LINKÖPING UNIVERSITY
Department of Mathematics
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Hand-in exercises for TATA34 Real Analysis, Honours Course, 2023

I will send out information by personal email during the course, so please send me an email to `anders.bjorn@liu.se` **with your name, preferred email address, programme and year.** Let me know if you decide to drop the course, and I won't bother you with further emails.

The examination of this course consists of six rounds of hand-in exercises as given below.

The exercises should be **solved individually** and you have to be prepared to demonstrate your solutions at the board during the exercise sessions held after each round (but for the last one).

Each exercise can give one point. For grade 3 it is enough to get 5 points in each round. For grades 4 and 5 it is enough with 6 resp. 7 points in each round, provided that among them there are at least 6 resp. 12 *-marked points in total.

If you are very close to fulfilling these requirements I will take an overall look after all six rounds, and perhaps give you (individually) some extra exercise(s) to solve in order for you to obtain a certain grade. This mainly applies for passing the course.

In addition, for passing the course you are **also required** to be present and demonstrate solutions at the board during at least three exercise sessions.

Instructions for the exercises

- For deadlines see the lecture plan. If for some reason you are not able to hand in the solutions by the deadline, you should notify **the examiner** as early as possible.
- If you need to to do a maxmin investigation in some problem, then please provide the details as required in the first year single-variable analysis course TATA41.
- Swedish-speaking students are encouraged to write their solutions in Swedish.
- You are allowed to discuss the problems with the other students taking the course, but you are **not** allowed to copy anyone else's solution. **You must formulate your own solutions, and understand them.**
- A good idea is to first solve a problem on **scrap paper**, and later write it down more carefully on another piece of paper.
- You should put an effort into writing **clearly** and **legibly**, and to carefully **justify** your arguments.
- You may of course write your solutions using **LaTeX** (or some similar program), but this is not at all necessary.
- The exercises should be **numbered** according to the list below, and **sorted** according to the **numbering**.
- There should be **space for my comments** on each page. You may solve several exercises on the same page, and use both sides of the paper, but there should always be space between the solutions and in the margins.

Continued on the next page.

- Each round should have a **cover page** on which you should write your name, programme and year. In addition you should tick off which exercises you have solved in a table of the following form: Mark solved exercises with X and unsolved with −. (The third line is meant for me to mark how many points you get on each exercise.)

1	2	3	4	5	6	7	8	9	10	11
X	X	−	X	X	X	−	−	X	X	X

- **Staple** the cover page together with your solutions in the upper left corner. (And make sure that no text is hidden on later pages, not even any exercise numbers.)
- The solutions should be **handed** into the course's mailbox, B-house, top floor, near entrance 21.
- Don't be afraid of the *-marked exercises, some of them are not so difficult.
- In all exercises (with several parts) it is ok to use the result in one part to show a later part, even if you haven't solved the first part, unless said otherwise.

Round 1

In this round all the involved numbers and sets are within $\mathbf{R}$*, unless explicitly said otherwise.*

In all exercises it is ok to use the result in one part to show a later part, even if you haven't solved the first part, unless said otherwise. This applies to all later rounds as well.

1. Let $A_1 \supset A_2 \supset A_3 \supset ....$ Decide if each of the following statements is true. If not give a counterexample. No proof needed if a statement is true. (Note that, in contrast to Abbott, when I write $A \subset B$ it is allowed that $A = B$.)
   (a) If each A_n is infinite (i.e. contains an infinite number of points), then $\bigcap_{n=1}^{\infty} A_n$ is infinite.
   (b) If each A_n is finite and nonempty, then $\bigcap_{n=1}^{\infty} A_n$ is nonempty.
2. Let $|x| = (\sum_{j=1}^{n} x_j^2)^{1/2}$ for $x = (x_1, \dots, x_n) \in \mathbf{R}^n$. Prove that $\big||a| - |b|\big| \leq |a - b|$ for $a, b \in \mathbf{R}^n$. *Hint*: Use the triangle inequality. You can get half credit if you only prove it for $n = 1$.
3. Let $x_1 = 1$ and define $x_{n+1} = (2x_n + 5)/3$ for $n = 1, 2, \dots$.
   (a) Use induction to prove that $x_n < 5$ for each n.
   (b) Show that the sequence is strictly increasing.
   (c) It follows from the monotone convergence theorem (MCT, SOMK) that the sequence has a limit $x = \lim_{n\to\infty} x_n$. Determine this limit.
4. Compute, without proofs, the suprema and infima of

