+b}{(x+a)^2+(0+b)^2} \\
&= \frac{b}{\pi}\left(\frac{1}{(x-a)^2+b^2} - \frac{1}{(x+a)^2+b^2}\right).
\end{aligned}
$$

A similar computation with $-\partial/\partial n = \partial/\partial x$ on the positive y-axis gives

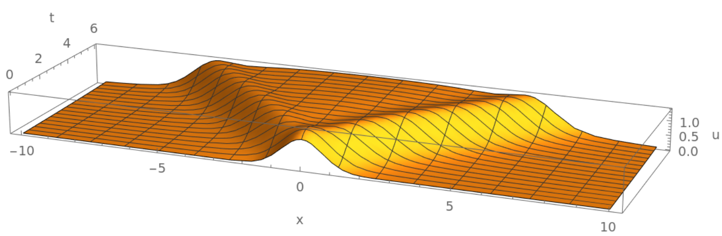
$$
-\frac{\partial G_{(a,b)}}{\partial n}(0,y) = \frac{\partial G_{(a,b)}}{\partial x}(0,y) = \frac{a}{\pi}\left(\frac{1}{a^2+(y-b)^2} - \frac{1}{a^2+(y+b)^2}\right).
$$

So the solution, for $a>0$ and $b>0$, is

$$
\begin{aligned}
u(a,b) &= \frac{b}{\pi}\int_0^\infty g(x)\left(\frac{1}{(x-a)^2+b^2} - \frac{1}{(x+a)^2+b^2}\right)dx \\
&\quad + \frac{a}{\pi}\int_0^\infty h(y)\left(\frac{1}{a^2+(y-b)^2} - \frac{1}{a^2+(y+b)^2}\right)dy.
\end{aligned}
$$

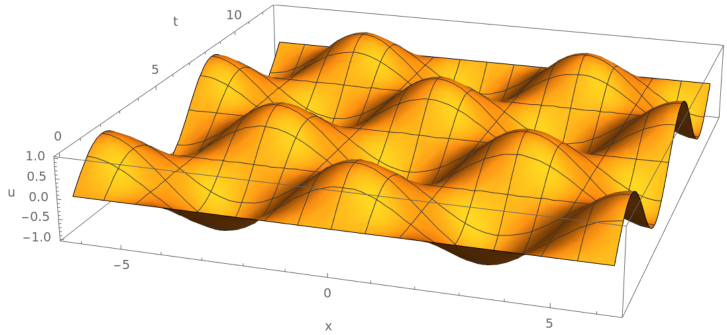
8.1

(a) Answer: $u(x,t) = \frac{1}{2}e^{-(x+ct)^2} + \frac{1}{2}e^{-(x-ct)^2} + \frac{1}{2c}\arctan(x+ct) - \frac{1}{2c}\arctan(x-ct)$. The graph of $u(x,t)$, for $c=1$:



(b) Answer: $u(x,t) = \frac{1}{2c}\sin(x+ct) - \frac{1}{2c}\sin(x-ct) = \frac{1}{c}\cos x \sin ct$. The graph of

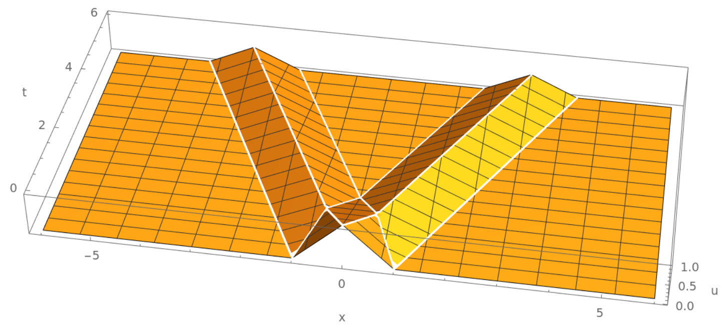
$u(x,t)$, for $c=1$:



(c) Answer: If we denote the given function for $u(x,0)$ by $\varphi(x)$, the solution is $u(x,t) = \frac{1}{2}\varphi(x+ct) + \frac{1}{2}\varphi(x-ct)$. More explicitly (for $t \geq 0$):

$$u(x,t) = \begin{cases} 0, & |x| \geq 1+ct, \\ 0, & ct \geq 1 \text{ and } |x| \leq -1+ct, \\ 1-|x|, & ct < \frac{1}{2} \text{ and } ct \leq |x| < 1-ct, \\ 1-ct, & |x+ct| < 1 \text{ and } |x-ct| < 1, \\ \frac{1}{2}(1-|x-ct|), & x+ct \geq 1 \text{ and } |x-ct| < 1, \\ \frac{1}{2}(1-|x+ct|), & x-ct \leq -1 \text{ and } |x+ct| < 1, \end{cases}$$

The graph of $u(x,t)$, for $c = 1/2$:



(d) One can use d'Alembert's formula, but it's perhaps just as easy to directly determine f and g such that $u(x,t) = f(x+ct) + g(x-ct)$ satisfies the initial conditions.

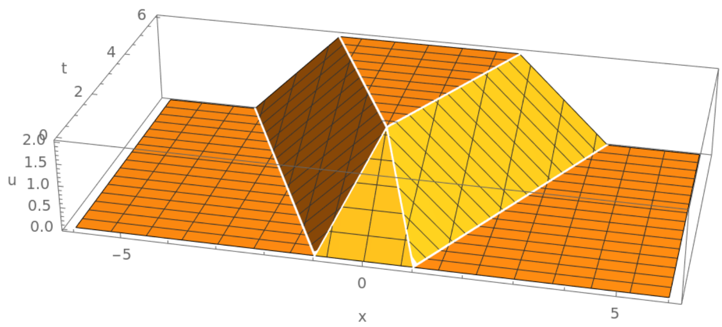
Answer: $u(x,t) = f(x+ct) - f(x-ct)$, where

$$f(x) = \begin{cases} -1/c, & x \leq -1, \\ x/c, & -1 < x < 1, \\ 1/c, & 1 \leq x. \end{cases}$$

More explicitly (for $t \geq 0$):

$$u(x,t) = \begin{cases} 0, & |x| \geq 1 + ct, \\ 1/c, & ct \geq 1 \text{ and } |x| \leq -1 + ct, \\ t, & ct \leq 1 \text{ and } |x| \leq 1 - ct, \\ \frac{1}{2c}(1 - x + ct), & x + ct > 1 \text{ and } |x - ct| < 1, \\ \frac{1}{2c}(1 + x + ct), & x - ct < -1 \text{ and } |x + ct| < 1. \end{cases}$$

The graph of $u(x,t)$, for $c = 1/2$:



8.2

(a) Answer: $u(x,t) = \frac{1}{2}t^2$.

(b) Answer: $u(x,t) = \frac{1}{2}\sin(x+ct) + \frac{1}{2}\sin(x-ct) + \frac{1}{2c^2}\big(2\cos x - \cos(x+ct) - \cos(x-ct)\big)$.

(c) Answer: $u(x,t) = \frac{1}{1-\omega^2}(\sin(\omega t) - \omega \sin t)\sin x$ in the generic case $\omega \neq 1$, but don't forget the resonant case $\omega = 1$ where the solution grows without bound: $u(x,t) = \frac{1}{2}(\sin t - t\cos t)\sin x$.

8.3

(a) Recall that Green's theorem is a special case of Stokes's theorem

$$\iint_\Omega (\nabla \times \mathbf{v}) \cdot \mathbf{n}\, dS = \int_{\partial\Omega} \mathbf{v} \cdot d\mathbf{r},$$

where the surface Ω lies in a coordinate plane, and the vector field $\mathbf{v}$ is parallel to that plane too. If that plane is the (x,y)-plane, and $\mathbf{v} = (A(x,y), B(x,y), 0)$, this becomes

$$\iint_\Omega (B_x - A_y)\, dx\, dy = \int_{\partial\Omega} (A\, dx + B\, dy),$$

with the boundary $\partial\Omega$ oriented so that Ω lies on its left side.

In our case, with t instead of y, and with $A = -u_t$ and $B = -c^2 u_x$, we get

$$\iint_D f(x,t)\, dx\, dt = \iint_D (u_{tt} - c^2 u_{xx})\, dx\, dt = \int_{\partial\Omega} (-u_t\, dx - c^2 u_x\, dt).$$

(b) Along the x-axis we have $dt = 0$, so the integral along that edge is

$$\int_{L_0} (-u_t\,dx - c^2 u_x\,dt) = \int_{x_0-ct_0}^{x_0+ct_0} (-u_t(x,0))\,dx = -\int_{x_0-ct_0}^{x_0+ct_0} \psi(x)\,dx.$$

(c) The edge L_1 from $(x_0+ct_0, 0)$ to (x_0, t_0) can be parametrized as $(x,t) = (x_0 + c(t_0 - s), s)$ where the parameter s runs from 0 to t_0. This gives $dx = -c\,ds$ and $dt = ds$, so that

$$\begin{aligned}
\int_{L_1} (-u_t\,dx - c^2 u_x\,dt) &= \int_0^{t_0} \bigl(c\,u_t(x_0 + c(t_0 - s), s) - c^2\,u_x(x_0 + c(t_0 - s), s)\bigr)\,ds \\
&= c\int_0^{t_0} \left(\frac{d}{ds} u(x_0 + c(t_0 - s), s)\right) ds \\
&= c\Bigl[u(x_0 + c(t_0 - s), s)\Bigr]_{s=0}^{t_0} \\
&= c\,u(x_0, t_0) - c\,u(x_0 + ct_0, 0) \\
&= c\,u(x_0, t_0) - c\varphi(x_0 + ct_0).
\end{aligned}$$

(d) Very similar to part (c) above.

(e) To obtain the final result, just compute

$$\iint_D f(x,t)\,dx dt = \int_{L_0} + \int_{L_1} + \int_{L_2} = \cdots,$$

divide by $2c$, and move some terms over to the other side.

8.4

(a) Differentiation gives $e_t = u_t u_{tt} + c^2 u_x u_{xt}$ and $e_{tt} = u_{tt}^2 + u_t u_{ttt} + c^2(u_{xt}^2 + u_x u_{xtt})$, and similarly $e_x = u_t u_{tx} + c^2 u_x u_{xx}$ and $e_{xx} = u_{tx}^2 + u_t u_{txx} + c^2(u_{xx}^2 + u_x u_{xxx})$. Since we are assuming $u \in C^3$, all mixed partial derivatives up to order three can be interchanged ($u_{xxt} = u_{txx}$, etc.). Thus,

$$\begin{aligned}
&e_{tt} - c^2 e_{xx} \\
&= u_{tt}^2 + u_t u_{ttt} + c^2(u_{xt}^2 + u_x u_{xtt}) - c^2\bigl(u_{tx}^2 + u_t u_{txx} + c^2(u_{xx}^2 + u_x u_{xxx})\bigr) \\
&= (u_{tt} + c^2 u_{xx})(u_{tt} - c^2 u_{xx}) + u_t(u_{tt} - c^2 u_{xx})_t + c^2 u_x (u_{tt} - c^2 u_{xx})_x,
\end{aligned}$$

which is identically zero if $u_{tt} - c^2 u_{xx}$ is.

(b) From the chain rule we have $u_x(x,t) = f'(x+ct) + g'(x-ct)$ and $u_t(x,t) = c\bigl(f'(x+ct) - g'(x-ct)\bigr)$, which gives $e(x,t) = \frac{1}{2}c^2\bigl(f'(x+ct)^2 + g'(x-ct)^2\bigr)$.

(c) The total energy is just the sum of the energy of the left-going wave and the energy of the right-going wave; there is no interaction between the two parts. Also note that part (b) gives an independent verification of part (a); the energy density must be a solution of the wave equation, since it's a sum of a function of $x + ct$ and a function of $x - ct$ (both of class C^2, if f and g are of class C^3).

8.5 Differentiate under the integral sign and use the PDE:

$$\begin{aligned}\frac{dE}{dt} &= \int_{-\infty}^{\infty} (u_t u_{tt} + c^2 u_x u_{xt})\,dx \\ &= \int_{-\infty}^{\infty} \big(u_t(c^2 u_{xx} - r u_t) + c^2 u_x u_{xt}\big)\,dx \\ &= -r \int_{-\infty}^{\infty} u_t^2\,dx + c^2 \Big[u_t u_x\Big]_{-\infty}^{\infty} \\ &= -r \int_{-\infty}^{\infty} u_t^2\,dx \le 0.\end{aligned}$$

9.1 Answer: $u(x,t)$ is given by d'Alembert's formula, with φ and ψ extended to *even* functions φ_{even} and ψ_{even} on $\mathbf{R}$. In order to give a classical solution, these extended functions need to be of class $C^2(\mathbf{R})$ and $C^1(\mathbf{R})$, respectively.

Remark. In more detail, the conditions for this to be the case are as follows. Considering the function ψ first, it needs to be of class $C^1(\mathbf{R}_+)$ to begin with, where $\mathbf{R}_+ = (0,\infty)$, and ψ and ψ' must extend to continuous functions on $[0,\infty)$ with $\psi'(0) = 0$. That is,

$$\lim_{x\to 0^+} \psi(x)$$

must exist (as a finite number), and

$$\lim_{x\to 0^+} \psi'(x)$$

must exist and be equal to zero. It then follows from the mean value theorem that the one-sided derivative

$$\psi'_+(0) = \lim_{h\to 0^+} \frac{\psi(h) - \psi(0)}{h}$$

exists and equals zero too, and by symmetry (the derivative $\psi'_{\text{even}}(x)$ for $x \ne 0$ is the odd extension of $\psi'(x)$ for $x > 0$) we get $\psi'_{\text{even}}(0) = 0$, so that ψ'_{even} is continuous on $\mathbf{R}$, and hence $\psi_{\text{even}} \in C^1(\mathbf{R})$, as desired. The same conditions must hold for the function φ, which must in addition be of class $C^2(\mathbf{R}_+)$ and such that φ'' extends to a continuous function on $[0,\infty)$, i.e.,

$$\lim_{x\to 0^+} \varphi''(x)$$

must exist (as a finite number).

(For comparison, the conditions for the odd extensions φ_{odd} and ψ_{odd} to be of class $C^2(\mathbf{R})$ and $C^1(\mathbf{R})$, respectively, are that φ, φ', φ'', ψ and ψ' are continuous on $\mathbf{R}_+$ and extend to continuous functions on $[0,\infty)$ with $\varphi(0) = \varphi''(0) = \psi(0) = 0$. Note that the condition $\varphi''(0) = 0$ is missing in Theorem 6.6 in David Rule's lecture notes, but it is needed. For a counterexample with that condition omitted, consider $\varphi(x) = x^2$ for $x > 0$. Then the odd extension is $\varphi_{\text{odd}}(x) = \operatorname{sgn}(x)\,x^2 = x|x|$, which does not have a second derivative at the origin.)

9.2

(a) Answer: $u(x,t) = \sum_{k=1}^{\infty} A_k \sin(kx)\cos(kct)$, where

$$\begin{aligned} A_k &= \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\sin(kx)\,dx \\ &= \frac{2}{\pi}\left[\frac{-\cos(kx)}{k}\cdot x(\pi-x) - \frac{-\sin(kx)}{k^2}\cdot(\pi - 2x) + \frac{\cos(kx)}{k^3}\cdot(-2)\right]_0^{\pi} \\ &= \frac{4\big(1-(-1)^k\big)}{\pi k^3}, \qquad k = 1,2,3,\ldots \end{aligned}$$

(b) Answer: $u(x,t) = t + \frac{1}{2} + \frac{1}{2}\cos(2x)\cos(2ct) - \frac{1}{3c}\cos(3x)\sin(3ct)$.

(c) Answer: $u(x,t) = \sum_{k=0}^{\infty} B_k \cos(\gamma_k x)\sin(\gamma_k ct)$, where $\gamma_k = k + \frac{1}{2}$ and

$$\begin{aligned} \gamma_k c B_k &= \frac{2}{\pi}\int_0^{\pi} (\pi^2 - x^2)\cos(\gamma_k x)\,dx \\ &= \frac{2}{\pi}\left[\frac{\sin(\gamma_k x)}{\gamma_k}\cdot(\pi^2 - x^2) - \frac{-\cos(\gamma_k x)}{\gamma_k^2}\cdot(-2x) + \frac{-\sin(\gamma_k x)}{\gamma_k^3}\cdot(-2)\right]_0^{\pi} \\ &= \frac{4(-1)^k}{\pi\gamma_k^3}, \qquad k = 0,1,2,\ldots \end{aligned}$$

9.3 The odd 2-periodic extensions of $\varphi(x) = \sin(nx)$ and $\psi(x) = 0$ (for $0 < x < \pi$) are simply $\varphi_{\text{ext}}(x) = \sin(nx)$ and $\psi_{\text{ext}}(x) = 0$ (for $x \in \mathbf{R}$), so d'Alembert's formula gives

$$u(x,t) = \tfrac{1}{2}\sin(n(x+ct)) + \tfrac{1}{2}\sin(n(x-ct)).$$

And this is equal to

$$u(x,t) = \sin(nx)\cos(nct),$$

which is what one would get from separation of variables.

