Linköpings universitet
Matematiska institutionen
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Utbildningskod: TATA27
Modul: TEN1

TATA27 Partiella differentialekvationer

Tentamen 2025-05-28 kl. 8.00–12.00

No aids allowed (except drawing tools, such as rulers, of course). You may write your answers in English or in Swedish, or some mixture thereof.

Each problem is marked *pass* (3 or 2 points) or *fail* (1 or 0 points). For grade $n \in \{3,4,5\}$ you need at least n passed problems and at least $3n-1$ points.

Solutions will be posted on the course webpage afterwards. Good luck!

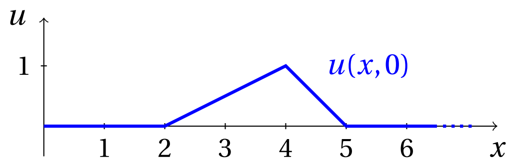
1. Find the solution $u(x,y)$ to the PDE $xu_x - yu_y = 0$ in the positive quadrant $x > 0$, $y > 0$, under the condition $u(x,2x) = \sin x$ for $x > 0$.

2. Consider the heat equation $u_t = u_{xx}$ on the interval $0 < x < \pi$, with the initial condition $u(x,0) = 1 - \cos 2x$. Find the solution $u(x,t)$ (for $t > 0$) in the following two cases:

   (a) With Neumann boundary conditions ($u_x = 0$ at both endpoints). (1p)

   (b) With Dirichlet boundary conditions ($u = 0$ at both endpoints). (2p)

3. Consider the wave equation $u_{tt} = 4u_{xx}$ on the half-line $x > 0$, with the boundary condition $u(0,t) = 0$ for all t. Suppose that the (weak) solution $u(x,t)$ looks as in the graph below when $t = 0$, and also that $u_t(x,0) = 0$. Draw graphs showing what $u(x,t)$ looks like when $t = 1$, $t = 2$ and $t = 3$.



4. Solve Laplace's equation $\Delta u(x,y,z) = 0$ in the unit ball $x^2 + y^2 + z^2 < 1$, with the boundary condition that $u(x,y,z) = x^3$ when $x^2 + y^2 + z^2 = 1$. (Hint: Seek a first-degree polynomial q such that $x^3 + (1 - r^2)\, q(x,y,z)$ is harmonic, where $r^2 = x^2 + y^2 + z^2$.)

5. Outline the main steps in (the basic variant of) the Finite Element Method for solving Poisson's equation $-\Delta u = f$ in a bounded domain $\Omega \subset \mathbf{R}^2$ with the Dirichlet condition $u = 0$ on $\partial\Omega$. (What is the "semi-weak formulation" that the method is based upon? In what form do we seek the approximate solution? What algebraic equations do we need to solve in order to find the unknown coefficients in that expression? Etc.)

6. Let $\Omega \subseteq \mathbf{R}^n$ be a connected nonempty open set, and suppose that $u \in C^2$ satisfies $\Delta u = u^2$ in Ω. Show that u cannot attain a local maximum in Ω unless $u \equiv 0$ there. (Hint: Subharmonic.)

Solutions for TATA27 2025-05-28

1. The PDE says that u is constant along the characteristics, which are the solution curves of the ODEs $\dot{x} = x$, $\dot{y} = -y$, namely $(x, y) = (x_0 e^t, y_0 e^{-t})$ with $x_0 > 0$ and $y_0 > 0$, or in other words the curves $xy = C$ in the positive quadrant. Thus $u(x, y) = f(xy)$, where the additional condition requires that $f(x \cdot 2x) = \sin x$ for $x > 0$, so that $f(\xi) = \sin \sqrt{\xi/2}$ for $\xi > 0$.

   Answer. $u(x, y) = \sin \sqrt{xy/2}$ for $x > 0$ and $y > 0$.

2. (a) Solvable by inspection! **Answer.** $u(x, t) = 1 - e^{-4t} \cos 2x$.

   (b) A basis of separated solutions is $u_n(x, t) = e^{-n^2 t} \sin nx$ for integers $n \geq 1$. To obtain the expansion of $u(x, t)$ in terms of $u_n(x, t)$, we need the expansion of $u(x, 0)$ in terms of $u_n(x, 0)$, namely $1 - \cos 2x = \sum_{n=1}^{\infty} c_n \sin nx$ for $0 < x < \pi$. Orthogonality gives