$$A = \left\{ \frac{n}{5n+2} : n = 1, 2, ... \right\} \quad \text{and} \quad B = \left\{ \frac{m}{2m+n} : m, n = 1, 2, ... \right\}.$$

   Is $\inf A \in A$?, $\sup A \in A$?, $\inf B \in B$?, $\sup B \in B$?
5. Let $A, B \subset \mathbf{R}$ be nonempty bounded sets. Decide if each of the following statements is true. If so give a short proof and otherwise a counterexample.
   (a) If $\sup A < \inf B$, then there exists $c \in \mathbf{R}$ such that $a < c < b$ for all $a \in A$ and $b \in B$.
   (b) If $c \in \mathbf{R}$ is such that $a < c < b$ for all $a \in A$ and $b \in B$, then $\sup A < \inf B$.

*6. Let $\mathbf{I} = \mathbf{R} \setminus \mathbf{Q}$ be the set of irrational numbers.
   (a) Show that if $a, b \in \mathbf{Q}$, then $a + b, ab \in \mathbf{Q}$.
   (b) Show that if $a \in \mathbf{Q} \setminus \{0\}$ and $b \in \mathbf{I}$, then $a + b, ab \in \mathbf{I}$.
   (c) What happens if $a, b \in \mathbf{I}$? Is it possible to have $a + b \in \mathbf{Q}$? How about $a + b \in \mathbf{I}$? How about $ab \in \mathbf{Q}$? How about $ab \in \mathbf{I}$? (For each question, give an example or prove none exists.)

7. Prove that the following numbers are irrational.
   (a) $\sqrt{7}$, (b) $\sqrt{10}$, (c) $\sqrt{2} + \sqrt{5}$.
8. (a) Show that $\sqrt[3]{5}$ and $\sqrt{2} + \sqrt{5}$ are algebraic.
   (b) Show that if $a \in \mathbf{A}$, $a \neq 0$, (i.e. a is algebraic) then $-a, 1/a \in \mathbf{A}$. (You are not allowed to use that $\mathbf{A}$ is a field.)

There are further questions on the next page.

*9. (a) Show that $(0,1) \sim \mathbf{R}$ (where $(0,1) = \{x : 0 < x < 1\}$).
(b) Show that $[0,1] \sim \mathbf{R}$ (where $[0,1] = \{x : 0 \leq x \leq 1\}$).
(c) Show that $(0,1) \times (0,1) \sim \mathbf{R}^2$.
(d) Show that $(0,1) \preceq (0,1) \times (0,1)$.
(e) Show that $(0,1) \times (0,1) \preceq (0,1)$. *Hint*: Use a mapping which maps $(x, y) \in (0,1) \times (0,1)$, where $x = 0.x_1x_2x_3...$ and $y = 0.y_1y_2y_3...$, to $z = 0.x_1y_1x_2y_2x_3y_3...$. Make sure your mapping is well-defined.
(f) Show that $(0,1) \times (0,1) \sim (0,1)$.
(g) Show that $\mathbf{R}^2 \sim \mathbf{R}$.

*10. Show that the set consisting of all finite subsets of $\mathbf{N}$ is countable.