9.4 Let $v(x,t) = u(x,t) + u(-x,t)$. Then $v_{tt}(x,t) = u_{tt}(x,t) + u_{tt}(-x,t)$ and $v_{xx}(x,t) = u_{xx}(x,t) + (-1)^2 u_{xx}(-x,t)$, so that v is a solution to the wave equation. Moreover, the initial values are $v(x,0) = 0$ and $v_t(x,0) = 0$, which implies that v is identically zero (by d'Alembert's formula).

10.1 We'll need the formula for the curl of the curl of a vector field $\mathbf{A} = (A_x, A_y, A_z)$, which can be obtained by brute force calculation (or looked up somewhere):

$$\begin{aligned}
\nabla \times (\nabla \times \mathbf{A}) &= \nabla \times \begin{pmatrix} \partial_y A_z - \partial_z A_y \\ \partial_z A_x - \partial_x A_z \\ \partial_x A_y - \partial_y A_x \end{pmatrix} \\
&= \begin{pmatrix} \partial_y(\partial_x A_y - \partial_y A_x) - \partial_z(\partial_z A_x - \partial_x A_z) \\ \partial_z(\partial_y A_z - \partial_z A_y) - \partial_x(\partial_x A_y - \partial_y A_x) \\ \partial_x(\partial_z A_x - \partial_x A_z) - \partial_y(\partial_y A_z - \partial_z A_y) \end{pmatrix} \\
&= \begin{pmatrix} \partial_x(\partial_y A_y + \partial_z A_z) - (\partial_y^2 + \partial_z^2) A_x \\ \dots \\ \dots \end{pmatrix} \\
&= \begin{pmatrix} \partial_x(\partial_x A_x + \partial_y A_y + \partial_z A_z) - (\partial_x^2 + \partial_y^2 + \partial_z^2) A_x \\ \dots \\ \dots \end{pmatrix} \\
&= \begin{pmatrix} \partial_x(\nabla \cdot \mathbf{A}) - \Delta A_x \\ \partial_y(\nabla \cdot \mathbf{A}) - \Delta A_y \\ \partial_z(\nabla \cdot \mathbf{A}) - \Delta A_z \end{pmatrix} = \nabla(\nabla \cdot \mathbf{A}) - \Delta \mathbf{A},
\end{aligned}$$

where the Laplace operator in the final expression acts on each component separately.

Then from Maxwell's equations in vacuum,

$$\begin{aligned}
\nabla \cdot \mathbf{E} = 0, \quad & \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \\
\nabla \cdot \mathbf{B} = 0, \quad & \nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t},
\end{aligned}$$

we find

$$\begin{aligned}
\frac{\partial^2 \mathbf{B}}{\partial t^2} &= \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{B}}{\partial t}\right) = \frac{\partial}{\partial t}\big(-\nabla \times \mathbf{E}\big) = -\nabla \times \frac{\partial \mathbf{E}}{\partial t} = -\nabla \times \left(\frac{1}{\mu_0 \varepsilon_0} \nabla \times \mathbf{B}\right) = \frac{-\nabla \times (\nabla \times \mathbf{B})}{\mu_0 \varepsilon_0} \\
&= \frac{-(\nabla(\nabla \cdot \mathbf{B}) - \Delta \mathbf{B})}{\mu_0 \varepsilon_0} = \frac{-(\nabla 0 - \Delta \mathbf{B})}{\mu_0 \varepsilon_0} = \frac{\Delta \mathbf{B}}{\mu_0 \varepsilon_0} = c^2 \Delta \mathbf{B},
\end{aligned}$$

and similarly $\frac{\partial^2 \mathbf{E}}{\partial t^2} = c^2 \Delta \mathbf{E}$, as was to be shown. (Note, however, that solving Maxwell's equations of course involves more than just solving the wave equation for the fields **E** and **B** separately, since they are coupled to each other.)

10.2

(a) It's the Euler–Poisson–Darboux equation $U_{tt} = U_{rr} + \frac{n-1}{r} U_r$.

(b) For $n = 3$ we get $(rU)_{tt} = (rU)_{rr}$, the one-dimensional wave equation for rU, so

$$rU(r,t) = f(r+t) + g(r-t)$$

for some functions f and g. If we want U to be an even function of $r \in \mathbf{R}$, it must take the form

$$U(r,t) = \begin{cases} \dfrac{h(t+r)-h(t-r)}{2r}, & r \neq 0, \\ h'(t), & r = 0, \end{cases}$$

for some function h.

Explanation: The function $V(r,t) = rU(r,t) = f(r+t) + g(r-t)$ is supposed to be odd as a function of r, hence so is $V_t(r,t) = f'(r+t) - g'(r-t)$. Suppose $V(r,0) = \beta(r)$ and $V_t(r,0) = \gamma(r)$, where β and γ are odd functions; this gives $f(r) + g(r) = \beta(r)$ and $f'(r) - g'(r) = \gamma(r)$, and thus $f(r) - g(r) = \Gamma(r) + C$, where Γ is some antiderivative of γ (which makes Γ an even function). Solving for f and g, we get $f(r) = \frac{1}{2}(\beta(r) + \Gamma(r) + C)$ and $g(r) = \frac{1}{2}(\beta(r) - \Gamma(r) - C)$, and hence

$$\begin{aligned} V(r,t) &= \frac{\big(\beta(r+t) + \Gamma(r+t) + C\big) + \big(\beta(r-t) - \Gamma(r-t) - C\big)}{2} \\ &= \big[\beta \text{ is odd and } \Gamma \text{ is even}\big] \\ &= \frac{\beta(t+r) + \Gamma(t+r) - \beta(t-r) - \Gamma(t-r)}{2} \\ &= \frac{h(t+r) - h(t-r)}{2}, \end{aligned}$$

where $h(s) = \beta(s) + \Gamma(s)$. For $r \neq 0$, division by r now gives the expression for $U(r,t)$ above. We automatically have $V(0,t) = \frac{1}{2}(h(t) - h(t)) = 0$, which is consistent with $V(0,t) = 0 \cdot U(0,t)$ for any finite value of $U(0,t)$, but the natural value to assign to $U(0,t)$ is of course the one determined by continuity: $U(0,t) = \lim_{r\to 0} \frac{V(r,t)}{r} = h'(t)$.

With initial data $U(r,0) = \varphi(r)$ and $U_t(r,0) = \psi(r)$ (both even functions of r), we get

$$h(s) = s\varphi(s) + \int s\psi(s)\, ds,$$

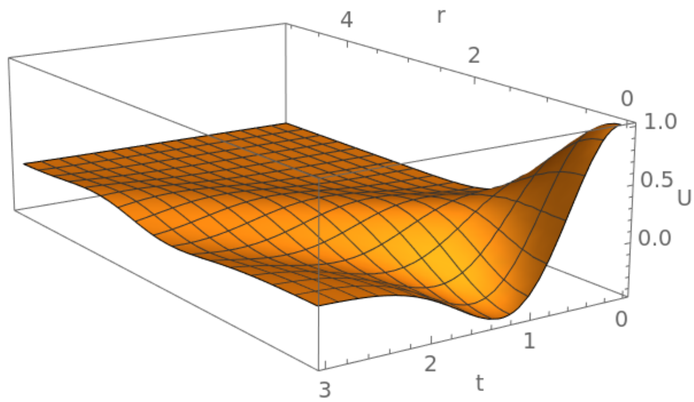
where the constant of integration can be chosen arbitrarily, since it cancels in the expression for U anyway.

(c) Since the initial conditions are spherically symmetric, so is the solution. With notation as in part (b), we find $h(s) = s\varphi(s) = s\,e^{-s^2}$, so $u(\mathbf{x},t) = U(|\mathbf{x}|,t)$ with

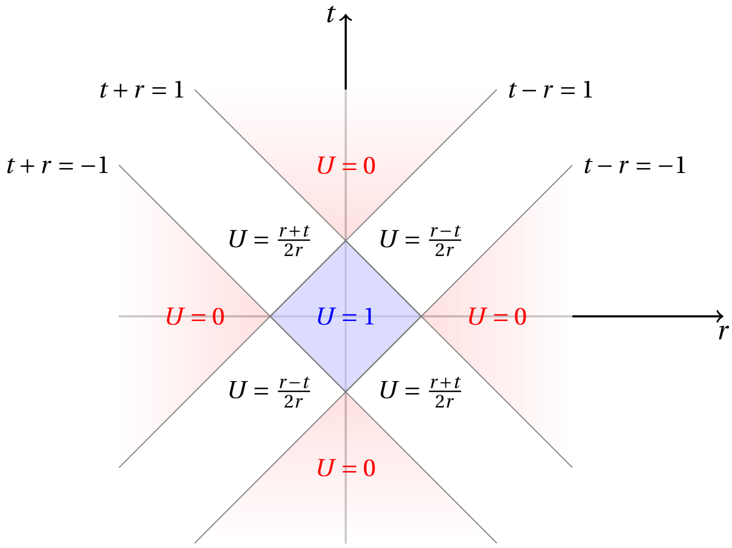
$$U(r,t) = \begin{cases} \dfrac{(t+r)\,e^{-(t+r)^2} - (t-r)\,e^{-(t-r)^2}}{2r}, & r \neq 0, \\ (1-2t^2)\,e^{-t^2}, & r = 0. \end{cases}$$

(Remark: Note that U is an even function of t as well, as it has to be when $u_t(\mathbf{x},0) = 0$.)

The graph of U, in the first quadrant:



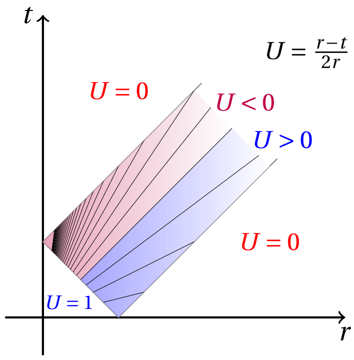
(d) In this case, $h(s) = s$ if $|s| \le 1$ and $h(s) = 0$ otherwise, so $u(\mathbf{x}, t) = U(|\mathbf{x}|, t)$ with $U(r, t)$ piecewise defined as follows:



(Remarks: Note that U is even with respect to r, and also with to t, so all essential information is in the first quadrant. The discontinuity in the initial condition travels to the right and to the left with speed 1, along the slanted lines in the picture. At $(r, t) = (0, \pm 1)$, the singularity of the solution becomes nastier than that of the initial data, since $U(r, \pm 1)$ has a $1/r$ type singularity at the origin. A similar focussing phenomenon lies behind the loss of regularity for the wave equation in general, where the solution may be less smooth than the initial data.)

Taking one last look at the behaviour of U in the strip in the first quadrant where it's nontrivial, we may note that $U(r, t) = \frac{r-t}{2r} = \frac{1}{2}(1 - \frac{t}{r})$ has the lines

$t/r = C$ as level curves, with corresponding values $U = (1-C)/2$, from which it is quite clear that $U(r,t) \to -\infty$ as $(r,t) \to (0,1)$ from within the strip:



10.3

(a) Let $v = u_t$. Then

$$v_t = (u_t)_{tt} = (u_{tt})_t \overset{(1)}{=} (c^2 \Delta u)_t \overset{(2)}{=} c^2 \Delta(u_t) = c^2 \Delta v,$$

where equality (1) holds since u satisfies the wave equation, and equality (2) holds since all third order partials commute if $u \in C^3$ ($u_{x_1 x_1 t} = u_{t x_1 x_1}$, etc.). So $v = u_t$ satisfies the wave equation too.

(b) Again let $v = u_t$. Then $v_t = u_{tt} = c^2 \Delta u$, so $v(\mathbf{x}, 0) = u_t(\mathbf{x}, 0) = \psi(\mathbf{x})$ and $v_t(\mathbf{x}, 0) = c^2 \Delta u(\mathbf{x}, 0) = c^2 \Delta 0 = 0$. (Note that taking partial derivatives with respect to some x_i and then letting $t = 0$ is the same thing as first letting $t = 0$ and then taking partials with respect to x_i.) That is, the initial conditions $u = 0$ and $u_t = \psi$ become $v = \psi$ and $v_t = 0$.

(c) If we know the solution formula for initial conditions $(u, u_t) = (0, \psi)$, then – according to part (b) – the time derivative of that expression gives the formula for the solution satisfying the initial conditions $(u, u_t) = (\psi, 0)$. And if we simply write φ instead of ψ, we get the solution satisfying the initial conditions $(u, u_t) = (\varphi, 0)$. The solution formula for general initial conditions $(u, u_t) = (\varphi, \psi)$ is just the sum of the solution with $(u, u_t) = (\varphi, 0)$ and the solution with $(u, u_t) = (0, \psi)$, since the wave equation is linear. This explains why the expression involving φ in the general solution formula is $\partial/\partial t$ of the expression involving ψ (but with φ replacing ψ, of course).

This is obvious in the case $n = 3$, where the solution formula takes the form

$$u(\mathbf{x}, t) = \frac{\partial}{\partial t}\Bigg(\underbrace{t \fint_{S(\mathbf{x},t)} \varphi \, dS}_{\text{same expression}}\Bigg) + \underbrace{t \fint_{S(\mathbf{x},t)} \psi \, dS}_{\text{as here}},$$

and similarly for $n = 2$, but it also holds for $n = 1$, since d'Alembert's formula can be written as

$$u(x,t) = \frac{\varphi(x+ct)+\varphi(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\,dy$$
$$= \frac{\partial}{\partial t}\underbrace{\left(\frac{1}{2c}\int_{x-ct}^{x+ct} \varphi(y)\,dy\right)}_{\text{same expression}} + \underbrace{\frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\,dy}_{\text{as here}}.$$

10.4

(a) Since the solution is unique, it is enough to verify that the proposed solution satisfies the PDE and the initial conditions. We will need to know how to differentiate an integral with respect to a parameter appearing both in the integrand and in the bounds of integration. Letting

$$G(a,b,c) = \int_a^b F(c,s)\,ds,$$

we have from the fundamental theorem of calculus and from the usual rule for differentiating under the integral sign (assuming $\partial F/\partial c$ is continuous) that

$$\frac{\partial G}{\partial a}(a,b,c) = -F(a,s), \quad \frac{\partial G}{\partial b}(a,b,c) = F(b,s), \quad \frac{\partial G}{\partial c}(a,b,c) = \int_a^b \frac{\partial F}{\partial c}(c,s)\,ds,$$

and then it follows from the multivariable chain rule that

$$\frac{d}{dt}\int_{\alpha(t)}^{\beta(t)} F(t,s)\,ds = \frac{d}{dt}G\big(\alpha(t),\beta(t),t\big)$$
$$= \frac{\partial G}{\partial a}\big(\alpha(t),\beta(t),t\big)\cdot\alpha'(t) + \frac{\partial G}{\partial b}\big(\alpha(t),\beta(t),t\big)\cdot\beta'(t) + \frac{\partial G}{\partial c}\big(\alpha(t),\beta(t),t\big)$$
$$= F(\beta(t),s)\,\beta'(t) - F(\alpha(t),s)\,\alpha'(t) + \int_{\alpha(t)}^{\beta(t)} \frac{\partial F}{\partial t}(t,s)\,ds.$$

Using this identity, we can differentiate the given expression

$$u(\mathbf{x},t) = \int_0^t v(\mathbf{x},t;s)\,ds$$

to obtain

$$u_t(\mathbf{x},t) = \underbrace{v(\mathbf{x},t;t)}_{=0\text{ by def.}} + \int_0^t v_t(\mathbf{x},t;s)\,ds = \int_0^t v_t(\mathbf{x},t;s)\,ds$$

and

$$u_{tt}(\mathbf{x},t) = \underbrace{v_t(\mathbf{x},t;t)}_{=f(\mathbf{x},t)\text{ by def.}} + \int_0^t \underbrace{v_{tt}(\mathbf{x},t;s)}_{=c^2\Delta v(\mathbf{x},t;s)}\,ds$$
$$= f(\mathbf{x},t) + c^2\Delta\int_0^t v(\mathbf{x},t;s)\,ds$$
$$= f(\mathbf{x},t) + c^2\Delta u(\mathbf{x},t),$$

so that u satisfies the inhomogeneous wave equation $u_{tt} - c^2 \Delta u = f$. And the initial conditions $u(\mathbf{x}, 0) = u_t(\mathbf{x}, 0) = 0$ are also satisfied, since letting $t = 0$ in the expressions for u and u_t above gives $\int_0^0 = 0$ in both cases.

