Linköpings universitet
Matematiska institutionen
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Utbildningskod: TATA27
Modul: TEN1

TATA27 Partiella differentialekvationer

Tentamen 2025-08-18 kl. 14.00–18.00

No aids allowed (except drawing tools, such as rulers, of course). You may write your answers in English or in Swedish, or some mixture thereof.

Each problem is marked *pass* (3 or 2 points) or *fail* (1 or 0 points). For grade $n \in \{3,4,5\}$ you need at least n passed problems and at least $3n-1$ points.

Solutions will be posted on the course webpage afterwards. Good luck!

1. Suppose $u(x,y)$ is harmonic on $\mathbf{R}^2$, with $u(x,y) = x^2$ when $x^2+y^2=4$. Determine $u(0,0)$.

2. Let $c>0$. Solve the wave equation $u_{tt} = c^2 u_{xx}$ for $0<x<\pi$ and $t>0$, with Dirichlet boundary conditions $u=0$ for $x=0$ and $x=\pi$ and initial conditions $u = \sin 2x$ and $u_t = \sin 3x$ for $t=0$.

3. (a) Let $\Omega \subset \mathbf{R}^n$ be a bounded nonempty open set and $I \subseteq \mathbf{R}$ an open interval. Suppose $u(\mathbf{x},t)$ satisfies the heat equation $u_t = \Delta u$ for $(\mathbf{x},t) \in \Omega \times I$, with the boundary condition $u(\mathbf{x},t) = 0$ for $(\mathbf{x},t) \in \partial\Omega \times I$. Show that $E(t) = \frac{1}{2}\int_\Omega u(\mathbf{x},t)^2\,dV$ (for $t \in I$) is a nonincreasing function.
   (It is assumed that we are looking at a nice enough domain Ω and nice enough functions u, so that we may differentiate under the integral sign and use the divergence theorem.)

   (b) Use part (a) to show that there is at most one solution $u(\mathbf{x},t)$ (within this class of nice enough functions) to the initial–boundary value problem for the heat equation on $\Omega \times \mathbf{R}_+$.

4. Solve $2x^2 y u_x - u_y = 0$ subject to the condition $u(x,0) = f(x)$ for $x \in \mathbf{R}$, where $f \in C^1(\mathbf{R})$ is a given function. It is enough if you compute the solution $u(x,y)$ in the region of the xy-plane where its values are actually determined by f. What region is that?

5. Prove the weak maximum principle for harmonic functions: If Ω is a *bounded* nonempty open set in $\mathbf{R}^n$, and if u is harmonic on Ω and continuous on $\overline{\Omega}$, then the maximum of u on $\overline{\Omega}$ (which exists by the extreme value theorem) is attained on the boundary $\partial\Omega$.

   (Reminder: It's useful to consider the function $v(\mathbf{x}) = u(\mathbf{x}) + \varepsilon\,|\mathbf{x}|^2$ for $\varepsilon > 0$.)

6. For the inviscid Burgers equation $u_t + uu_x = 0$, find a weak solution $u(x,t)$ for $t>0$ with the initial condition

$$u(x,0) = \begin{cases} 0, & x \le 0, \\ x, & 0 < x < 1, \\ 0, & 1 < x. \end{cases}$$

   (Recall the Rankine–Hugoniot condition for constructing a weak solution by gluing strong solutions u^L and u^R along a curve $x = g(t)$: the shock velocity $g'(t)$ must equal the average of u^L and u^R on the curve.)

Solutions for TATA27 2025-08-18

1. By the mean value property for harmonic functions, $u(0,0)$ is the average of the known values $u(x,y) = x^2$ on the circle $(x,y) = (2\cos\varphi, 2\sin\varphi)$:

$$u(0,0) = \frac{1}{2\pi}\int_0^{2\pi} (2\cos\varphi)^2\,d\varphi = \frac{1}{\pi}\int_0^{2\pi} (1+\cos 2\varphi)\,d\varphi = 2.$$

   (We can also verify this by seeking the solution on the form $u(x,y) = x^2 + c(4-x^2-y^2)$, $c \in \mathbf{R}$, which satisfies the boundary condition by construction and is harmonic iff $c = 1/2$. Thus $u(x,y) = 2 + \frac{1}{2}(x^2 - y^2)$, so $u(0,0) = 2$.)

   Answer. $u(0,0) = 2$.

2. A function with the sought properties can be found by inspection, and theory says that the solution is unique.

   Answer. $u(x,t) = \sin 2x \cos 2ct + \frac{1}{3c}\sin 3x \sin 3ct$.

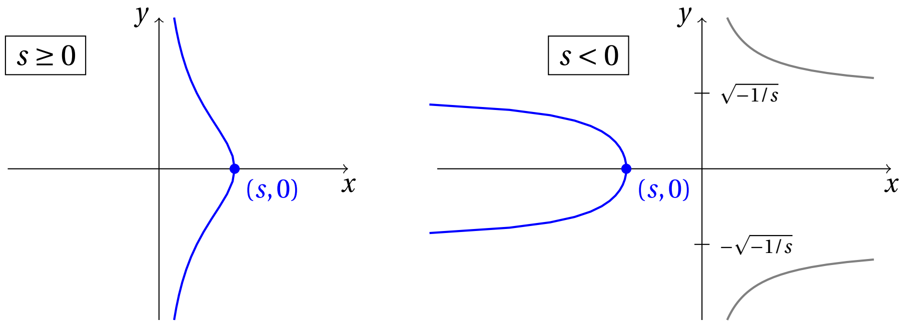
3. (a) Using that $u_t = \Delta u$ and that $u = 0$ for $\mathbf{x} \in \partial\Omega$, we have

$$\begin{aligned} E'(t) &= \tfrac{d}{dt}\int_\Omega \tfrac{1}{2}u^2\,dV = \int_\Omega \tfrac{1}{2}\tfrac{\partial}{\partial t}(u^2)\,dV = \int_\Omega u u_t\,dV = \int_\Omega u\Delta u\,dV \\ &= \int_\Omega \big(\nabla\cdot(u\nabla u) - \nabla u\cdot\nabla u\big)\,dV = \underbrace{\int_{\partial\Omega} u\nabla u\cdot\mathbf{n}\,dS}_{=0} - \underbrace{\int_\Omega |\nabla u|^2\,dV}_{\le 0} \le 0 \end{aligned}$$