*11. Let $a_n = 1/n$. Then we know that $a_n \to 0$, but is this completely obvious. In this exercise we'll look at how this follows from our axioms. (You don't have to show that the sequence is decreasing.)
(a) Let $i = \inf\{a_1, a_2, ...\}$, which exists by the supremum axiom (SA, Axiom of Completness). As 0 is a lower bound for the set, we see that $i \geq 0$. Show that the archimedean property (AP, AE) $\Rightarrow i = 0$. (Hence we know that SA $\Rightarrow$ AP $\Rightarrow$ $i = 0$.) *Hint*: Consider the contrapositive implication.
(b) Let $a = \lim_{n\to\infty} a_n$, which exists by the monotone convergence theorem (MCT, SOMK). By the order limit theorem, $a \geq 0$. Show that $a = 0$ without using AP. *Hint*: Assume $a > 0$, then we may use $\varepsilon = a$ in the definition of limits.

Round 2 *In this round, all the involved numbers and sets are within* $\mathbf{R}$.

1. Verify, using the (ε,N)-definition of limit, that $\lim\limits_{n\to\infty} \dfrac{3n+2}{5n+3} = \dfrac{3}{5}$.
2. Let $x_n \geq 0$.
   (a) Show that if $x_n \to 0$, then $\sqrt{x_n} \to 0$.
   (b) Show that if $x_n \to x$, then $\sqrt{x_n} \to \sqrt{x}$.
3. Let $(x_n)_{n=1}^{\infty}$ be a sequence such that $x_n \to a$ and $x_n \to b$, as $n \to \infty$. Show that $a = b$.
4. What happens if we reverse the order of the (first two) quantifiers in the definition of limits? We get the following (nonstandard) definition.
   Definition: A sequence $(x_n)_{n=1}^{\infty}$ *conges* to x if $\exists \varepsilon > 0\ \forall N \in \mathbf{N}\ \forall n \geq N\ |x_n - x| < \varepsilon$.
   (a) Give an example of a congent sequence (and state what it conges to).
   (b) Does there exist a congent sequence which is divergent?
   (c) Can a sequence cong to two different values?
   (d) Exactly what is described by this weird definition? (It is enough to give a description.)
5. Let
$$x_1 = 5 \qquad \text{and} \qquad x_{n+1} = \frac{1}{2}\left(x_n + \frac{3}{x_n}\right), \quad n = 1, 2, \ldots.$$
   Show that $x_n \to \sqrt{3}$. *Hint*: Prove that $x_n > \sqrt{3}$ for each n, and then that $(x_n)_{n=1}^{\infty}$ is decreasing.

*6. Let $(a_n)_{n=1}^{\infty}$ be a bounded sequence and define the *limit superior*
$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} x_n, \quad \text{where } x_n = \sup\{a_k : k \geq n\}.$$
   (a) Show that $(x_n)_{n=1}^{\infty}$ converges.
   (b) Provide a reasonable definition for $\liminf_{n\to\infty} a_n$.
   (c) Prove that $\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n$, and show with an example that the inequality can be strict.
   (d) Show that $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = a$ if and only if $\lim_{n\to\infty} a_n = a$.

7. Let (a_n) and (b_n) be Cauchy sequences, and let $c_n = |a_n - b_n|$. Is (c_n) always a Cauchy sequence? Show this or give a counterexample.
8. (a) Show that if $\sum_{n=1}^{\infty} a_n$ is absolutely convergent then also $\sum_{n=1}^{\infty} a_n^2$ is absolutely convergent.
   (b) Is it true that if $\sum_{n=1}^{\infty} a_n$ is convergent, then also $\sum_{n=1}^{\infty} a_n^2$ is convergent? Give a proof or a counterexample.
   (c) Assume that both $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ diverge. Can it happen that $\sum_{n=1}^{\infty} x_n y_n$ converges? Give an example or prove that none exist.

There are further questions on the next page.

*9. The *Leibniz' criterium* says that if $(a_n)_{n=1}^{\infty}$ is a decreasing sequence tending to 0, then

$$\sum_{n=1}^{\infty}(-1)^{n-1}a_n$$

converges. Prove this in the following three ways:

(a) Prove that the partial sums

$$s_n = \sum_{k=1}^{n}(-1)^{k-1}a_k$$

form a Cauchy sequence, and use this to prove Leibniz' criterium.