(b) Think of the inhomogeneous wave equation (for a vibrating string or membrane, for example) like this:

$$\underbrace{\frac{\partial^2 u}{\partial t^2}}_{\text{acceleration}} = \underbrace{c^2 \Delta u + f}_{\text{force / mass}}$$

where the force terms on the right-hand side are what's causing the acceleration; the term $c^2 \Delta u$ comes from the internal tension, while the term f represents some external force. Now, applying an external force producing the extra acceleration $f(\mathbf{x}, s)$ at the point $\mathbf{x}$ during an infinitesimal time interval from s to $s + ds$ ought to add an extra $f(\mathbf{x}, s)\, ds$ to the velocity $u_t(\mathbf{x}, s)$, so the contribution to the wave's future caused by that external force (acting at all points $\mathbf{x}$ during that time interval) ought to be simply ds times a wave starting off from zero with initial velocity $f(\mathbf{x}, s)$ at time s. And integrating these contributions for $0 \le s \le t$ should give the total wave at time t.

(c) For $n = 1$, d'Alembert's formula (with $\varphi = 0$, $\psi = f(\cdot, s)$, and $t - s$ instead of t) gives

$$v(x, t; s) = \frac{1}{2c} \int_{x-c(t-s)}^{x+c(t-s)} f(y, s)\, dy,$$

so we get a result which should be familiar already:

$$u(x, t) = \int_0^t \left(\frac{1}{2c} \int_{x-c(t-s)}^{x+c(t-s)} f(y, s)\, dy \right) ds = \frac{1}{2c} \iint_D f(y, s)\, dy ds,$$

where D is the triangle in the (y, s)-plane with corners at (x, t) and $(x \pm ct, 0)$.

For $n = 3$, Kirchhoff's formula (with $\varphi = 0$, $\psi = f(\cdot, s)$, and $t - s$ instead of t) gives

$$v(\mathbf{x}, t; s) = \frac{1}{4\pi c(t-s)} \int_{S(\mathbf{x}, c(t-s))} f(\mathbf{y}, s)\, dS(\mathbf{y}),$$

so that

$$\begin{aligned}
u(\mathbf{x}, t) &= \int_0^t \left(\frac{1}{4\pi c(t-s)} \int_{S(\mathbf{x}, c(t-s))} f(\mathbf{y}, s)\, dS(\mathbf{y}) \right) ds \\
&= \begin{bmatrix} r = c(t-s) \\ dr = -c\, ds \end{bmatrix} = -\int_{ct}^0 \left(\frac{1}{4\pi c^2 r} \int_{S(\mathbf{x}, r)} f(\mathbf{y}, t - \tfrac{r}{c})\, dS(\mathbf{y}) \right) dr \\
&= \frac{1}{4\pi c^2} \int_0^{ct} \left(\int_{S(\mathbf{x}, r)} \frac{f(\mathbf{y}, t - \frac{r}{c})}{r}\, dS(\mathbf{y}) \right) dr \\
&= \frac{1}{4\pi c^2} \int_{B(\mathbf{x}, ct)} \frac{f(\mathbf{y}, t - \frac{1}{c}|\mathbf{y} - \mathbf{x}|)}{|\mathbf{y} - \mathbf{x}|}\, dV(\mathbf{y}).
\end{aligned}$$

11.1

(a) Answer: $u(x,t) = \dfrac{\pi}{2} + \sum_{k=1}^{\infty} A_k \cos(kx) e^{-k^2 t}$ (don't forget the constant term!), where

$$\begin{aligned}
A_k &= \frac{2}{\pi}\int_0^\pi x\cos(kx)\,dx \\
&= \frac{2}{\pi}\left[\frac{\sin(kx)}{k}\cdot x - \frac{-\cos(kx)}{k^2}\cdot 1\right]_0^\pi \\
&= \frac{2\big((-1)^k - 1\big)}{\pi k^2}, \qquad k = 1, 2, 3, \ldots
\end{aligned}$$

(b) Let $v(x,t) = u(x,t) - x/\pi$. Then $v_t = v_{xx}$ with $v(0,t) = v(\pi,t) = 0$ for $t > 0$, $v(x,0) = \sin(x/2) - x/\pi$ for $0 \le x \le \pi$.

Answer: $u(x,t) = \dfrac{x}{\pi} + \sum_{k=1}^{\infty} A_k \sin(kx) e^{-k^2 t}$, where

$$\begin{aligned}
A_k &= \frac{2}{\pi}\int_0^\pi \left(\sin(x/2) - \frac{x}{\pi}\right)\sin(kx)\,dx \\
&= \frac{2}{\pi}\int_0^\pi \frac{\cos(kx - x/2) - \cos(kx + x/2)}{2}\,dx + \frac{2}{\pi^2}\int_0^\pi x\cdot(-\sin(kx))\,dx \\
&= \frac{1}{\pi}\left[\frac{\sin(k-\frac{1}{2})x}{k-\frac{1}{2}} - \frac{\sin(k+\frac{1}{2})x}{k+\frac{1}{2}}\right]_0^\pi + \frac{2}{\pi^2}\left[\frac{\cos(kx)}{k}\cdot x - \frac{\sin(kx)}{k^2}\cdot 1\right]_0^\pi \\
&= \frac{1}{\pi}\left(\frac{(-1)^{k-1}}{k-\frac{1}{2}} - \frac{(-1)^k}{k+\frac{1}{2}}\right) + \frac{2}{\pi^2}\cdot\frac{(-1)^k}{k}\cdot\pi \\
&= \frac{(-1)^{k-1}}{\pi}\left(\frac{2k}{k^2 - \frac{1}{4}} - \frac{2}{k}\right) = \frac{(-1)^{k-1}}{2\pi k(k^2 - \frac{1}{4})}, \qquad k = 1, 2, 3, \ldots
\end{aligned}$$

(c) Let $v(x,t) = u(x,t) - e^{-t}x/\pi$. Then $v_t = v_{xx} + e^{-t}x/\pi$ with $v(0,t) = v(\pi,t) = 0$ for $t > 0$, $v(x,0) = 0$ for $0 \le x \le \pi$. We seek a solution of the form

$$v(x,t) = \sum_{k=1}^{\infty} A_k(t)\sin(kx),$$

and for $0 < x < \pi$ we have (reusing some calculations from part (b) above)

$$\frac{e^{-t}x}{\pi} = \sum_{k=1}^{\infty} B_k(t)\sin(kx),$$

where

$$B_k = \frac{e^{-t}}{\pi}\cdot\frac{2}{\pi}\int_0^\pi x\sin(kx)\,dx = \frac{e^{-t}}{\pi}\cdot\frac{2(-1)^{k-1}}{k} = b_k e^{-t}, \qquad b_k = \frac{2(-1)^{k-1}}{k\pi}.$$

The boundary conditions for v are already taken care of, and from the PDE and the initial condition we find

$$A_k'(t) = -k^2 A_k(t) + b_k e^{-t}, \qquad A_k(0) = 0,$$

that is,

$$\frac{d}{dt}\big(A_k(t)\, e^{k^2 t}\big) = b_k e^{-t} e^{k^2 t}, \qquad A_k(0) = 0.$$

Integration gives

$$A_k(t) = \begin{cases} b_1\, t e^{-t}, & k = 1, \\ b_k\, e^{-k^2 t}\, \dfrac{e^{(k^2-1)t} - 1}{k^2 - 1}, & k \geq 2. \end{cases}$$

Answer: $u(x,t) = \dfrac{e^{-t} x}{\pi} + \dfrac{2}{\pi}\, t e^{-t} \sin x + \displaystyle\sum_{k=2}^{\infty} \frac{2(-1)^{k-1}\big(e^{-t} - e^{-k^2 t}\big)}{(k^2-1)k\pi} \sin(kx).$

11.2

(a) The assumptions imply that the function $w = u - v$ satisfies $\Delta w - w_t = g - f \geq 0$ on Ω_∞ and $w \leq 0$ on Γ_T. According to the weak maximum principle for subsolutions of the heat equation, the maximum of w on $\overline{\Omega_T}$ is attained on Γ_T, so $w \leq 0$ (i.e., $u \leq v$) throughout $\overline{\Omega_T}$, as was to be shown.

(b) A short computation shows that the function $u(x,t) = (1 - e^{-t}) \sin x$ satisfies $u_t = u_{xx} + \sin x$, With $\Omega = (0,\pi)$ and $f(x,t) = g(x,t) = \sin x$, it is then easily verified that all hypotheses from part (a) are satisfied for any value $T > 0$. So the conlusion holds too: $u(x,t) \leq v(x,t)$ on any $\overline{\Omega_T}$, and hence on $\overline{\Omega_\infty}$, as desired.

11.3

(a) $u(x,t) = C \sin(nx)\, e^{-n^2 t}$.

(b) Solutions do not depend continuously on the given data; for example, with $C_n = 1/n$ in part (a) we obtain a sequence of solutions $u_n(x,t)$ such that $u_n(x,0)$ tends to the zero function $\varphi(x) = 0$ (uniformly) as $n \to \infty$, but $u_n(x,t)$ doesn't (for any fixed $t < 0$).

There is also trouble with existence; it can be shown that even if $u(x,0)$ isn't smooth (it could be just continuous, or not even that), the solution $u(x,t)$ of the forward heat equation is of class C^∞ as a function of x for any fixed $t > 0$. So if the given function $u(x,0)$ in the backwards problem is not smooth, it's impossible for it to have arisen by heat flow from some function $u(x,t)$ for $t < 0$.

Curiously enough, uniqueness is not a problem; there is at most one solution (see Evans, Section 2.3, Theorem 11).

11.4

(a) If $u \geq 0$, then $\Delta u - u_t \geq cu \geq 0$, so that u is a subsolution to the heat equation, and the result follows from the usual weak maximum principle for subsolutions.

(b) Note that the inequality $u_t \leq \Delta u - cu$ can be written as $(ue^{ct})_t \leq \Delta(ue^{ct})$, so that $v = ue^{ct}$ is a subsolution. Taking $v(x,t) = -1-2t-x^2$ (which actually is a solution to the heat equation $v_t = v_{xx}$, not just a subsolution) on the interval $\Omega = (-1,1) \subset \mathbf{R}$, say, we have $u(x,t) = (-1-2t-x^2)\,e^{-ct}$ (which satisfies $u_t = u_{xx} - cu$). For T large enough, $\max_{\overline{\Omega_T}} u$ is attained at the point $(x,t) = (0,T)$ and not on Γ_T.

12.1

(a,b,c) Easy.

(d) Just calculate:

$$\begin{aligned}
u_t(x,y) &= c^2 u(x,y) + e^{-cx+c^2t}\big(-2cf_x(x-2ct,t) + f_t(x-2ct,t)\big),\\
u_x(x,y) &= -cu(x,y) + e^{-cx+c^2t} f_x(x-2ct,t),\\
u_{xx}(x,y) &= -cu_x(x,y) + (-c)e^{-cx+c^2t} f_x(x-2ct,t) + e^{-cx+c^2t} f_{xx}(x-2ct,t)\\
&= [f_{xx} = f_t]\\
&= c^2u(x,y) - 2ce^{-cx+c^2t} f_x(x-2ct,t) + e^{-cx+c^2t} f_t(x-2ct,t),
\end{aligned}$$

which shows that $u_t(x,y) = u_{xx}(x,y)$, as claimed.

(e) To make life a little easier, let's use the abbrevations $T = 1+4ct$ and $E = \exp(-cx^2/T)$, and also write just f instead of $f(x/T, t/T)$ (and similarly for the derivatives of f). Then $u = T^{-1/2}Ef$, and we compute

$$\begin{aligned}
u_t &= -2cT^{-3/2}Ef + T^{-1/2}(4c^2x^2/T^2)Ef + T^{-1/2}E\big(-4cxf_x/T^2 + f_t/T^2\big)\\
&= -2cT^{-3/2}Ef + T^{-5/2}E\big(4c^2x^2f + f_t - 4cxf_x\big),\\
u_x &= T^{-1/2}(-2cx/T)Ef + T^{-1/2}Ef_x/T\\
&= T^{-3/2}E\big(f_x - 2cxf\big),\\
u_{xx} &= T^{-3/2}(-2cx/T)E\big(f_x - 2cxf\big) + T^{-3/2}E\big(f_{xx}/T - 2cf - 2cxf_x/T\big)\\
&= T^{-5/2}E\Big(-2cx\big(f_x - 2cxf\big) + f_{xx} - 2cxf_x\Big) - 2cT^{-3/2}Ef\\
&= [f_{xx} = f_t] = T^{-5/2}E\big(-4cxf_x + 4c^2x^2f + f_t\big) - 2cT^{-3/2}Ef,
\end{aligned}$$

and again we find that $u_t = u_{xx}$.

(f) Similar to (e).

12.2

(a) Easy.

(b) $p_0(x,t) = 1$, $p_1(x,t) = x$, $p_2(x,t) = x^2 + 2t$, $p_3(x,t) = x^3 + 6xt$, $p_4(x,t) = x^4 + 12x^2t + 12t^2$, $p_5(x,t) = x^5 + 20x^3t + 60xt^2$, and in general

$$p_n(x,t) = n! \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{t^k}{k!} \frac{x^{n-2k}}{(n-2k)!}.$$

(c) —

12.3 Thinking of the inhomogeneous heat equation as

$$\underbrace{\frac{\partial u}{\partial t}}_{\substack{\text{rate of}\\\text{temperature}\\\text{change}}} = \underbrace{D\Delta u}_{\text{Fick's law}} + \underbrace{f(\mathbf{x},t)}_{\substack{\text{external}\\\text{heat source}}},$$

we see that the contribution from the extra source term $f(\mathbf{x},s)$ at the point $\mathbf{x}$ during an infinitesimal time interval from s to $s+ds$ ought to add an extra $f(\mathbf{x},s)\,ds$ to $u(\mathbf{x},s)$, so the contribution to the future temperature caused by that external heat source (acting at all points $\mathbf{x}$ during that time interval) ought to be simply ds times the temperature distribution which starts off from the initial value $f(\mathbf{x},s)$ at time s. And integrating these contributions for $0 \le s \le t$ should give the total contribution at time t from the external source. So our conjecture is that the solution should be

$$u(\mathbf{x},t) = \int_0^t v(\mathbf{x},t;s)\,ds,$$

where $v(\mathbf{x},t;s)$ is the solution of the following initial value problem (starting at time s) for the usual homogeneous heat equation, with the source term f appearing in the initial condition instead:

$$\begin{aligned} v_t(\mathbf{x},t;s) &= D\Delta v(\mathbf{x},t;s) \qquad \text{for } t > s,\\ v(\mathbf{x},s;s) &= f(\mathbf{x},s). \end{aligned}$$

Like in the answer to exercise 10.4, we differentiate the expression

$$u(\mathbf{x},t) = \int_0^t v(\mathbf{x},t;s)\,ds$$

to obtain, as desired,

$$\begin{aligned} u_t(\mathbf{x},t) &= \underbrace{v(\mathbf{x},t;t)}_{=f \text{ by def.}} + \int_0^t \underbrace{v_t(\mathbf{x},t;s)}_{=D\Delta v(\mathbf{x},t;s)}\,ds\\ &= f(\mathbf{x},t) + D\Delta \int_0^t v(\mathbf{x},t;s)\,ds\\ &= f(\mathbf{x},t) + D\Delta u(\mathbf{x},t). \end{aligned}$$

12.4

(a) Let's write $u(x,t) = g(p)$, where $p = x/\sqrt{t} = xt^{-1/2}$. Then

$$\begin{aligned} u_t &= g'(p)\,p_t = -\tfrac{1}{2}xt^{-3/2}g'(p), \\ u_x &= g'(p)\,p_x, \\ u_{xx} &= g''(p)\,p_x^2 + g'(p)\,p_{xx} = (t^{-1/2})^2 g''(p) + 0g'(p) = t^{-1}g''(p), \end{aligned}$$

so that $u_t = u_{xx}$ becomes $-\frac{1}{2}xt^{-3/2}g'(p) = t^{-1}g''(p)$, which is equivalent to

$$g''(p) + \tfrac{1}{2}pg'(p) = 0.$$

(b) Multiplication by the integrating factor $e^{\frac{1}{4}p^2}$ gives $\left(e^{\frac{1}{4}p^2}g'(p)\right)' = 0$, so that

$$g'(p) = C\,e^{-\frac{1}{4}p^2}$$

and hence

$$g(p) = D\operatorname{erf}(p/2) + E,$$

where the *error function* is defined as

$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\xi^2}\,d\xi.$$

(The constant $2/\sqrt{\pi}$ is included in the definition in order to make $\operatorname{erf} x \to \pm 1$ as $x \to \pm\infty$.) So

$$u(x,t) = g\big(x/\sqrt{t}\big) = D\operatorname{erf}\big(x/\sqrt{4t}\big) + E$$

for $t > 0$, and to satisfy the initial condition we must take $D = E = \frac{1}{2}$:

$$u(x,t) = \begin{cases} \frac{1}{2}\operatorname{erf}\big(x/\sqrt{4t}\big) + \frac{1}{2}, & t > 0, \\ H(x), & t = 0. \end{cases}$$

(Note that the limit of $\operatorname{erf}\big(x/\sqrt{4t}\big)$ as $t \to 0^+$ depends on whether $x < 0$, $x = 0$ or $x > 0$.)