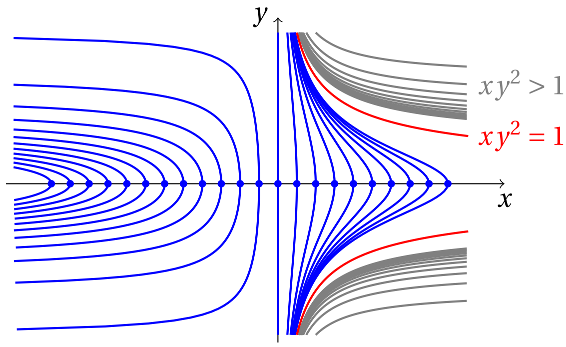
   for all $t \in I$, which implies that E is nonincreasing.

   (b) If u_1 and u_2 both solve the same initial–boundary value problem, then $u = u_1 - u_2$ is like in part (a) with $I = \mathbf{R}_+$, so that $E(t)$ is nondecreasing for $t > 0$, and moreover $u = 0$ at $t = 0$, so that $E(0) = 0$. As usual when talking about solutions to initial–boundary value problems, it is assumed that u_1 and u_2 are continuous on $\overline{\Omega} \times [0,\infty)$, and therefore E is continuous on $[0,\infty)$. Since $E(t)$ is nonnegative (being the integral of $\frac{1}{2}u^2$), it follows that $E(t) = 0$ for all $t \ge 0$, which implies that u is identically zero, so that $u_1 = u_2$.

4. For a fixed $s \in \mathbf{R}$, the characteristic curve $\big(x(t), y(t)\big)$ starting at the point $(s,0)$ is given by the ODEs $\dot{x} = 2x^2 y$ and $\dot{y} = -1$ with initial conditions $x(0) = s$ and $y(0) = 0$. This is readily solved (but note that the case $s = 0$ may need separate treatment) to give $x(t) = \frac{s}{1+st^2}$ and $y(t) = -t$. Note that for $s \ge 0$ the parameter t can take any real value, but if $s < 0$ then we only have $|t| < \sqrt{-1/s}$; values with $|t| > \sqrt{-1/s}$ parametrize two characteristic curves which are not connected to our starting point $(s,0)$:



For a given point $(x, y) \in \mathbf{R}^2$, we may invert the relations $(x, y) = \big(\frac{s}{1+st^2}, -t\big)$ to find $t = -y$ and $s = \frac{x}{1-xy^2}$, provided that $xy^2 \neq 1$. Thus, the point (x, y) lies on a characteristic curve corresponding to the value $s = \frac{x}{1-xy^2}$, where the value of u is (locally) constant since the right-hand side of the PDE is zero. However, it is only for $xy^2 < 1$ that this characteristic curve is actually connected to $(s,0)$, where we know the value $u = f(s) = f\big(\frac{x}{1-xy^2}\big)$:



Answer. $u(x, y) = f\big(\frac{x}{1-xy^2}\big)$ for $xy^2 < 1$.

(Whether or not this can be continued into the region $xy^2 \ge 1$ to form a global solution $u \in C^1(\mathbf{R}^2)$ depends on how $f(x)$ behaves as $x \to +\infty$, but that was outside the scope of the question.)

5. See the course materials.

6. We want to construct a weak solution by gluing the "left" and "right" strong solutions $u^L(x,t) = u^R(x,t) = 0$ to the "middle" strong solution $u^M(x,t) = \frac{x}{t+1}$, which can be found from the initial values $u^M(x,0) = x$ by thinking of the solution as consisting of "particles" with the property that a particle at elevation h travels with velocity h; indeed, a particle starting out at $(x,u) = (h,h)$ when $t = 0$ will be at $(x,u) = (h + th, h)$ at time $t > 0$, so that the slope of the line has changed from 1 to $\frac{1}{t+1}$.

   We can immediately glue x^L and x^M (continuously, i.e., without a shock) along the line $x = 0$. For gluing x^M and x^R, we seek a curve $x = g(t) > 0$, with $g(0) = 1$, such that the Rankine–Hugoniot condition

$$g'(t) = \frac{x^M\big(g(t),t\big) + x^R\big(g(t),t\big)}{2} = \frac{\frac{g(t)}{t+1} + 0}{2}$$

   holds for $t > 0$. Since $g(t) > 0$, we can write this as $\frac{g'(t)}{g(t)} = \frac{1/2}{t+1}$ and integrate (using $g(0) = 1$) to obtain $\ln g(t) = \frac{1}{2}\ln(t+1)$, i.e., $g(t) = \sqrt{t+1}$.

   Answer. For $t > 0$, a weak solution is

$$u(x,t) = \begin{cases} 0, & x \le 0, \\ \frac{x}{t+1}, & 0 < x < \sqrt{t+1}, \\ 0, & \sqrt{t+1} < x. \end{cases}$$