(b) Use Cantor's encapsulation theorem for intervals to prove Leibniz' criterium

(c) Show that the subsequences $(s_{2n})_{n=1}^{\infty}$ and $(s_{2n+1})_{n=1}^{\infty}$ are monotone, and use this to prove Leibniz' criterium.

10. Assume that $\sum_{k=1}^{\infty} a_k = A \in \mathbf{R}$ and $\sum_{k=1}^{\infty} b_k = B \in \mathbf{R}$. If one wants to sum the double series $\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} a_k b_j$ it may depend on the order of summation. If we sum over squares this leads to $\lim_{n\to\infty}\sum_{k=1}^{n}\sum_{j=1}^{n} a_k b_j$, while if we sum over triangles we obtain

$$\lim_{n\to\infty}\sum_{k=1}^{n} c_k, \quad \text{where } c_k = \sum_{j=1}^{k-1} a_j b_{k-j}.$$

(a) Show that if we sum over squares we always get

$$\lim_{n\to\infty}\sum_{k=1}^{n}\sum_{j=1}^{n} a_k b_j = AB.$$

(b) Show, by giving an example, that it can happen that the sum over triangles $\sum_{k=1}^{\infty} c_k$ diverges. *Hint*: Consider $a_k = b_k = (-1)^k/\sqrt{k}$. If you make other choices you need to show that $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ are convergent.

*11. (See above for the definition of lim sup.)

(a) Show that $\sum_{k=1}^{\infty} a_k$ is absolutely convergent if $\limsup_{k\to\infty} |a_k|^{1/k} < 1$.

(b) Show that $\sum_{k=1}^{\infty} a_k$ is divergent if $\limsup_{k\to\infty} |a_k|^{1/k} > 1$.

(c) Give an example where $\limsup_{k\to\infty} |a_k|^{1/k} = 1$ and $\sum_{k=1}^{\infty} a_k$ is absolutely convergent.

(d) Give an example where $\limsup_{k\to\infty} |a_k|^{1/k} = 1$ and $\sum_{k=1}^{\infty} a_k$ is conditionally convergent.

(e) Give an example where $\limsup_{k\to\infty} |a_k|^{1/k} = 1$ and $\sum_{k=1}^{\infty} a_k$ is divergent.

Round 3

1. Let
$$a_n = (-1)^n + \frac{2}{n}, \quad n = 1, 2, 3, \dots, \qquad \text{and} \qquad A = \{a_n : n = 1, 2, 3, \dots\}.$$
   (a) Calculate $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$. *Only answers required in* (a), (b), (e) *and* (f).
   (b) Determine the limit points of A.
   (c) Is A open? Why?
   (d) Is A closed? Why?
   (e) Has A any isolated points? If so which?
   (f) Determine $\overline{A}$.
2. For each of the following sets determine if it is open, closed or neither (with respect to $\mathbf{R}$). If a set isn't open, find a point in it without a neighbourhood within the set. If a set isn't closed, find a limit point which isn't in the set. No further justification has to be given.
   (a) $\mathbf{Q}$, (b) $\mathbf{N}$, (c) $\{x \in \mathbf{R} : x \neq 0\}$,
   (d) $\{1 + 1/2 + 1/3 + \dots + 1/n : n = 1, 2, \dots\}$, (e) $\{1 + 1/4 + 1/9 + \dots + 1/n^2 : n = 1, 2, \dots\}$.
3. (a) Show that $\overline{A \cup B} = \overline{A} \cup \overline{B}$. *In both parts you may assume that the sets are subsets of* $\mathbf{R}$.
   (b) Is the corresponding identity true for infinite unions? Give a proof or a counterexample.
4. Decide which of the following statements are true. For each one give a proof or a counterexample.
   (a) If $A \subset \mathbf{R}$ is open and $\mathbf{Q} \subset A$, then $A = \mathbf{R}$.
   (b) Every nonempty open set $A \subset \mathbf{R}$ contains a rational number.
   (c) Every infinite closed set $A \subset [0, 1]$ contains a rational number.
   (d) Every infinite countable set $A \subset [0, 1]$ has a limit point.
   (e) Every infinite countable set $A \subset [0, 1]$ has an isolated point.
5. Which of the following sets are compact? For the noncompact ones, show why it isn't sequentially compact, i.e. give an example of a sequence in the set without convergent subsequence with limit in the set.
   (a) $\mathbf{N}$, (b) $\mathbf{Q} \cap [0, 1]$, (c) $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$, (d) $\{1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\}$,
   (e) $\{1 + 1/4 + 1/9 + \dots + 1/n^2 : n = 1, 2, \dots\}$.
6. For the sets which were noncompact in the previous exercise, find an open cover without finite subcover.