For $x = 0$, we have $u(0,t) = \frac{1}{2}$ for all $t > 0$, so the weaker initial condition that we required at the origin is obviously satisfied. For $x_0 > 0$, we should check that $u(x,t) \to 1$ as $(x,t) \to (x_0, 0)$ (with $t > 0$) in the two-variable sense, not just as $t \to 0^+$ with a fixed $x = x_0$, But this is clear, since for any M we can find a (small) half-disk centered at $(x_0, 0)$ such that $x/\sqrt{4t} > M$ there; that is, $x/\sqrt{4t} \to \infty$ as $(x,t) \to (x_0, 0)$. Similarly for $x_0 < 0$, one checks just as easily that $u(x,t) \to 0$, since $x/\sqrt{4t} \to -\infty$.

(c) We have $u_x(x,0) = H'(x) = \delta(x)$ in the sense of distributions. (In the classical sense, $u_x(x,0) = 0$ for $x \neq 0$, while $u_x(0,0)$ is undefined.) For $t > 0$, we have

$$\begin{aligned} u_x(x,t) &= \frac{\partial}{\partial x}\left(\frac{1}{2}\operatorname{erf}\bigl(x/\sqrt{4t}\bigr) + \frac{1}{2}\right) = \frac{1}{2}\operatorname{erf}'\bigl(x/\sqrt{4t}\bigr)\cdot\frac{1}{\sqrt{4t}} \\ &= \frac{1}{2}\cdot\frac{1}{\sqrt{4t}}\cdot\frac{2}{\sqrt{\pi}}\exp\left(-\left(\frac{x}{\sqrt{4t}}\right)^2\right) = \frac{1}{\sqrt{4\pi t}}\exp\left(-\frac{x^2}{4t}\right) = S(x,t), \end{aligned}$$

the well-known "source solution" of the heat equation, i.e., the solution with the Dirac delta $\delta(x)$ as initial data. (Note that since u solves the heat equation for $t > 0$, so does u_x, since $(u_x)_t = (u_t)_x = (u_{xx})_x = (u_x)_{xx}$.)

12.5

(a) Extend φ to an odd function and use the solution formula for the whole real line.

(b) Extend φ to an even function.

(c) The verifications are straightforward. Any multiple of $v(x,t)$ can be added to the solution $u(x,t)$ in part (a) without disturbing the initial and boundary conditions, at least away from the origin. But v is not bounded near the origin, since $v(x,x^2) = \frac{1}{x|x|}e^{-1/4} \to \infty$ as $x \to 0^+$. To get uniqueness in part (a), we can for example require φ to extend continuously to $\varphi(0) = 0$ and require the solution u to be continuous and bounded on $[0,\infty)\times[0,\infty)$.

13.1

(a) The chain rule gives

$$\begin{aligned} u_x &= u_r r_x + u_s s_x, \\ u_{xx} &= (u_r)_x r_x + u_r r_{xx} + (u_s)_x s_x + u_s s_{xx} \\ &= (u_{rr} r_x + u_{rs} s_x) r_x + u_r r_{xx} + (u_{sr} r_x + u_{ss} s_x) s_x + u_s s_{xx}, \end{aligned}$$

and so on, which in the end gives

$$\begin{aligned} \widetilde{A} &= \begin{pmatrix} r_x & r_y \end{pmatrix}\begin{pmatrix} A & B \\ B & C \end{pmatrix}\begin{pmatrix} r_x \\ r_y \end{pmatrix}, \\ \widetilde{B} &= \begin{pmatrix} r_x & r_y \end{pmatrix}\begin{pmatrix} A & B \\ B & C \end{pmatrix}\begin{pmatrix} s_x \\ s_y \end{pmatrix}, \\ \widetilde{C} &= \begin{pmatrix} s_x & s_y \end{pmatrix}\begin{pmatrix} A & B \\ B & C \end{pmatrix}\begin{pmatrix} s_x \\ s_y \end{pmatrix}, \\ \widetilde{D} &= A\,r_{xx} + 2B\,r_{xy} + C\,r_{yy} + D\,r_x + E\,r_y, \\ \widetilde{E} &= A\,s_{xx} + 2B\,s_{xy} + C\,s_{yy} + D\,s_x + E\,s_y, \\ \widetilde{F} &= F, \\ \widetilde{G} &= G. \end{aligned}$$

(More precisely, $\widetilde{F}(r(x,y),s(x,y)) = F(x,y)$, or, in terms of the inverse transformation, $\widetilde{F}(r,s) = F(x(r,s),y(r,s))$. Similarly for all the other coefficients.)

(b) The quickest way is probably to notice that

$$\begin{pmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{B} & \widetilde{C} \end{pmatrix} = \begin{pmatrix} r_x & r_y \\ s_x & s_y \end{pmatrix} \begin{pmatrix} A & B \\ B & C \end{pmatrix} \begin{pmatrix} r_x & s_x \\ r_y & s_y \end{pmatrix}$$

and take determinants on both sides:

$$\widetilde{A}\widetilde{C} - \widetilde{B}^2 = (r_x s_y - s_x r_y)^2 (AC - B^2).$$

The Jacobian determinant $r_x s_y - s_x r_y$ is nonzero, since (by assumption) the change of variables $r = r(x,y)$, $s = s(x,y)$ is invertible with differentiable inverse, so that (by the chain rule)

$$\begin{pmatrix} r_x & r_y \\ s_x & s_y \end{pmatrix} \begin{pmatrix} x_r & x_s \\ y_r & y_s \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(c) At every point in the domain in question, the matrix $\left(\begin{smallmatrix} A & B \\ B & C \end{smallmatrix}\right)$ has exactly one zero eigenvalue, with corresponding eigenvector $\mathbf{v}$ (unique up to scaling). Choose some $\mathbf{v}(x,y)$ at each point, compute the flow of the rotated vector field,

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} v_1(x,y) \\ v_2(x,y) \end{pmatrix},$$

and eliminate the time parameter to obtain the family of characteristic curves on the form $s(x,y) = C$ for some function s, which will have the property that ∇s is proportional to $\mathbf{v}$ at each point, so that $\left(\begin{smallmatrix} A & B \\ B & C \end{smallmatrix}\right)\nabla s = \mathbf{0}$, and thus $\widetilde{B} = \widetilde{C} = 0$, according to the formulas in part (a). Let the new variables be $(r,s) = (r(x,y), s(x,y))$ where r is any function that makes the Jacobian determinant nonzero (typically just take $r(x,y) = x$ or $r(x,y) = y$).

13.2

(a) The coefficents are $A(x,y) = 0$, $B(x,y) = -xy$ and $C(x,y) = y^2$, so $AC - B^2 = -x^2y^2$, which is negative if $x \neq 0$ and $y \neq 0$, so the PDE is indeed hyperbolic away from the coordinate axes. The quadratic form

$$Q_{(x,y)}(\mathbf{v}) = A(x,y)\, v_1^2 + 2B(x,y)\, v_1 v_2 + C(x,y)\, v_2^2 = \big(-2x\, v_1 + y v_2\big) y v_2$$

is zero for $\mathbf{v} = (1,0)$ and for $\mathbf{v} = (y, 2x)$ (for example). In the first case, we see immediately that $\mathbf{v} = (1,0) = \nabla r(x,y)$ for $r(x,y) = x$. In the second case, we compute the flow of the rotated vector field:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -v_2 \\ v_1 \end{pmatrix} = \begin{pmatrix} -2x \\ y \end{pmatrix} \qquad \Longleftrightarrow \qquad \begin{cases} x(t) = x_0 e^{-2t}, \\ y(t) = y_0 e^{t}. \end{cases}$$

Eliminating t, we find that the trajectories have the form $xy^2 = C$, so we take $s(x,y) = xy^2$. (As a verification, we can check that $\nabla s = (y^2, 2xy)$ is proportional to the vector field $\mathbf{v} = (y, 2x)$ that we started with, as it should be. It is also possible to eliminate t right away by writing $dy/dx = \dot{y}/\dot{x} = y/(-2x)$; this gives $-2\int dy/y = \int dx/x$, so that $-2\ln y = \ln x + C$, and again we find that xy^2 is constant.) In terms of the new variables $(r,s) = (x, xy^2)$ the PDE becomes

$$\begin{aligned}
0 &= y^2\, u_{yy} - 2xy\, u_{xy} + 2x\, u_x \\
&= y^2\,(4x^2y^2\, u_{ss} + 2x\, u_s) - 2xy\,(2xy\, u_{rs} + 2xy^3\, u_{ss} + 2y\, u_s) + 2x\,(u_r + y^2\, u_s) \\
&= -4x^2y^2\, u_{rs} + 2x\, u_r \\
&= -4rs\, u_{rs} + 2r\, u_r.
\end{aligned}$$

In the first qudrant, x and y are positive, hence so are r and s, so we can divide the equation by $-4rs$ and continue with the help of the integrating factor $\exp(-\frac{1}{2}\ln s) = \frac{1}{\sqrt{s}}$:

$$u_{rs} + \frac{-1}{2s}\, u_r = 0 \quad \Longleftrightarrow \quad \left(\frac{u_r}{\sqrt{s}}\right)_s = 0 \quad \Longleftrightarrow \quad \frac{u_r}{\sqrt{s}} = f(r),$$

where f is an arbitrary C^1-function. Then $f = F'$, where F is an arbitrary C^2-function, and one more integration gives

$$u = F(r)\sqrt{s} + G(r),$$

where G is another arbitrary C^2-function. Now it only remains to go back to the original variables:

$$u = F(x)\sqrt{xy^2} + G(xy^2) = \underbrace{F(x)\sqrt{x}}_{=H(x)}\, y + G(xy^2).$$

Answer: $u(x,y) = G(xy^2) + y\,H(x)$, where G and H are arbitrary C^2-functions.

(b) The coefficents are $A(x,y) = xy$, $B(x,y) = \frac{1}{2}(x^2 - y^2)$ and $C(x,y) = -xy$, so $AC - B^2 = -x^2y^2 - \frac{1}{4}(x^2-y^2)^2 = -\frac{1}{4}(x^2+y^2)^2$, which is negative if $(x,y) \neq (0,0)$, so the PDE is indeed hyperbolic away from the origin. The quadratic form

$$\begin{aligned}
Q_{(x,y)}(\mathbf{v}) &= A(x,y)\, v_1^2 + 2B(x,y)\, v_1 v_2 + C(x,y)\, v_2^2 \\
&= xy\, v_1^2 + (x^2 - y^2)\, v_1 v_2 - xy\, v_2^2 \\
&= (y\, v_1 + x\, v_2)(x\, v_1 - y\, v_2)
\end{aligned}$$

is zero for $\mathbf{v} = (2x, -2y) = \nabla(x^2 - y^2)$ and for $\mathbf{v} = (2y, 2x) = \nabla(xy)$ (for example), so that $(r,s) = (x^2 - y^2, 2xy)$ are characteristic coordinates. (This change of variables is invertible under the assumption $x > 0$; note that $r + is = (x + iy)^2$, so that $x + iy$ is then the principal complex square root of $r + is$.) Then

$$\begin{aligned}
u_x &= 2x\,u_r + 2y\,u_s,\\
u_y &= -2y\,u_r + 2x\,u_s,\\
u_{xx} &= 4x^2\,u_{rr} + 8xy\,u_{rs} + 4y^2\,u_{ss} + 2u_r,\\
u_{xy} &= -4xy\,u_{rr} + 4(x^2 - y^2)\,u_{rs} + 4xy\,u_{ss} + 2u_s,\\
u_{yy} &= 4y^2\,u_{rr} - 8xy\,u_{rs} + 4x^2\,u_{ss} - 2u_r,
\end{aligned}$$

so that the PDE becomes

$$\begin{aligned}
4s &= 8xy\\
&= xy\,(u_{xx} - u_{yy}) + (x^2 - y^2)\,u_{xy} + \frac{(y^3 - 3x^2y)\,u_x + (3xy^2 - x^3)\,u_y}{x^2 + y^2}\\
&= xy\big(4(x^2 - y^2)(u_{rr} - u_{ss}) + 16xy\,u_{rs} + 4u_r\big)\\
&\quad + (x^2 - y^2)\big(-4xy\,(u_{rr} - u_{ss}) + 4(x^2 - y^2)\,u_{rs} + 2u_s\big)\\
&\quad + \frac{(y^3 - 3x^2y)\,(2x\,u_r + 2y\,u_s) + (3xy^2 - x^3)\,(-2y\,u_r + 2x\,u_s)}{x^2 + y^2}\\
&= \big(4(x^2 - y^2) + 16x^2y^2\big)u_{rs} + \left(4xy + \frac{2x(y^3 - 3x^2y) - 2y(3xy^2 - x^3)}{x^2 + y^2}\right)u_r\\
&\quad + \left(2(x^2 - y^2) + \frac{2y(y^3 - 3x^2y) + 2x(3xy^2 - x^3)}{x^2 + y^2}\right)u_s\\
&= 4(x^2 + y^2)^2 u_{rs} = 4(r^2 + s^2)\,u_{rs},
\end{aligned}$$

and hence

$$\begin{aligned}
(u_r)_s = \frac{s}{r^2 + s^2} \quad &\Longleftrightarrow \quad u_r = \frac{1}{2}\ln(r^2 + s^2) + f'(r)\\
&\Longleftrightarrow \quad u = \frac{s}{2}\ln(r^2 + s^2) - s + r\arctan\frac{s}{r} + f(r) + g(s).
\end{aligned}$$

Answer: $u(x, y) = (x^2 - y^2)\ln(x^2 + y^2) + y^2 - x^2 + 2xy\arctan\frac{x^2 - y^2}{2xy} + f(x^2 - y^2) + g(2xy)$, where f and g are arbitrary C^2-functions.

13.3

(a) The coefficents are $A(x, y) = 4y^2$, $B(x, y) = -2y$ and $C(x, y) = 1$, so $AC - B^2 = 0$, and the PDE is indeed parabolic everywhere. The quadratic form

$$\begin{aligned}
Q_{(x,y)}(\mathbf{v}) &= A(x, y)\,v_1^2 + 2B(x, y)\,v_1 v_2 + C(x, y)\,v_2^2\\
&= 4y^2\,v_1^2 - 4y\,v_1 v_2 + v_2^2\\
&= (2y\,v_1 - v_2)^2
\end{aligned}$$

is zero for $\mathbf{v} = (1, 2y) = \nabla(x + y^2)$, so we can (for example) take $r = x - y^2$ and $s = y$ as characteristic coordinates. The PDE becomes

$$\begin{aligned}6s = 6y &= 4y^2 u_{xx} - 4y u_{xy} + u_{yy} - 2u_x \\ &= 4y^2 u_{rr} - 4y\big(2y\, u_{rr} + u_{rs}\big) + \big(4y^2\, u_{rr} + 4y\, u_{rs} + u_{ss} + 2u_r\big) - 2u_r \\ &= u_{ss},\end{aligned}$$

so that $u = s^3 + s\, f(r) + g(r)$.

Answer: $u(x, y) = y^3 + y f(x + y^2) + g(x + y^2)$, where f and g are arbitrary C^2-functions.