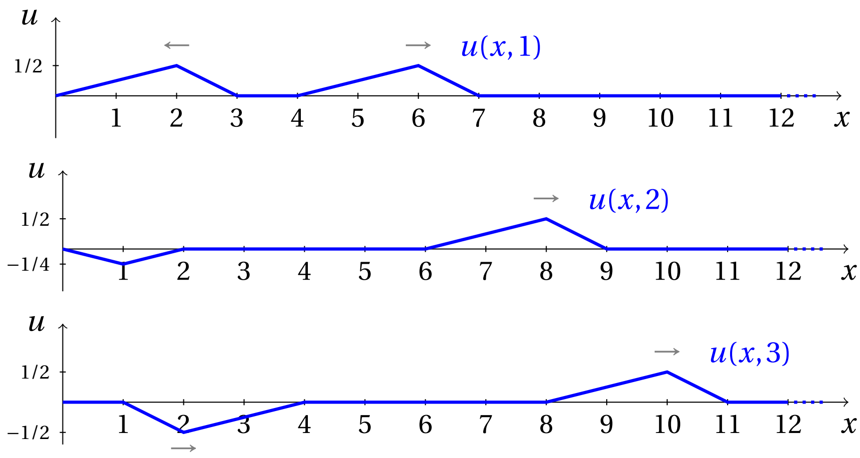
$$\int_0^{\pi} \bigl(1 - \cos 2x\bigr) \sin nx \, dx = c_n \int_0^{\pi} \sin^2 nx \, dx,$$

   so that

$$\begin{aligned} c_n &= \frac{2}{\pi} \int_0^{\pi} \bigl(\sin nx - \tfrac{1}{2} \sin(n+2)x - \tfrac{1}{2} \sin(n-2)x\bigr) dx \\ &= \begin{cases} 0, & n = 2, \\ \frac{2}{\pi}\bigl(1 - (-1)^n\bigr)\bigl(\frac{1}{n} - \frac{1/2}{n+2} - \frac{1/2}{n-2}\bigr), & 1 \leq n \neq 2 \end{cases} \\ &= \begin{cases} 0, & n \text{ even}, \\ \frac{16}{\pi n (4 - n^2)}, & n \text{ odd}. \end{cases} \end{aligned}$$

   Answer. $u(x, t) = \sum_{n=1}^{\infty} c_n u_n(x, t) = \sum_{\text{odd } n \geq 1} \dfrac{16 e^{-n^2 t} \sin nx}{\pi n (4 - n^2)}$.

3. Extend the initial function $u(x,0)$ to an odd function on the whole line, and then use d'Alembert's formula, i.e., split the function into the sum of two equal halves and let them move left and right, respectively, with the wave speed $c = 2$. At time $t = 1$, the upside-down wave coming in from the left starts interfering with the left-moving positive wave coming from the right, giving the impression that this wave is bouncing against the interval's endpoint $x = 0$:



4. The function $u(x,y,z) = x^3 + (1-r^2)(ax+by+cz+d)$ satisfies the boundary condition $u|_{r=1} = x^3$ by construction, and it is harmonic if

$$\begin{aligned} 0 = \Delta u &= \Delta\big(x^3 + (1-x^2-y^2-z^2)\,(ax+by+cz+d)\big) \\ &= \cdots = 6x - 10ax - 10by - 10cz - 6d, \end{aligned}$$

which gives $a = 3/5$ and $b = c = d = 0$.

Answer. $u(x,y,z) = x^3 + (1-x^2-y^2-z^2)\cdot\frac{3}{5}x = \frac{2}{5}x^3 - \frac{3}{5}xy^2 - \frac{3}{5}xz^2 + \frac{3}{5}x.$

5. Multiply the PDE by a test function φ which is zero on the boundary $\partial\Omega$, and integrate by parts (using the divergence theorem) to obtain the semi-weak formulation $\int_\Omega \nabla u \cdot \nabla\varphi \, dx\, dy = \int_\Omega f\varphi \, dx\, dy$. Triangulate Ω and label the interior nodes in the triangulation by $1,\ldots,N$. Define the basis function φ_k to be a "tent function": continuous, equal to one at node number k, equal to zero at all other nodes (including the boundary nodes), and piecewise affine (given by a first-degree polynomial on each triangle). Then the approximate solution is $u = \sum_{k=1}^{N} c_j\varphi_j$, where the coefficients c_j are determined by requiring this expression to satisfy the semi-weak formulation with $\varphi = \varphi_i$ for $1 \le i \le N$; in other words, they are given by the linear system $\sum_{j=1}^{N} K_{ij}c_j = f_i$ for $1 \le i \le N$, where $K_{ij} = \int_\Omega \nabla\varphi_i \cdot \nabla\varphi_j \, dx\, dy$ and $f_i = \int_\Omega f\varphi_i \, dx\, dy$.

6. Since $\Delta u = u^2$ we have $\Delta u \geq 0$, i.e., u is subharmonic. Suppose u attains a local maximum at an interior point of Ω. According to the strong maximum principle for subharmonic functions, this can only happen if u is constant in that connected component of the domain, which is this case is all of Ω since it was assumed to be connected. So u is constant on Ω, which implies that $\Delta u = 0$, and then the PDE $\Delta u = u^2$ shows that the constant value of u is actually zero. Q.E.D.