*7. Let X be a metric space. Let $a \in X$, and let $(a_n)_{n=1}^{\infty}$ be a sequence in X such that every subsequence has a subsubsequence converging to a. Show that $\lim_{n\to\infty} a_n = a$.
   Hint: Assume that $a_n \not\to a$. You can get 0.7p if you only consider this for $X = \mathbf{R}$.

   There are further questions on the next page.

8. Decide which of the following statements are true. For each one either give a proof or a counterexample. For full credit you should consider if the statement is true in arbitrary metric spaces, but partial credit will be given if you only consider subsets of $\mathbf{R}$.
   (a) An arbitrary intersection of compact sets is compact.
   (b) An arbitrary union of compact sets is compact.
   (c) If A is arbitrary and K compact, then $A \cap K$ is compact.
   (d) If $F_1 \supset F_2 \supset F_3 \supset \ldots$ is a decreasing sequence of nonempty closed sets, then $\bigcap_{n=1}^{\infty} F_n \neq \varnothing$.
   (e) A finite set is always compact.
   (f) A countable set is always compact.

*9. Let (X, d_X) and (Y, d_Y) be nonempty metric spaces. Let

$$d_{X \times Y}((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2).$$

   (a) Show that $(X \times Y, d_{X \times Y})$ is a metric space, i.e. that $d_{X \times Y}$ is a metric on $X \times Y$.
   (b) Show that $X \times Y$ is complete if and only if both X and Y are complete.

10. A set is *clact* if every cover by closed sets has a finite subcover. (This is a nonstandard definition.) Describe all clact subsets of $\mathbf{R}$. Prove that your description is correct.

*11. Repeat the Cantor construction starting with the interval $A_0 = B_0 = [0, 1]$. This time, however, remove the open middle quarter of each interval in A_{n-1} when constructing A_n, and the open middle interval of length 4^{-n} of each interval in B_{n-1} when constructing B_n. The set $A = \bigcap_{n=0}^{\infty} A_n$ is a selfsimilar Cantor set, whereas $B = \bigcap_{n=0}^{\infty} B_n$ is a nonselfsimilar Cantor set.
   (a) Draw A_1, A_2, B_1 and B_2, with explicit coordinates using fractions. $A_1 = B_1$ but how can you compare A_2 and B_2? (Is $A_2 = B_2$, $A_2 \subset B_2$, $A_2 \supset B_2$ or none of these?)
   (b) Draw the first two intervals (closest to 0) of A_3 and B_3, with explicit coordinates using fractions. How can you compare A_3 and B_3?
   (c) Draw the first two intervals (closest to 0) of A_4 and B_4, with explicit coordinates using fractions. How can you compare A_4 and B_4?
   (d) Are A and B compact? Perfect? Do they have any interior points?
   (e) What is the total length of the removed intervals when constructing A resp. B?
   (f) What is the length (i.e. outer measure) of A resp. B? What are their dimensions? (We haven't all the tools available for this, but give your best guesses, and try to explain them.)