(b) Differentiation of the answer from part (a) gives

$$u_y(x, y) = 3y^2 + f(x + y^2) + y \cdot 2y f'(x + y^2) + 2y g'(x + y^2),$$

so the given conditions $u(x, 0) = x^2$ and $u_y(x, 0) = \sin x$ amount to

$$\begin{aligned}0^2 + 0\, f(x) + g(x) &= x^2, \\ 3 \cdot 0^2 + f(x) + 0\, f'(x) + 0\, g'(x) &= \sin x,\end{aligned}$$

so that $f(x) = \sin x$ and $g(x) = x^2$, and hence $f(x + y^2) = \sin(x + y^2)$ and $g(x + y^2) = (x + y^2)^2$.

Answer: $u(x, y) = y^3 + y \sin(x + y^2) + (x + y^2)^2$.

13.4

(a) Elliptic in the upper half-plane $y > 0$, parabolic on the line $y = 0$, hyperbolic in the lower half-plane $y < 0$.

(b) For $y < 0$, the quadratic form

$$Q_{(x,y)}(\mathbf{v}) = A(x, y)\, v_1^2 + 2B(x, y)\, v_1 v_2 + C(x, y)\, v_2^2 = y\, v_1^2 + v_2^2$$

is zero for $\mathbf{v} = (1, \pm\sqrt{-y})$. The flows of the rotated vector fields are given by $\dot{x} = \mp\sqrt{-y}$ and $\dot{y} = 1$, so $dy/dx = \dot{y}/\dot{x} = \mp(-y)^{-1/2}$ and hence $\frac{2}{3}(-y)^{3/2} = \pm x + C$, so we can (for example) take $r = x + \frac{2}{3}(-y)^{3/2}$ and $s = x - \frac{2}{3}(-y)^{3/2}$ as characteristic coordinates. (Draw a picture of the coordinate grid!)

Expressed in these coordinates, the PDE becomes

$$\begin{aligned}0 &= y\, u_{xx} + u_{yy} \\ &= y\big(u_{rr} + 2u_{rs} + u_{ss}\big) + \tfrac{1}{2}(-y)^{-1/2}(u_r - u_s) - y\big(u_{rr} - 2u_{rs} + u_{ss}\big) \\ &= 4y\, u_{rs} + \tfrac{1}{2}(-y)^{-1/2}(u_r - u_s).\end{aligned}$$

Answer:

$$u_{rs} = \frac{u_r - u_s}{6(r - s)}.$$

Remark: We can make the left-hand side look like the usual wave equation by letting $r = w + z$ and $s = w - z$ (which is the same as letting $w = x$ and $z = \frac{2}{3}(-y)^{3/2}$ in terms of the original variables). This gives

$$u_{ww} - u_{zz} = \frac{1/3}{z} u_z,$$

which is nothing but our old friend the Euler–Poisson–Darboux

$$u_{tt} = u_{rr} + \frac{\beta}{r} u_r$$

(with $z = \frac{2}{3}(-y)^{3/2} > 0$ playing the role of r and $w = x$ playing the role of t), but with the parameter value $\beta = 1/3$ instead of $\beta = n - 1$ as we had in the method of spherical means.

(c) For $y > 0$, we have $u_x = u_w$, $u_{xx} = u_{ww}$, $u_y = y^{1/2} u_z$ and $u_{yy} = \frac{1}{2} y^{-1/2} u_z + y^{1/2} \cdot y^{1/2} u_{zz}$, so the PDE obtains a canonical form where the principal part is given by the Laplace operator:

$$0 = u_{xx} + \frac{u_{yy}}{y} = u_{ww} + \tfrac{1}{2} y^{-3/2} u_z + u_{zz} = u_{ww} + u_{zz} + \frac{1/3}{z} u_z.$$

This is the so-called *elliptic* Euler–Poisson–Darboux equation with parameter $\beta = 1/3$.

14.1

(a) For any test function φ, we have

$$\begin{aligned} \langle H', \varphi \rangle &= -\langle H, \varphi' \rangle = -\int_{\mathbf{R}} H(x)\, \varphi'(x)\, dx = -\int_0^\infty \varphi'(x)\, dx = -\Big[\varphi(x)\Big]_0^\infty \\ &= \varphi(0) = \langle \delta, \varphi \rangle, \end{aligned}$$

which is precisely what the statement "$H' = \delta$" means.

(b) For any test function φ, we have

$$\langle f\delta, \varphi \rangle = \langle \delta, f\varphi \rangle = f(0)\, \varphi(0) = f(0)\, \langle \delta, \varphi \rangle = \langle f(0)\, \delta, \varphi \rangle,$$

which is precisely what the statement "$f\delta = f(0)\,\delta$" means. Next, $\langle f\delta', \varphi \rangle = \langle \delta', f\varphi \rangle = -\langle \delta, (f\varphi)' \rangle = -\langle \delta, f'\varphi + f\varphi' \rangle = -f'(0)\, \varphi(0) - f(0)\, \varphi'(0) = -f'(0)\, \langle \delta, \varphi \rangle - f(0) \langle \delta, \varphi' \rangle = -f'(0)\, \langle \delta, \varphi \rangle + f(0) \langle \delta', \varphi \rangle = \langle -f'(0)\, \delta + f(0) \delta', \varphi \rangle$, so that

$$f\delta' = f(0)\, \delta' - f'(0)\, \delta,$$

and similarly we find

$$f\delta'' = f(0)\, \delta'' - 2f'(0)\, \delta' + f''(0)\, \delta.$$

(c) Answer: $u'(x) = -\operatorname{sgn}(x)\, e^{-|x|}$, $u''(x) = e^{-|x|} - 2\delta(x)$.

(d) For any test function φ,

$$\begin{aligned}\langle (fT)', \varphi\rangle &= -\langle fT, \varphi'\rangle = -\langle T, f\varphi'\rangle = -\langle T, (f\varphi)' - f'\varphi\rangle \\ &= -\langle T, (f\varphi)'\rangle + \langle T, f'\varphi\rangle = \langle T', f\varphi\rangle + \langle f'T, \varphi\rangle \\ &= \langle fT', \varphi\rangle + \langle f'T, \varphi\rangle = \langle fT' + f'T, \varphi\rangle.\end{aligned}$$

(e) Answer: $T(x) = e^{3x}(H(x) + C)$, where H is the Heaviside function. (Or, more precisely: $T = T_g$ where $g(x) = e^{3x}(H(x) + C)$.)

(f) We can use the theorem about solutions to the heat equation, which (among other things) says that the function

$$u(x,t) = \begin{cases} \displaystyle\int_{\mathbf{R}} S(x-y,t)\,\varphi(y)\,dy, & t > 0, \\ \varphi(x), & t = 0, \end{cases}$$

is continuous at $(x,0)$ for every x where φ is continuous. Just take $x = 0$ in this theorem, and use that $S(-y,t) = S(y,t)$.

We can also give a direct proof (which is simpler than the proof of that theorem) if we use that φ is a test function, not just a continuous function. Since φ is smooth, it admits a first-order Maclaurin expansion $\varphi(x) = \varphi(0) + x\,B(x)$, where B is bounded near the origin, and since φ has compact support, B has compact support too, and therefore B is bounded, say $|B(x)| \le M$ for all $x \in \mathbf{R}$. Then, letting $z = x/\sqrt{4t}$, so that $dx = \sqrt{4t}\,dz$, we have for $t > 0$ that

$$\begin{aligned}\int_{\mathbf{R}} S(x,t)\,\varphi(x)\,dx &= \int_{\mathbf{R}} S(x,t)\,\big(\varphi(0) + x\,B(x)\big)\,dx \\ &= \frac{1}{\sqrt{4\pi t}}\int_{\mathbf{R}} \exp\left(-\frac{x^2}{4t}\right)\big(\varphi(0) + x\,B(x)\big)\,dx \\ &= \frac{1}{\sqrt{\pi}}\int_{\mathbf{R}} e^{-z^2}\big(\varphi(0) + \sqrt{4t}\,z\,B(\sqrt{4t}\,z)\big)\,dz \\ &= \varphi(0)\frac{1}{\sqrt{\pi}}\int_{\mathbf{R}} e^{-z^2}\,dz + \sqrt{4t}\cdot\frac{1}{\sqrt{\pi}}\int_{\mathbf{R}} e^{-z^2}\,z\,B(\sqrt{4t}\,z)\,dz \\ &= \varphi(0) + g(t),\end{aligned}$$

where

$$\big|g(t)\big| \le \sqrt{4t}\cdot\frac{M}{\sqrt{\pi}}\int_{\mathbf{R}} e^{-z^2}\,|z|\,dz \to 0, \qquad t \to 0^+.$$

14.2

(a) Answer: $L^*\varphi = -\varphi_t - \varphi_{xx}$.

(b) Answer: $L^*\varphi = \varphi_{xx} + \varphi_{yy}$.

(c) Answer: $L^*\varphi = (xy\varphi)_{xx} - \varphi_{xyy} = 2y\varphi_x + xy\varphi_{xx} - \varphi_{xyy}$.

14.3 What we need to show is that $u(x,t) = f(x-ct)$ satisfies

$$0 = \iint_{\mathbf{R}^2} u(\varphi_{tt} - c^2\varphi_{xx})\,dx\,dt$$

for all test functions φ. With the change of variables $y = x - ct$, $s = t$ (whose Jacobian matrix is the identity matrix, so that $dx\,dt = dy\,ds$), we get

$$\begin{aligned}
&\iint_{\mathbf{R}^2} f(x-ct)\Big(\varphi_{tt}(x,t) - c^2\varphi_{xx}(x,t)\Big)\,dx\,dt \\
&= \iint_{\mathbf{R}^2} f(y)\Big(\varphi_{tt}(y+cs,s) - c^2\varphi_{xx}(y+cs,s)\Big)\,dy\,ds \\
&= \int_{\mathbf{R}} f(y)\left(\int_{\mathbf{R}}\Big(\varphi_{tt}(y+cs,s) - c^2\varphi_{xx}(y+cs,s)\Big)\,ds\right)dy \\
&= \int_{\mathbf{R}} f(y)\left(\int_{\mathbf{R}}\Big(c\varphi_{tx}(y+cs,s) + \varphi_{tt}(y+cs,s) - c^2\varphi_{xx}(y+cs,s) - c\varphi_{xt}(y+cs,s)\Big)\,ds\right)dy \\
&= \int_{\mathbf{R}} f(y)\left(\int_{\mathbf{R}}\left(\frac{d}{ds}\varphi_t(y+cs,s) - c\frac{d}{ds}\varphi_x(y+cs,s)\right)ds\right)dy \\
&= \int_{\mathbf{R}} f(y)\Big[\varphi_t(y+cs,s) - c\varphi_x(y+cs,s)\Big]_{s=-\infty}^{\infty}\,dy = \int_{\mathbf{R}} f(y)\cdot 0\,dy = 0,
\end{aligned}$$

since φ_t and φ_x have compact support when φ is a test function.

14.4

(a) Answer: For $0 < t < 1$ the wave gradually steepens,

$$u(x,t) = \begin{cases} 1, & x \le t, \\ \frac{1-x}{1-t}, & t < x < 1, \\ 0, & x \ge 1, \end{cases}$$

and for $t \ge 1$ we have a shock wave travelling with velocity $\frac{1}{2}$ (according to the Rankine–Hugoniot condition),

$$u(x,t) = 1 - H\big(x - \tfrac{1}{2}t - \tfrac{1}{2}\big) = \begin{cases} 1, & x < \frac{1}{2}t + \frac{1}{2}, \\ 0, & x > \frac{1}{2}t + \frac{1}{2}, \end{cases}$$

where H is the Heaviside function.

(b) For initial data $u_0(x) = 0$ and $u_0(x) = x - 1$ we have the strong solutions $u(x,t) = 0$ and $u(x,t) = \frac{x-1}{1+t}$, respectively, and we can glue them together along a curve $x = g(t)$ provided that the Rankine–Hugoniot condition

$$g'(t) = \frac{u^L(g(t),t) + u^R(g(t),t)}{2} = \frac{0 + \frac{g(t)-1}{1+t}}{2}$$

is satisfied. This is an ODE for g that we can solve using the integrating factor $\exp(-\frac{1}{2}\ln(1+t)) = (1+t)^{-1/2}$, with the initial condition $g(0) = 0$ (since the shock is at the origin initially), and the result is $g(t) = 1 - \sqrt{1+t}$.

Answer: For $t \geq 0$,

$$u(x,t) = \begin{cases} 0, & x < 1 - \sqrt{1+t}, \\ \dfrac{1-x}{1+t}, & x > 1 - \sqrt{1+t}. \end{cases}$$

14.5

(a) For all test functions φ,

$$\iint_{\mathbf{R}^2} \left(-u\varphi_t - \tfrac{1}{2}u^2\varphi_x + au\varphi\right) dx\,dt = 0.$$

(b) The jump condition arises from integration by parts in the terms containing φ_t and φ_x, and this does not involve the new term $au\varphi$.

(c) The ODEs for the characteristic curves are

$$\begin{aligned} dx/d\tau &= z, \\ dt/d\tau &= 1, \\ dz/d\tau &= az, \end{aligned}$$

with initial values $(x,t,z) = (x_0, 0, u_0(x_0))$ at $\tau = 0$. We see that $t(\tau) = \tau$, so we can work with t instead of τ, and first solve $dz/dt = az$ and then integrate $dx/dt = z(t)$.

Answer: $(x(t), t, z(t))$ where $x(t) = x_0 + u(x_0) \cdot \frac{1}{a}(1 - e^{-at})$ and $z(t) = u(x_0)\, e^{-at}$.

(d) Answer: For $t \geq 0$,

$$\begin{cases} 0, & x \leq 0, \\ \dfrac{ax}{(a+1)e^{at} - 1}, & 0 < x < 1 + \frac{1}{a}(1 - e^{-at}), \\ e^{-at}, & x \geq 1 + \frac{1}{a}(1 - e^{-at}). \end{cases}$$

(e) For initial data $u_0(x) = 1$ and $u_0(x) = 0$ we have the strong solutions $u(x,t) = e^{-at}$ and $u(x,t) = 0$, respectively, and we can glue them together along a curve $x = g(t)$ provided that the Rankine–Hugoniot condition

$$g'(t) = \frac{u^L(g(t),t) + u^R(g(t),t)}{2} = \frac{e^{-at} + 0}{2}$$

is satisfied. Integrating this, with the initial condition $g(0) = 0$ (since the shock is at the origin initially), we find $g(t) = \frac{1}{2a}(1 - e^{-at})$.

Answer: For $t \geq 0$,

$$u(x,t) = \begin{cases} e^{-at}, & x < \frac{1}{2a}(1-e^{-at}), \\ 0, & x > \frac{1}{2a}(1-e^{-at}). \end{cases}$$

(f) To avoid shock formation, we need to make sure that the projected characteristics in the (x,t)-plane don't cross. A rather obvious sufficient condition for this is that u_0 is non-decreasing. But we can do a little better, if we use the fact that the *envelope* of a family of curves $F(x,t;\beta)=0$ is obtained by eliminating the parameter β from the equations

$$F(x,t;\beta)=0, \qquad \frac{\partial F}{\partial \beta}(x,t;\beta)=0.$$

In our case here, the (projected) characteristics from part (c) constitute a curve family parametrized by x_0, and the equations for the envelope are

$$x_0 + u(x_0)\cdot\frac{1}{a}(1-e^{-at}) - x = 0, \qquad 1 + u'(x_0)\cdot\frac{1}{a}(1-e^{-at}) = 0,$$

where the second equation is $\partial/\partial x_0$ of the first one. Since $a>0$, we have $0 < \frac{1}{a}(1-e^{-at}) < \frac{1}{a}$ for all $t>0$, so if $u_0'(x) \geq -a$ the system has no solution, implying that no characteristics cross.

Answer: $u_0'(x) \geq -a$.

(So if the initial slope is negative, as long as it's not too steep, the damping will prevent the shock formation that would have taken place in the undamped situation.)