Round 4

1. Use the (ε,δ)-definition of limit to show:
   (a) $\lim_{x\to 1} x^4 - 1 = 0$,
   (b) $\lim_{x\to 2} x^2 + x + 1 = 7$.
2. Prove or disprove that the following functions are uniformly continuous:
   (a) e^x on $(0,1)$, (b) e^x on $(1,\infty)$, (c) $\ln x$ on $(0,1)$, (d) $\ln x$ on $(1,\infty)$.
3. Define for each of the following sets A a function $f : \mathbf{R} \to \mathbf{R}$ which is discontinuous at every point in A but continuous at every point in A^c: (It is enough if you define them.)
   (a) $A = \mathbf{Z}$, (b) $A = [0,1]$, (c) $A = (0,1)$, (d) $A = \{1, \frac{1}{2}, \frac{1}{3}, ...\}$.

*4. (a) Prove that Dirichlet's function $\chi_{\mathbf{Q}}$ isn't continuous at any point. Here

$$\chi_{\mathbf{Q}}(x) = \begin{cases} 1, & \text{if } x \in \mathbf{Q}, \\ 0, & \text{otherwise.} \end{cases}$$

   (b) Show that Thomae's function t, see Section 4.1, isn't continuous at any rational point.
   (c) Show that t is continuous at every irrational point. *Hint*: Study the set $\{x \in \mathbf{R} : t(x) \geq \varepsilon\}$.

5. Let X and Y be metric spaces and $f : X \to Y$ be continuous. Show that $f^{-1}(F)$ is closed if F is closed. You can get half credit if you only consider continuous functions $f : \mathbf{R} \to \mathbf{R}$.
6. Let $f : \mathbf{R} \to \mathbf{R}$ be a function satisfying

$$f(x+y) = f(x) + f(y) \quad \text{for all } x, y \in \mathbf{R}.$$

   (a) Show that $f(0) = 0$ and that $f(-x) = -f(x)$ for all $x \in \mathbf{R}$.
   (b) Let $k = f(1)$. Show that $f(x) = kx$ for all $x \in \mathbf{N}$; for all $x \in \mathbf{Z}$; and for all $x \in \mathbf{Q}$.
   (c) Show that if f is continuous at 0, then f is continuous at every point in $\mathbf{R}$ and $f(x) = kx$ for all $x \in \mathbf{R}$. (That is, every additive function which is continuous at 0 is a linear function.)
7. Give for each part an example of such an f and a Cauchy sequence (x_n) such that $(f(x_n))$ isn't a Cauchy sequence, or explain why this is impossible to fulfill.
   (a) With a continuous $f : (0,1) \to \mathbf{R}$.
   (b) With a continuous $f : [0,1] \to \mathbf{R}$.
   (c) With a continuous $f : [0,\infty) \to \mathbf{R}$.
   (d) With a uniformly continuous $f : (0,1) \to \mathbf{R}$.
   (e) With a uniformly continuous $f : [0,1] \to \mathbf{R}$.
   (f) With a uniformly continuous $f : [0,\infty) \to \mathbf{R}$.

There are further questions on the next page.

*8. A function $f : A \to \mathbf{R}$ is *Lipschitz* with Lipschitz constant M if $|f(x)-f(y)| \le M|x-y|$ for all $x, y \in A$.

(a) Show that if f is differentiable on a compact interval $[a, b]$ and moreover f' is continuous on $[a, b]$, then f is Lipschitz on $[a, b]$. (At the end points a and b the derivative is defined using one-sided limits.)

(b) What happens in (a) if we replace $[a, b]$ by $\mathbf{R}$? (Give proof or counterexample.)

(c) Show that if f is Lipschitz on $(0, 1]$, then f has a continuous extension $g : [0, 1] \to \mathbf{R}$, i.e. such that $g(x) = f(x)$ for $x \in (0, 1]$.

(d) Is g in (c) always Lipschitz on $[0, 1]$? (Give proof or counterexample.)

9. Show that a metric space X is connected if and only if X and $\varnothing$ are the only sets which are both open and closed.

*10. Let (X, d_X) be a nonempty metric space, and $I = [0, 1]$. Let

$$d_{X\times I}((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + |y_1 - y_2|.$$

(a) Show that if X is pathconnected then $X \times I$ is pathconnected.

(b) Show that if $X \times I$ is pathconnected then X is pathconnected.

(c) Show that if X is connected then $X \times I$ is connected.

(d) Show that if $X \times I$ is connected then X is connected.

Hint: The projections $P_X : X \times I \to X$, $P_X(x, t) = x$, and $P_I : X \times I \to I$, $P_I(x, t) = t$, which are clearly Lipschitz (you don't need to prove that) and the fibers $X \times \{t\} = \{(x, t) : x \in X\}$, $t \in I$, and $\{x\} \times I$, $x \in X$, may be of some help.

*11. Let X be a metric space. Show that if a set $E \subset X$ is connected, then so is $\overline{E}$? You can get 0.7p if you only consider this for $X = \mathbf{R}$.

Round 5

If you need to to do a maxmin investigation in some problem, then please provide the details as required in the first year single-variable analysis course TATA41.

You are **NOT** *allowed to use fixed point theorems that have not been* **proved** *at the lectures.*

1. For each of the following sets E show that the statement "if $f : E \to E$ is continuous, then f has a fixed point" is true, or give a counterexample if it is false.
   (a) $E = [0,1]$, (b) $E = (0,1)$, (c) $E = \mathbf{R}$, (d) $E = [0,1] \cup [2,3]$.
2. Let $f : [1,4] \to \mathbf{R}$ be a differentiable function such that $f(1) = 0$, $f(2) = 1$ and $f(4) = 8$. Show that (make sure that you give careful references to the key theorems you use)
   (a) f has a fixed point,
   (b) there is some x such that $f'(x) = 1$,
   (c) there is some x such that $f'(x) = 2$.
3. Let
$$f_n(x) = \frac{nx}{1 + nx^3}, \quad x \geq 0, \ n = 1, 2, \dots, \qquad \text{and} \qquad f(x) = \lim_{n\to\infty} f_n(x).$$
   (a) Determine f.
   (b) Is the convergence uniform on $(0,1)$? Show this or show that it isn't.
   (c) Is the convergence uniform on $(1,\infty)$? Show this or show that it isn't.
4. Let for $x \in [0,\infty)$,
$$f_n(x) = \frac{x^2}{1 + x^n}, \quad n = 1, 2, \dots, \qquad \text{and} \qquad f(x) = \lim_{n\to\infty} f_n(x).$$
   (a) Determine f.
   (b) Explain how we know that f_n does not tend to f uniformly on $[0,\infty)$.
   (c) On which closed subintervals I of $[0,\infty)$ does $f_n \to f$ uniformly? Prove that the convergence is indeed uniform on each of these subintervals.
5. Let (r_n) be a sequence of positive numbers such that $\lim_{n\to\infty} r_n = 0$. Let $f : \mathbf{R} \to \mathbf{R}$ be continuous and define $f_n(x) = f(x + r_n)$.
   (a) Show that $f_n \to f$ pointwise.
   (b) Show that if f is uniformly continuous, then $f_n \to f$ uniformly.
   (c) Give an example of a continuous f such that f_n doesn't tend to f uniformly (and show this).