14.6 Just compute: $u = -2\mu v_x/v$ gives

$$u_x = -2\mu\left(\frac{v_{xx}}{v} - \frac{v_x^2}{v^2}\right),$$

$$u_{xx} = -2\mu\left(\frac{v_{xxx}}{v} - \frac{v_x v_{xx}}{v^2} - \left(\frac{2v_x v_{xx}}{v^2} - \frac{2v_x^3}{v^3}\right)\right) = -2\mu\left(\frac{v_{xxx}}{v} - \frac{3v_x v_{xx}}{v^2} + \frac{2v_x^3}{v^3}\right)$$

and

$$u_t = -2\mu\left(\frac{v_{xt}}{v} - \frac{v_x v_t}{v^2}\right) = \left[v_t = \mu v_{xx}\right] = -2\mu^2\left(\frac{v_{xxx}}{v} - \frac{v_x v_{xx}}{v^2}\right),$$

so that

$$\begin{aligned} u_t + u u_x &= -2\mu^2\left(\frac{v_{xxx}}{v} - \frac{v_x v_{xx}}{v^2}\right) + (-2\mu)^2\frac{v_x}{v}\left(\frac{v_{xx}}{v} - \frac{v_x^2}{v^2}\right) \\ &= -2\mu^2\left(\frac{v_{xxx}}{v} - \frac{v_x v_{xx}}{v^2} - 2\frac{v_x}{v}\left(\frac{v_{xx}}{v} - \frac{v_x^2}{v^2}\right)\right) \\ &= \mu\cdot(-2\mu)\left(\frac{v_{xxx}}{v} - \frac{3v_x v_{xx}}{v^2} + \frac{2v_x^3}{v^3}\right) = \mu u_{xx}, \end{aligned}$$

as was to be shown.

15.1

(a) $u(x,t) = \sin(\pi x)\, e^{-\pi^2 t}$.

(b) Initial value: $U(1,0) = u(\frac{1}{2},0) = \sin\frac{\pi}{2} = 1$. Solution: $U(1,m) = (1-8\tau)^m$.

(c) Initial values: $U(1,0) = U(3,0) = \frac{1}{\sqrt{2}}$ and $U(2,0) = 1$. That is, $U(k,0) = \sin\frac{k\pi}{4}$ for $k \in \{1,2,3\}$.

A general fact, that we can use below as well, is that if $h = 1/K$, then we know from separation of variables that the solution with initial values $U(k,0) = X(k) = \sin\frac{jk\pi}{K}$ is $U(k,m) = X(k)\,\xi^m$, where

$$X(k) \cdot \frac{\xi^{m+1} - \xi^m}{\tau} = \frac{X(k+1) - 2X(k) + X(k-1)}{h^2} \cdot \xi^m,$$

so that

$$\xi - 1 = \frac{\tau}{h^2} \cdot \frac{X(k+1) - 2X(k) + X(k-1)}{X(k)} = \cdots = 2K^2\tau\left(\cos\frac{j\pi}{K} - 1\right).$$

that is,

$$\xi = 1 + 2Q, \qquad \text{where} \quad Q = K^2\tau\left(\cos\frac{j\pi}{K} - 1\right).$$

In our case here, we have $K = 4$ and $j = 1$, so the solution is

$$U(k,m) = \left(1 + 32\tau\left(\tfrac{1}{\sqrt{2}} - 1\right)\right)^m \sin\frac{k\pi}{4},$$

for $k \in \{1,2,3\}$ and $m \geq 0$.

(d) In the setup from part (c), we still have $K = 4$ but now $j = 3$, so the solution is

$$U(k,m) = \left(1 + 32\tau\left(-\tfrac{1}{\sqrt{2}} - 1\right)\right)^m \sin\frac{3k\pi}{4}.$$

(e) We could use the formulas with $K = 4$ and $j = 4$, or simply notice that the initial values $U(k,0) = \sin\frac{4k\pi}{4}$ are simply $U(1,0) = U(2,0) = U(3,0) = 0$, so the solution is trivially

$$U(k,m) = 0.$$

(f) Here we could use the formulas with $K = 4$ and $j = 5$, but we can also notice that the initial values $U(k,0) = \sin\frac{5k\pi}{4}$ are $U(1,0) = \sin\frac{5\pi}{4} = -\frac{1}{\sqrt{2}}$, $U(2,0) = \sin\frac{10\pi}{4} = -1$ and $U(3,0) = \sin\frac{15\pi}{4} = -\frac{1}{\sqrt{2}}$, the negatives of the initial values in part (c), so the solution here is the negative of the solution in part (c):

$$U(k,m) = -\left(1 + 32\tau\left(\tfrac{1}{\sqrt{2}} - 1\right)\right)^m \sin\frac{k\pi}{4}.$$

(Note that in parts (e) and (f), our numerical scheme is clearly doing a very bad job! We would need to take a smaller h in order to handle the oscillations in the initial data.)

15.2 For the Crank–Nicolson scheme with $h = 1/K$, the solution with initial values $U(k,0) = X(k) = \sin\frac{jk\pi}{K}$ is $U(k,m) = X(k)\,\xi^m$, where

$$X(k)\cdot\frac{\xi^{m+1}-\xi^m}{\tau} = \frac{1}{2}\,\frac{X(k+1)-2X(k)+X(k-1)}{h^2}\cdot\xi^m + \frac{1}{2}\,\frac{X(k+1)-2X(k)+X(k-1)}{h^2}\cdot\xi^{m+1},$$

so that

$$\frac{\xi-1}{\xi+1} = \frac{\tau}{2h^2}\cdot\frac{X(k+1)-2X(k)+X(k-1)}{X(k)} = K^2\tau\left(\cos\frac{j\pi}{K}-1\right),$$

that is,

$$\xi = \frac{1+Q}{1-Q}, \qquad \text{where} \quad Q = K^2\tau\left(\cos\frac{j\pi}{K}-1\right).$$

Answers:

(b) $U(1,m) = \left(\frac{1-4\tau}{1+4\tau}\right)^m$.

(c) $U(k,m) = \left(\frac{1+16\tau\left(\frac{1}{\sqrt{2}}-1\right)}{1-16\tau\left(\frac{1}{\sqrt{2}}-1\right)}\right)^m \sin\frac{k\pi}{4}$.

(d) $U(k,m) = \left(\frac{1+16\tau\left(-\frac{1}{\sqrt{2}}-1\right)}{1-16\tau\left(-\frac{1}{\sqrt{2}}-1\right)}\right)^m \sin\frac{3k\pi}{4}$.

(e) $U(k,m) = 0$.

(f) The negative of the answer in part (c).

15.3 One way is to add an extra column of grid points $(-1,m)$, and to require that

$$\frac{U(0,m)-U(-1,m)}{h} = g(m\tau)$$

and

$$\frac{U(0,m+1)-U(0,m)}{\tau} = \frac{U(1,m)-2U(0,m)+U(-1,m)}{h^2}.$$

15.4 Comparing A to

$$LR = \begin{pmatrix} 1 & & & & \\ l_1 & 1 & & & \\ & l_2 & 1 & & \\ & & l_3 & 1 & \\ & & & l_4 & 1 \end{pmatrix} \begin{pmatrix} m_1 & r_1 & & & \\ & m_2 & r_2 & & \\ & & m_3 & r_3 & \\ & & & m_4 & r_4 \\ & & & & m_5 \end{pmatrix}$$
$$= \begin{pmatrix} m_1 & r_1 & & & \\ m_1 l_1 & l_1 r_1 + m_2 & r_2 & & \\ & m_2 l_2 & l_2 r_2 + m_3 & r_3 & \\ & & m_3 l_3 & l_3 r_3 + m_4 & r_4 \\ & & & m_4 l_4 & l_4 r_4 + m_5 \end{pmatrix},$$

we see that $r_i = b_i$ for all i, while m_i and l_i are easily determined successively in terms of $\{a_j, b_j, c_j\}$ from the equations

$$\begin{aligned} m_1 &= a_1, \\ m_1 l_1 &= c_1, \\ l_1 b_1 + m_2 &= a_2, \\ m_2 l_2 &= c_2, \\ l_2 b_2 + m_3 &= a_3, \\ m_3 l_3 &= c_3, \end{aligned}$$

and so on. So the factorization requires one loop through the matrix (essentially n steps). Then the system $A\mathbf{x} = LR\mathbf{x} = \mathbf{d}$ can be split into two subproblems: first solve $L\mathbf{y} = \mathbf{d}$ for $\mathbf{y}$ (which amounts to another loop of size n, since L is bidiagonal so that it's just a matter of back-substitution) and then solve $R\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$ (yet another loop of size n).

(**Remark.** Compare this to Gaussian elimination with a full matrix A, which requires $O(n^3)$ operations.)

15.5

(a) The FEM solution is

$$u(x) = c_1\varphi_1(x) + c_2\varphi_2(x),$$

where φ_1 and φ_2 are the piecewise linear basis functions that are equal to 1 at $\frac{1}{3}$ and at $\frac{2}{3}$, respectively, and equal to 0 at the other mesh points. The equations that determine the coefficients c_1 and c_2 are

$$\int_0^1 u'(x)\,\varphi_i'(x)\,dx = \int_0^1 f(x)\,\varphi_i(x)\,dx, \qquad i \in \{1,2\},$$

or more explicitly

$$\int_0^1 \big(c_1\varphi_1'(x) + c_2\varphi_2'(x)\big)\,\varphi_1'(x)\,dx = \int_0^1 f(x)\,\varphi_1(x)\,dx,$$
$$\int_0^1 \big(c_1\varphi_1'(x) + c_2\varphi_2'(x)\big)\,\varphi_2'(x)\,dx = \int_0^1 f(x)\,\varphi_2(x)\,dx,$$

or even more explicitly

$$\begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix},$$

where

$$\begin{aligned}
K_{11} &= \int_0^1 \varphi_1'(x)^2\,dx = \int_0^{1/3} 3^2\,dx + \int_{1/3}^{2/3} (-3)^2\,dx = 6,\\
K_{12} &= \int_0^1 \varphi_1'(x)\,\varphi_2'(x)\,dx = \int_{1/3}^{2/3} (-3)\cdot 3\,dx = -3,\\
K_{22} &= \int_0^1 \varphi_2'(x)^2\,dx = \int_{1/3}^{2/3} 3^2\,dx + \int_{2/3}^{1} (-3)^2\,dx = 6
\end{aligned}$$

and

$$\begin{aligned}
f_1 &= \int_0^1 f(x)\,\varphi_1(x)\,dx = \int_0^{1/3} 3x\,f(x)\,dx + \int_{1/3}^{2/3} (2-3x)\,f(x)\,dx,\\
f_2 &= \int_0^1 f(x)\,\varphi_2(x)\,dx = \int_{1/3}^{2/3} (3x-1)\,f(x)\,dx + \int_{2/3}^{1} (3-3x)\,f(x)\,dx.
\end{aligned}$$

(b) With $U(k) = u(k/3)$, the standard finite difference approximation is

$$\begin{aligned}
U(0) &= 0,\\
-\frac{U_0 - 2U_1 + U_2}{(1/3)^2} &= f(\tfrac{1}{3}),\\
-\frac{U_1 - 2U_2 + U_3}{(1/3)^2} &= f(\tfrac{2}{3}),\\
U(3) &= 0,
\end{aligned}$$

that is,

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} U(1) \\ U(2) \end{pmatrix} = \frac{1}{9} \begin{pmatrix} f(\frac{1}{3}) \\ f(\frac{2}{3}) \end{pmatrix}.$$

To compare with the FEM solution in part (a), note that $c_1 = U(1) = u(\frac{1}{3})$ and $c_2 = U(2) = u(\frac{2}{3})$, so that the FEM equations

$$\begin{pmatrix} 6 & -3 \\ -3 & 6 \end{pmatrix} \begin{pmatrix} U(1) \\ U(2) \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}.$$

If the integrals f_1 and f_2 are computed by approximating f with its value at the respective mesh point,

$$\begin{aligned}
f_1 &\approx \int_0^1 \varphi_1(x)\,f(\tfrac{1}{3})\,dx = \tfrac{1}{3} f(\tfrac{1}{3}),\\
f_2 &\approx \int_0^1 \varphi_2(x)\,f(\tfrac{2}{3})\,dx = \tfrac{1}{3} f(\tfrac{2}{3}),
\end{aligned}$$

we see that FEM and finite differences give the same approximate solution in this case.

(c) With these boundary conditions, the FEM solution is

$$u(x) = c_1\varphi_1(x) + c_2\varphi_2(x) + c_3\varphi_3(x),$$

where φ_k is the basis function at $x = k/3$, with the coefficients determined by the requirement that

$$\int_0^1 u'(x)\,\varphi_i'(x)\,dx = \int_0^1 f(x)\,\varphi_i(x)\,dx, \qquad i \in \{1,2,3\}.$$

Similar computations as in part (a) lead to the equations

$$\begin{pmatrix} 6 & -3 & 0 \\ -3 & 6 & -3 \\ 0 & -3 & 6 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}, \qquad f_k = \int_0^1 f(x)\,\varphi_k(x)\,dx,$$

where the simplest approximation to f_k is $f_k \approx \frac{1}{3} f(\frac{k}{3})$.

(d) For the FEM solutions, we get different stiffnesses:

$$\begin{aligned}
K_{11} &= \int_0^1 \varphi_1'(x)^2\,dx = \int_0^{1/3} 3^2\,dx + \int_{1/3}^{1/2} (-6)^2\,dx = 9, \\
K_{12} &= \int_0^1 \varphi_1'(x)\,\varphi_2'(x)\,dx = \int_{1/3}^{1/2} (-6)\cdot 6\,dx = -6, \\
K_{13} &= \int_0^1 \varphi_1'(x)\,\varphi_3'(x)\,dx = 0, \\
K_{22} &= \int_0^1 \varphi_2'(x)^2\,dx = \int_{1/3}^{1/2} 6^2\,dx + \int_{1/2}^1 (-2)^2\,dx = 8, \\
K_{23} &= \int_0^1 \varphi_2'(x)\,\varphi_3'(x)\,dx = \int_{1/2}^1 (-2)\cdot 2\,dx = -2, \\
K_{33} &= \int_0^1 \varphi_3'(x)^2\,dx = \int_{1/2}^1 2^2\,dx = 2,
\end{aligned}$$

so that with $u(0) = u(1) = 0$ we get

$$u(x) = c_1\varphi_1(x) + c_2\varphi_2(x), \qquad \begin{pmatrix} 9 & -6 \\ -6 & 8 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix},$$

and with $u(0) = u'(1) = 0$ we get

$$u(x) = c_1\varphi_1(x) + c_2\varphi_2(x) + c_3\varphi_3(x), \qquad \begin{pmatrix} 9 & -6 & 0 \\ -6 & 8 & -2 \\ 0 & -2 & 2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix},$$

where the simplest approximations to $f_k = \int_0^1 f(x)\,\varphi_k(x)\,dx$ are $f_1 \approx \frac{1}{4} f(\frac{1}{3})$, $f_2 \approx \frac{1}{3} f(\frac{1}{2})$ and $f_3 \approx \frac{1}{4} f(1)$.

To derive a finite difference approximation of the second derivative with an irregular mesh, we compute

$$
\begin{aligned}
u(x+h) &= u(x) + u'(x)h + \tfrac{1}{2}u''(x)h^2 + O(h^3),\\
u(x-k) &= u(x) - u'(x)k + \tfrac{1}{2}u''(x)k^2 + O(k^3),
\end{aligned}
$$

and eliminate the $u'(x)$ terms to find

$$
u''(x) \approx \frac{k\big(u(x+h)-u(x)\big) + h\big(u(x-k)-u(x)\big)}{\tfrac{1}{2}hk(h+k)}.
$$

With our particular mesh, this gives

$$
\begin{aligned}
u''(\tfrac{1}{3}) &\approx \frac{\tfrac{1}{3}\big(u(\tfrac{1}{2})-u(\tfrac{1}{3})\big) + \tfrac{1}{6}\big(u(0)-u(\tfrac{1}{3})\big)}{1/72},\\
u''(\tfrac{1}{2}) &\approx \frac{\tfrac{1}{6}\big(u(1)-u(\tfrac{1}{2})\big) + \tfrac{1}{2}\big(u(\tfrac{1}{3})-u(\tfrac{1}{2})\big)}{1/36},
\end{aligned}
$$

so that the finite difference approximation scheme is

$$
\begin{aligned}
U(0) &= 0,\\
-\frac{\tfrac{1}{3}\big(U(2)-U(1)\big) + \tfrac{1}{6}\big(U(0)-U(1)\big)}{1/72} &= f(\tfrac{1}{3}),\\
-\frac{\tfrac{1}{6}\big(U(3)-U(2)\big) + \tfrac{1}{2}\big(U(1)-U(2)\big)}{1/36} &= f(\tfrac{1}{2}),\\
U(3) &= 0,
\end{aligned}
$$

that is,

$$
\begin{pmatrix} 6 & -4 \\ -3 & 4 \end{pmatrix}\begin{pmatrix} U(1) \\ U(2) \end{pmatrix} = \frac{1}{6}\begin{pmatrix} f(\tfrac{1}{3}) \\ f(\tfrac{2}{3}) \end{pmatrix}.
$$

We see that the finite difference scheme again gives the same result as FEM (with the given approximations for f_1 and f_2), since

$$
\begin{pmatrix} 9 & -6 \\ -6 & 8 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \tfrac{1}{4}f(\tfrac{1}{3}) \\ \tfrac{1}{3}f(\tfrac{1}{2}) \end{pmatrix} \qquad \Longleftrightarrow \qquad \begin{pmatrix} 6 & -4 \\ -3 & 4 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} f(\tfrac{1}{3}) \\ f(\tfrac{2}{3}) \end{pmatrix}.
$$

15.6 Denote the position vectors of the nodes by $\mathbf{a} = (a_1, a_2)$, $\mathbf{b} = (b_1, b_2)$ and $\mathbf{c} = (c_1, c_2)$, and let $\mathbf{v} = \mathbf{b} - \mathbf{a}$ and $\mathbf{w} = \mathbf{c} - \mathbf{a}$ be the edge vectors from node a to node b and c, respectively. The angle between $\mathbf{v}$ and $\mathbf{w}$ is α, so

$$
|\mathbf{v}|\,|\mathbf{w}|\cos\alpha,
$$

and the area of the parallelogram that they span, which is also twice the area of the triangle T, is

$$
|\mathbf{v}|\,|\mathbf{w}|\sin\alpha.
$$

and this is also the absolute value of the determinant

$$\begin{aligned}\Delta &= \begin{vmatrix} v_1 & w_1 \\ v_2 & w_2 \end{vmatrix} = \begin{vmatrix} b_1 - a_1 & c_1 - a_1 \\ b_2 - a_2 & c_2 - a_2 \end{vmatrix} \\ &= a_1 b_2 + b_1 c_2 + c_1 a_2 - a_1 c_2 - b_1 a_2 - c_1 b_2 \\ &= \begin{vmatrix} 1 & a_1 & a_2 \\ 1 & b_1 & b_2 \\ 1 & c_1 & c_2 \end{vmatrix}.\end{aligned}$$

Note that the value of Δ is unchanged under cyclic permutations of the vectors **a**, **b** and **c**. Next, we derive a formula for the restriction of the basis function φ_a to the triangle T. Since it's piecewise linear, we have $\varphi_a(x, y) = K + Lx + My$ on T, and since it's equal to 1 at node a and equal to 0 at nodes b and c, we have

$$\begin{aligned} K + La_1 + Ma_2 &= 1, \\ K + Lb_1 + Mb_2 &= 0, \\ K + Lc_1 + Mc_2 &= 0, \end{aligned}$$

where we can solve for (K, L, M) and insert back into the formula for φ_a, which after a bit of calculation shows that

$$\varphi_a(x, y) = \frac{1}{\Delta} \begin{vmatrix} x - c_1 & b_1 - c_1 \\ y - c_2 & b_2 - c_2 \end{vmatrix}$$

for $(x, y) \in T$. It follows that the gradient (which is a piecewise constant vector) equals

$$\nabla\varphi_a(x, y) = \frac{1}{\Delta} \begin{pmatrix} b_2 - c_2 \\ -(b_1 - c_1) \end{pmatrix}$$

for (x, y) in the interior of T. The formulas for φ_b and φ_c on T are obtained by cyclic permuations of the vectors **a**, **b** and **c**, so on T we have

$$\nabla\varphi_b \cdot \nabla\varphi_c = \frac{1}{\Delta} \begin{pmatrix} c_2 - a_2 \\ -(c_1 - a_1) \end{pmatrix} \cdot \frac{1}{\Delta} \begin{pmatrix} a_2 - b_2 \\ -(a_1 - b_1) \end{pmatrix} = \frac{(\mathbf{c} - \mathbf{a}) \cdot (\mathbf{a} - \mathbf{b})}{\Delta^2} = -\frac{\mathbf{v} \cdot \mathbf{w}}{\Delta^2},$$

and thus

$$\begin{aligned} K_{bc}^T &= \int_T \nabla\varphi_b \cdot \nabla\varphi_c \, dx\,dy = -\operatorname{area}(T) \frac{\mathbf{v} \cdot \mathbf{w}}{\Delta^2} = -\tfrac{1}{2} |\Delta| \frac{\mathbf{v} \cdot \mathbf{w}}{\Delta^2} = -\frac{\mathbf{v} \cdot \mathbf{w}}{2|\Delta|} \\ &= -\frac{|\mathbf{v}|\,|\mathbf{w}| \cos\alpha}{2|\mathbf{v}|\,|\mathbf{w}| \sin\alpha} = -\tfrac{1}{2} \cot\alpha. \end{aligned}$$

Similarly,

$$\begin{aligned} K_{aa}^T &= \int_T \nabla\varphi_a \cdot \nabla\varphi_a \, dx\,dy = \operatorname{area}(T) \frac{(\mathbf{b} - \mathbf{c}) \cdot (\mathbf{b} - \mathbf{c})}{\Delta^2} \\ &= \frac{\big((\mathbf{b} - \mathbf{a}) + (\mathbf{a} - \mathbf{c})\big) \cdot (\mathbf{b} - \mathbf{c})}{2|\Delta|} \\ &= -\frac{(\mathbf{a} - \mathbf{b}) \cdot (\mathbf{b} - \mathbf{c}) + (\mathbf{b} - \mathbf{c}) \cdot (\mathbf{c} - \mathbf{a})}{2|\Delta|} \\ &= -(K_{ac}^T + K_{ab}^T) = \tfrac{1}{2} \cot\beta + \tfrac{1}{2} \cot\gamma. \end{aligned}$$

16.1

(a) Green's second identity says that $\int_\Omega \big((\Delta u)v - u(\Delta v)\big)\,dV = \int_{\partial\Omega}\big((\partial u/\partial n)v - u(\partial v/\partial n)\big)\,dV$, which equals zero if u and v are zero on $\partial\Omega$.

(b) Green's first identity says that $\int_{\partial\Omega} \nabla\cdot(u\Delta v)\,dV = \int_\Omega \big(\nabla u\cdot\nabla v + u\Delta v\big)\,dV$. Taking $u = v$, with $u = 0$ on the boundary, we obtain $\int_\Omega(-\Delta u)u\,dV = \int_\Omega |\nabla u|^2\,dV \geq 0$, with equality only if u is constant, i.e., if $u \equiv 0$ in Ω (since it's zero on the boundary). So if $u \not\equiv 0$, then we have strict inequality, as was to be shown.

(c) Using $-\Delta u_i = \lambda_i u_i$ and the symmetry proved in part (a), we have $\lambda_i \int_\Omega u_i u_j\,dV = \int_\Omega(-\Delta u_i)u_j\,dV = \int_\Omega u_i(-\Delta u_j)\,dV = \lambda_j \int_\Omega u_i u_j\,dV$, so that $(\lambda_i - \lambda_j)\int_\Omega u_i u_j\,dV = 0$. Since $\lambda_i - \lambda_j \neq 0$ by assumption, the integral must be zero.

(d) If $-\Delta u = \lambda u$ and $u \not\equiv 0$, then $\lambda \int_\Omega u^2\,dV = \int_\Omega (\lambda u)\,u\,dV = \int_\Omega(-\Delta u)\,u\,dV > 0$ by the positive definiteness proved in part (b). Since obviously $\int_\Omega u^2\,dV > 0$, it follows that $\lambda > 0$.

(e) The operator $-\Delta$ with Neumann conditions is still symmetric, but only positive *semidefinite*, since $u \equiv C$ satisfies $\partial u/\partial n = 0$ for any value of C, not just $C = 0$. Orthogonality of the eigenspaces still holds, but the eigenvalues only satisfy $\lambda \geq 0$, not $\lambda > 0$. (The lowest eigenvalue is $\lambda_0 = 0$, with eigenfunction $u_0 \equiv 1$).

16.2 The answer is

$$u(x,t) = \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} C_{km}\sin(kx)\sin(my)e^{-(k^2+m^2)t},$$

where

$$C_{km} = \frac{\iint_\Omega x(\pi - x)\sin^2 y\cdot\sin(kx)\sin(my)\,dx\,dy}{\iint_\Omega \big(\sin(kx)\sin(my)\big)^2\,dx\,dy} = \underbrace{\left(\frac{2}{\pi}\int_0^\pi x(\pi-x)\sin(kx)\,dx\right)}_{=A_k}\underbrace{\left(\frac{2}{\pi}\int_0^\pi \sin^2 y\sin(my)\,dy\right)}_{=B_m},$$

where we get

$$A_k = \begin{cases} \dfrac{8}{\pi k^3}, & \text{if } k \text{ is odd,} \\ 0 & \text{if } k \text{ is even,} \end{cases}$$

from exercise 9.2, and where we compute

$$B_m = \begin{cases} \dfrac{-8}{\pi(m^3 - 4m)}, & \text{if } m \text{ is odd,} \\ 0 & \text{if } m \text{ is even,} \end{cases}$$

using

$$\sin^2 y \sin(my) = \tfrac{1}{2}\sin(my) - \tfrac{1}{4}\sin((m+2)y) - \tfrac{1}{4}\sin((m-2)y).$$

(Note that the case $m = 2$ needs separate treatment when computing the antiderivative.)

16.3

(a) We write $u(r\cos\varphi, r\sin\varphi) = R(r)\,\Phi(\varphi)$. Then everything is like for the disk, except that we get $-\Phi''(\varphi) = \gamma\Phi(\varphi)$ with Dirichlet conditions $\Phi(0) = \Phi(\beta) = 0$ instead of 2π-periodicity, so that (up to a constant) $\Phi(\varphi) = \sin(n\pi x/\beta)$ and $\gamma = (n\pi/\beta)^2$ where $m \geq 1$ is an integer. And then the Bessel equation that we get for $Q(\rho) = R(\rho/\sqrt{\lambda})$ will have parameter $n\pi/\beta$ instead of n. The radial part will be $R(r) = J_{n\pi/\beta}(\sqrt{\lambda}\,r)$, where λ is determined by the boundary condition $R(a) = 0$. (The Bessel functions are defined not just for integer values of the parameter, but actually for any complex value.) If we write μ_{nk} for the kth positive zero of $J_{n\pi/\beta}$, then the eigenfunctions and their corresponding eigenvalues are

$$u_{nk}(r\cos\varphi, r\sin\varphi) = J_{n\pi/\beta}(\mu_{nk}\,r/a)\,\sin(n\pi x/\beta), \qquad \lambda_{nk} = (\mu_{nk}/a)^2,$$

for integers $n \geq 1$ and $k \geq 1$.

(b) —

16.4 The minimal value of $I(w)$ is the smallest eigenvalue of the one-dimensional Laplacian $-d^2/dx^2$ with boundary conditions $w(0) = w(1) = 1$, namely $\lambda_1 = \pi^2$. (The eigenfunctions are $w_m(x) = \sin(m\pi x)$ with $\lambda_m = (m\pi)^2$, for integers $m \geq 1$.) This minimum is attained for $w(x) = \pm 2\sin x$.

16.5

(a) From the expression $\Delta u = u_{rr} + \frac{1}{r^2}\,u_{\theta\theta} + \frac{1}{r^2\sin^2\theta}\,u_{\varphi\varphi} + \frac{2}{r}\,u_r + \frac{\cos\theta}{r^2\sin\theta}\,u_\theta$, we obtain

$$R''\Theta\Phi + \frac{1}{r^2}R\Theta''\Phi + \frac{1}{r^2\sin^2\theta}R\Theta\Phi'' + \frac{2}{r}R'\Theta\Phi + \frac{\cos\theta}{r^2\sin\theta}R\Theta'\Phi = -\lambda R\Theta\Phi,$$

that is,

$$\frac{R''}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} + \frac{1}{r^2\sin^2\theta}\frac{\Phi''}{\Phi} + \frac{2}{r}\frac{R'}{R} + \frac{\cos\theta}{r^2\sin\theta}\frac{\Theta'}{\Theta} = -\lambda.$$

This can be rearranged to

$$r^2\left(\frac{R''}{R} + \frac{2}{r}\frac{R'}{R} + \lambda\right) = -\left(\frac{\Theta''}{\Theta} + \frac{1}{\sin^2\theta}\frac{\Phi''}{\Phi} + \frac{\cos\theta}{\sin\theta}\frac{\Theta'}{\Theta}\right),$$

where the left-hand side is independent of θ and φ, while the right-hand side is independent of r; hence, both sides must be equal to some constant γ, so that we get one equation for the radial part $R(r)$ and one involving the angles:

$$R'' + \frac{2}{r}R' + \left(\lambda - \frac{\gamma}{r^2}\right)R = 0, \qquad \sin^2\theta\left(\frac{\Theta''}{\Theta} + \frac{\cos\theta}{\sin\theta}\frac{\Theta'}{\Theta} + \gamma\right) = -\frac{\Phi''}{\Phi}.$$

And in the angular equation, both sides must be equal to some constant α, so that we get further separation into one equation for $\Theta(\theta)$ and one for $\Phi(\varphi)$:

$$\Theta'' + \frac{\cos\theta}{\sin\theta}\Theta' + \left(\gamma - \frac{\alpha}{\sin^2\theta}\right)\Theta = 0, \qquad \Phi'' = -\alpha\Phi.$$

The boundary conditions are that $R(r)$ must be finite at $r = 0$, $R(a) = 0$, $\Theta(\theta)$ must be finite at $\theta = 0$ and $\theta = \pi$, and $\Phi(\varphi)$ must be 2π-periodic; this last condition implies that $\alpha = m^2$ for some integer $m \geq 0$, and that $\Phi(\varphi) = A\cos(m\varphi) + B\sin(m\varphi)$ (or just $\Phi(\varphi) = A$ if $m = 0$).

(b) With $R(r) = Q(\sqrt{\lambda}\,r)\,r^{-1/2}$ we get $R'(r) = \sqrt{\lambda}\,Q'(\sqrt{\lambda}\,r)\,r^{-1/2} - \frac{1}{2}Q(\sqrt{\lambda}\,r)\,r^{-3/2}$ and $R''(r) = \lambda Q''(\sqrt{\lambda}\,r)\,r^{-1/2} - \sqrt{\lambda}\,Q'(\sqrt{\lambda}\,r)\,r^{-3/2} + \frac{3}{4}Q(\sqrt{\lambda}\,r)\,r^{-5/2}$, so the ODE for R becomes

$$\begin{aligned}
0 &= R'' + \frac{2}{r}R' + \left(\lambda - \frac{\gamma}{r^2}\right)R = 0 \\
&= \frac{\lambda r^2 Q''(\sqrt{\lambda}\,r) - \sqrt{\lambda}\,r Q'(\sqrt{\lambda}\,r) + \frac{3}{4}Q(\sqrt{\lambda}\,r)}{r^{5/2}} \\
&\quad + \frac{2}{r}\,\frac{\sqrt{\lambda}\,r Q'(\sqrt{\lambda}\,r) - \frac{1}{2}Q}{r^{3/2}} + \frac{\lambda r^2 - \gamma}{r^2}\,\frac{Q(\sqrt{\lambda}\,r)}{r^{1/2}} \\
&= r^{-5/2}\Big(\rho^2 Q''(\rho) + \rho Q'(\rho) + \big(\rho^2 - \gamma - \tfrac{1}{4}\big)Q(\rho)\Big).
\end{aligned}$$

Comparison to Bessel's equation $\rho^2 Q''(\rho) + \rho Q'(r) + (\rho^2 - n^2)Q(\rho) = 0$ shows that $n = \sqrt{\gamma + \frac{1}{4}}$. This isn't necessarily an integer, but the Bessel functions J_n are defined for all complex values of n, so this is not a problem. Thus, the radial radial solution will have the form

$$R(r) = \frac{J_n(\sqrt{\lambda}\,r)}{\sqrt{r}}, \qquad n = \sqrt{\gamma + \tfrac{1}{4}}.$$

The boundary condition $R(a) = 0$ implies that $\sqrt{\lambda}a$ must be a zero of that Bessel function J_n, and this links the values of λ and γ. (But we haven't yet determined what values γ is allowed to take; see below.)

(c) If we write $\Theta(\theta) = Z(\cos\theta)$ for some function $Z(z)$, then $\Theta'(\theta) = -\sin\theta\, Z'(\cos\theta)$ and $\Theta''(\theta) = \sin^2\theta\, Z''(\cos\theta) - \cos\theta\, Z'(\cos\theta)$, so the ODE for Θ becomes

$$\begin{aligned}
0 &= \Theta''(\theta) + \frac{\cos\theta}{\sin\theta}\Theta'(\theta) + \left(\gamma - \frac{m^2}{\sin^2\theta}\right)\Theta(\theta) \\
&= \sin^2\theta\, Z''(\cos\theta) - \cos\theta\, Z'(\cos\theta) \\
&\quad + \frac{\cos\theta}{\sin\theta}\big(-\sin\theta\big)Z'(\cos\theta) + \left(\gamma - \frac{m^2}{\sin^2\theta}\right)Z(\cos\theta) \\
&= (1 - z^2)Z''(z) - 2z\,Z'(z) + \left(\gamma - \frac{m^2}{1 - z^2}\right)Z(z), \qquad -1 < z < 1.
\end{aligned}$$

This is called the *(associated) Legendre equation*, and it turns out that it has solutions that extend nicely to $z = \pm 1$ (corresponding to $\theta = 0$ and $\theta = \pi$) if and only if $\gamma = l(l+1)$ where $l \geq 0$ is an integer, and $0 \leq m \leq l$. These solutions, denoted $Z(z) = P_l^m(z)$, are certain polynomials in z of degree $l - m$, multiplied by $(1 - z^2)^{m/2}$. (For more details, see for example Strauss's book, Sections 10.3 and 10.6).

To summarize, the eigenfunctions that we have found (and it can be shown that this is a complete set) are constructed as follows: Pick integers $0 \leq m \leq l$. Let $\Theta(\theta) = P_l^m(\cos\theta)$. Let $\Phi(\varphi) = \cos(m\varphi)$ or $\sin(m\varphi)$ (or just $\Phi(\varphi) = 1$ if $m = 0$). And finally pick some integer $j \geq 1$ and let $R(r) = J_n(\sqrt{\lambda} r)/\sqrt{r}$, where

$$n = \sqrt{\gamma + \tfrac{1}{4}} = \sqrt{l(l+1) + \tfrac{1}{4}} = l + \tfrac{1}{2},$$

and where $\sqrt{\lambda} a$ is equal to the jth positive zero of the Bessel function $J_n = J_{l+1/2}$; let's denote this zero by μ_{lj}, so that $\lambda = (\mu_{lj}/a)^2$. Then $u = R(r)\,\Theta(\theta)\,\Phi(\varphi)$ satisfies $-\Delta u = \lambda u$ in the ball, and $u = 0$ on the boundary sphere, so it's an eigenfunction with the eigenvalue $\lambda = (\mu_{lj}/a)^2$. (Note the degeneracy; the eigenvalue doesn't depend on m, so we can form a linear combination of eigenfunctions with the same l and j but different m, to obtain another eigenfunction with the same eigenvalue.)

17.1

(a) Not dispersive, since $\omega = -ik^2$ isn't real when k is real.

(b) Not dispersive, since $w = ck$ makes $\frac{d^2\omega}{dk^2}$ identically zero.

(c) Not dispersive, since $\omega = \pm ck$ makes $\frac{d^2\omega}{dk^2}$ identically zero (in both cases).

(d) Dispersive, with $\omega = ck - bk^3$, so that $c_{\text{phase}} = c - bk^2$ and $c_{\text{group}} = c - 3bk^2$.

(e) Dispersive, with $\omega = \pm ck/(1 + b^2k^2)^{1/2}$, so that $c_{\text{phase}} = \pm c/(1 + b^2k^2)^{1/2}$ and $c_{\text{group}} = \pm(1 - b^2k^2)/(1 + b^2k^2)^{3/2}$.

(f) Dispersive, with $\omega = \pm(c^2k^2 + m^2)^{1/2}$, so that

$$c_{\text{phase}} = \pm\frac{(c^2k^2 + m^2)^{1/2}}{k} = \pm c\left(1 + \frac{m^2}{c^2k^2}\right)^{1/2}$$

and

$$c_{\text{group}} = \pm c^2 k (c^2k^2 + m^2)^{-1/2} = \pm c\left(1 + \frac{m^2}{c^2k^2}\right)^{-1/2}.$$

(g) Not dispersive, since ω isn't real when k is real.

17.2 If $\omega = ck + d$, then a wave packet takes the form

$$u(x,t) = \int_0^\infty F(k)\, e^{i(kx-\omega t)}\, dk = \int_0^\infty F(k)\, e^{i(kx-ckt-dt)}\, dk$$
$$= e^{-idt} \int_0^\infty F(k)\, e^{ik(x-ct)}\, dk = e^{-idt} f(x-ct),$$

so even if its shape changes (due to the factor e^{-idt}, the wave packet as a whole still travels with velocity c, and doesn't really *disperse*.

17.3 Rewriting as suggested, we obtain

$$u(x,t) \approx \cos\Big(\big(kx - \omega(k)t\big) - \delta\big(x - \omega'(k)t\big)\Big) + \cos\Big(\big(kx - \omega(k)t\big) + \delta\big(x - \omega'(k)t\big)\Big)$$
$$= 2\cos\Big(kx - \omega(k)t\Big)\cos\Big(\delta\big(x - \omega'(k)t\big)\Big)$$
$$= 2\cos\Big(k\big(x - \tfrac{\omega(k)}{k}\, t\big)\Big)\cos\Big(\delta\big(x - \omega'(k)t\big)\Big),$$

so u looks like a rapidly oscillating wave with wave number k, moving with the phase velocity $c_{\text{phase}} = \omega(k)/k$, multiplied by a slowly oscillating amplitude factor with wave number δ, moving with the group velocity $c_{\text{group}} = \omega'(k)$.

17.4 Hints: The ODE for g is $g'''(p) = p\, g'(p)$, so $y(p) = g'(p)$ satisfies the Airy equation $y''(p) = p\, y(p)$, whose general solution is $y(p) = C\,\mathrm{Ai}(p) + D\,\mathrm{Bi}(p)$, where $\mathrm{Bi}(p) \nearrow \infty$ as $p \to \infty$, while Ai is bounded on $\mathbf{R}$ and satisfies $\int_{-\infty}^0 \mathrm{Ai}(p)\, dp = \frac{2}{3}$ and $\int_0^\infty \mathrm{Ai}(p)\, dp = \frac{1}{3}$ (Abramowitz & Stegun, *Handbook of Mathematical Functions*, formulas 10.4.82–83).

Answer:

$$u(x,t) = \frac{2}{3} + \int_0^{x/(3t)^{1/3}} \mathrm{Ai}(r)\, dr.$$

Remark: Note that differentiating this with respect to x gives

$$\frac{1}{(3t)^{1/3}}\,\mathrm{Ai}\left(\frac{x}{(3t)^{1/3}}\right),$$

which is the fundamental solution of this PDE (the solution with the Dirac delta $\delta(x) = H'(x)$ as initial data).

17.5 The Laplace equation implies $(ik)^2 Z(z) + Z''(z) = 0$, so that

$$Z(z) = C\cosh(kz) + D\sinh(kz).$$

The boundary condition at the bottom gives $Z'(-h) = 0$, so $D/C = \tanh(kh)$, and hence

$$Z(z) = C\,(\cosh(kz) + \tanh(kh)\sinh(kz)).$$

And finally, the boundary condition at the surface gives

$$0 = -\omega^2 Z(0) + g Z'(0) = C\big(-\omega^2 + gk\tanh(kh)\big).$$

Answer: $\omega = \pm\sqrt{gk\tanh(kh)}$. For small k, we can approximate $\tanh(kh)$ with kh to obtain $\omega \approx \pm\sqrt{gh}\,k$, and as $k \to \infty$, $\tanh(kh) \to 1$, so that $\omega \approx \pm\sqrt{gk}$.

17.6 Deriving the ODE should hopefully present no problems. Integration gives

$$-cf(\xi) + \tfrac{1}{2}f(\xi)^2 + f''(\xi) = A,$$

where we must have $A = 0$ since f and f'' were assumed to vanish at infinity. Multiplying by $2f'(\xi)$ and integrating again gives

$$-cf(\xi)^2 + \tfrac{1}{3}f(\xi)^3 + f'(\xi)^2 = B,$$

where we must have $B = 0$ since f and f' were assumed to vanish at infinity. So

$$f'(\xi) = \pm f(\xi)\sqrt{c - \tfrac{1}{3}f(\xi)},$$

which is a separable ODE. The constant solution $f = 0$ is not very interesting here, so we discard that case. For $0 < f(\xi) < 3c$, on the other hand, we get

$$\int \frac{df}{f\sqrt{c - \frac{1}{3}f}} = \pm\int d\xi$$

With the substitution $f = 3cy$, we can compute

$$\begin{aligned}
\pm(\xi - x_0) &= \frac{1}{\sqrt{c}}\int \frac{dy}{y\sqrt{1-y}} \\
&= \left[\text{let } s = \sqrt{1-y}, \text{ use partial fractions, etc.}\right] \\
&= \frac{1}{\sqrt{c}}\ln\frac{1-\sqrt{1-y}}{1+\sqrt{1-y}},
\end{aligned}$$

so that

$$\frac{1-\sqrt{1-y}}{1+\sqrt{1-y}} = e^{\pm\sqrt{c}(\xi - x_0)} =: E.$$

This gives

$$\sqrt{1-y} = \frac{1-E}{1+E}$$

and hence

$$1 - y = \frac{1+E^2+2E}{1+E^2-2E} = 1 - \frac{4E}{(1+E)^2},$$

so that

$$f(\xi) = 3cy = 3c\cdot\frac{4E}{(1+E)^2} = \frac{3c}{\left(\dfrac{1+E}{2\sqrt{E}}\right)^2} = \frac{3c}{\left(\dfrac{E^{1/2}+E^{-1/2}}{2}\right)^2} = \frac{3c}{\cosh^2\left(\frac{\sqrt{c}}{2}(\xi - x_0)\right)}.$$

The computation above has a potential problem with division by zero at $\xi = x_0$, where $c - \frac{1}{3}f = 0$, so we should perhaps give a bit more justification. The function $f(\xi)$ satisfies the ODE $f'(\xi) = +f(\xi)\sqrt{c - \frac{1}{3}f(\xi)}$ for $\xi < x_0$, and the ODE $f'(\xi) = -f(\xi)\sqrt{c - \frac{1}{3}f(\xi)}$ for $\xi > x_0$ (according to our calulations, which are fine in those cases). Morover, it satisfies *both* these ODEs at the point $\xi = x_0$, since $f = 3c$ and $f' = 0$ there. So the original ODE $-cf(\xi)^2 + \frac{1}{3}f(\xi)^3 + f'(\xi)^2 = 0$ is indeed satisfied for all $\xi \in \mathbf{R}$.

17.7

(a) We seek $f(\xi)$ such that $u = f(x - ct)$ is a solution, which requires that

$$(-c)^2 f''(\xi) - f''(\xi) + \sin f(\xi) = 0.$$

Multiply by $2f'(\xi)$ and integrate:

$$(c^2 - 1)f'(\xi)^2 - 2\cos f(\xi) = A.$$

By assumption, $\cos f(\xi) \to 1$ as $\xi \to \pm\infty$, and we can't have $c^2 - 1 = 0$ since that would imply that f is constant, so $f'(\xi)^2 \to (2 + A)/(c^2 - 1)$ as $\xi \to \pm\infty$. If the right-hand side were nonzero, then f could not have finite limits at $\pm\infty$, so $A = -2$, and

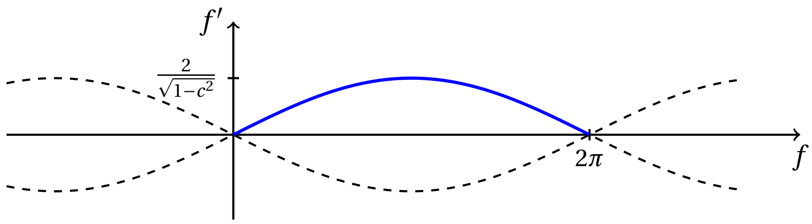
$$f'(\xi)^2 = \frac{2(1 - \cos f(\xi))}{1 - c^2} = \frac{4\sin^2\left(\frac{1}{2}f(\xi)\right)}{1 - c^2};$$

here we see that we must have $|c| < 1$ in order for a solution of this type to exist. Thinking of the relation

$$f'(\xi) = \pm\frac{2}{\sqrt{1 - c^2}}\left|\sin\left(\tfrac{1}{2}f(\xi)\right)\right|$$

in terms of the (f, f') phase plane, we see that the only curve coming from the point $(0, 0)$ and going to the point $(2\pi, 0)$ is

$$f' = \frac{2}{\sqrt{1 - c^2}}\sin(\tfrac{1}{2}f), \qquad 0 < f < 2\pi.$$



The solution following this curve is given by separation of variables:

$$\frac{1}{\sqrt{1 - c^2}}\int d\xi = \int \frac{df}{2\sin(\frac{1}{2}f)},$$

which gives

$$\frac{\xi - x_0}{\sqrt{1-c^2}} = \ln\tan(\tfrac{1}{4}f),$$

where $0 < \frac{1}{4}f < \frac{\pi}{2}$.

Answer: $u(x,t) = 4\arctan\left(e^{(x-ct-x_0)/\sqrt{1-c^2}}\right)$, for $-1 < c < 1$.

(b) Just compute: $u = 4\arctan\frac{T}{X}$ gives

$$u_t = \frac{4}{1+\frac{T^2}{X^2}} \cdot \frac{T_t}{X} = \frac{4XT_t}{X^2+T^2}, \qquad u_{tt} = 4X \cdot \frac{T_{tt}(X^2+T^2) - T_t \cdot 2TT_t}{(X^2+T^2)^2}$$

and

$$u_x = \frac{4}{1+\frac{T^2}{X^2}} \cdot \frac{-TX_x}{X^2} = \frac{-4X_xT}{X^2+T^2}, \qquad u_{xx} = -4T \cdot \frac{X_{xx}(X^2+T^2) - X_x \cdot 2XX_x}{(X^2+T^2)^2}.$$

Moreover, the identity $\sin(4\alpha) = 2\sin(2\alpha)\cos(2\alpha) = 4\sin\alpha\cos\alpha(\cos^2\alpha - \sin^2\alpha)$, with $\alpha = \arctan\frac{T}{X}$ so that $\sin\alpha = X/\sqrt{X^2+T^2}$ and $\cos\alpha = T/\sqrt{X^2+T^2}$, gives

$$\sin u = 4 \cdot \frac{X}{\sqrt{X^2+T^2}} \cdot \frac{T}{\sqrt{X^2+T^2}} \cdot \left(\frac{X^2}{X^2+T^2} - \frac{T^2}{X^2+T^2}\right) = \frac{4XT(X^2-T^2)}{(X^2+T^2)^2}.$$

Inserting this into the sine–Gordon equation $u_{tt} - u_{xx} + \sin u = 0$ and cancelling the common factor $4/(X^2+T^2)^2$ gives the first equation, and then the second equation is obtained through division by XT.

(c) With $T(t) = \sqrt{1-\omega^2}\cos(\omega t)$ and $X(x) = \omega\cosh\left(\sqrt{1-\omega^2}\,x\right)$ we have $T_{tt} = -\omega^2 T$ and $X_{xx} = (1-\omega^2)X$, and moreover from $\cos^2 + \sin^2 = 1 = \cosh^2 - \sinh^2$ we get

$$T_t^2 = (1-\omega^2)\omega^2\sin^2(\omega t) = \omega^2(1-\omega^2)\left(1 - \frac{T^2}{1-\omega^2}\right) = \omega^2(1-\omega^2) - \omega^2 T^2$$

and

$$X_x^2 = \omega^2(1-\omega^2)\sinh^2\left(\sqrt{1-\omega^2}\,x\right) = (1-\omega^2)\omega^2\left(\frac{X^2}{\omega^2} - 1\right) = (1-\omega^2)X^2 - \omega^2(1-\omega^2),$$

so that the condition from part (b) is fulfilled:

$$\begin{aligned}
&\frac{T_{tt}}{T}X^2 + XX_{xx} + X^2 - 2X_x^2 + \frac{X_{xx}}{X}T^2 + TT_{tt} - T^2 - 2T_t^2 \\
&= -\omega^2X^2 + (1-\omega^2)X^2 + X^2 - 2(1-\omega^2)X^2 + 2\omega^2(1-\omega^2) \\
&\quad + (1-\omega^2)T^2 - \omega^2T^2 - T^2 + 2\omega^2T^2 - 2\omega^2(1-\omega^2) \\
&= 0.
\end{aligned}$$