*6. We can construct the Cantor function f (a.k.a. the Devil's staircase) in the following way. Let $f_0(x) = x$ for $0 \leq x \leq 1$. Define recursively, for $n = 1, 2, \dots$,
$$f_n(x) = \begin{cases} \frac{1}{2} f_{n-1}(3x), & 0 \leq x < \frac{1}{3}, \\ \frac{1}{2}, & \frac{1}{3} \leq x \leq \frac{2}{3}, \\ \frac{1}{2} + \frac{1}{2} f_{n-1}(3x-2), & \frac{2}{3} < x \leq 1. \end{cases}$$
   Let $f(x) = \lim_{n\to\infty} f_n(x)$.
   (a) Draw the graphs of f_0, f_1 and f_2. (You may draw them all in one diagram.)
   (b) Show that $f_n \to f$ uniformly.
   (c) Show that f is continuous and increasing and that $f(0) = 0$ and $f(1) = 1$.
   (d) Show that $f'(x) = 0$ for all $x \in [0,1] \setminus C$, where C is the (standard ternary) Cantor set.

There are further questions on the next page.

7. Let
$$f_n(x) = \frac{x}{1+nx^4}, \qquad n = 1,2,\dots, \qquad \text{and} \qquad f(x) = \lim_{n\to\infty} f_n(x).$$
   (a) Calculate f_n'.
   (b) For each n find the maximum and minimum values of f_n on $\mathbf{R}$.
   (c) Use this to show that (f_n) converges uniformly on $\mathbf{R}$.
   (d) Determine f.
   (e) Determine for which x, $f'(x) = \lim_{n\to\infty} f_n'(x)$.
8. (Make sure that you give careful references to the key theorems you use.) Let
$$f(x) = \sum_{n=1}^{\infty} \frac{1}{x^4+n^4}.$$
   (a) Show that f is a continuous function defined on $\mathbf{R}$.
   (b) Show that f' exists and is continuous.
   (c) Does f'' exist?, and if so is it continuous? Show these facts or explain why they fail.

(*)9. *Part* (b) *is *-marked and can give* 0.5 *points.*
   Let $M > 0$ and let $f_n : [0,1] \to \mathbf{R}$ be a sequence of Lipschitz functions with common Lipschitz constant M, i.e. $|f_n(x) - f_n(y)| \le M|x-y|$. Assume that $f_n \to f$ pointwise, where $f : [0,1] \to \mathbf{R}$.
   (a) Show that f is Lipschitz continuous.
   *(b) Show that it follows that $f_n \to f$ uniformly. *Hint*: Use Arzelà–Ascoli's theorem together with an exercise from an earlier round.

*10. Let $I \subset \mathbf{R}$ be an interval. A function $f : I \to \mathbf{R}$ is *lower semicontinuous* (nedåt halvkontinuerlig) if $f(x) \le \liminf_{y\to x} f(y)$ for all $x \in I$, where
$$\liminf_{y\to x} f(y) = \lim_{r\to 0+} \inf\{f(y) : y \in (I \cap (x-r, x+r)) \setminus \{x\}\}.$$
   Equivalently, f is lower semicontinuous if $f^{-1}((z,\infty))$ is relatively open in I for every $z \in \mathbf{R}$ (you don't need to show this equivalence). Next, let $f_n : [0,1] \to [0,1]$ be an increasing sequence (i.e. $f_{n+1} \ge f_n$) of continuous function, and let $f(x) = \lim_{n\to\infty} f_n(x)$.
   (a) Show that f is lower semicontinuous using the $\liminf$-definition.
   (b) Show that $f^{-1}((z,\infty)) = \bigcup_{n=1}^{\infty} f_n^{-1}((z,\infty))$ if $z \in \mathbf{R}$.
   (c) Use (b) to show that f is lower semicontinuous using the f^{-1}-definition.
   (d) Give an example showing that f doesn't have to be continuous.

*11. Let $\{r_1, r_2, r_3, \dots\}$ be an enumeration of $\mathbf{Q}$. Let
$$f_n(x) = \begin{cases} 1/n^2, & \text{if } x > r_n, \\ 0, & \text{if } x \le r_n, \end{cases} \qquad \text{and} \qquad f(x) = \sum_{n=1}^{\infty} f_n(x).$$
   (a) Show that f is strictly increasing.
   (b) Show that f is continuous at every $x \in \mathbf{R} \setminus \mathbf{Q}$.
   (c) Show that f is discontinuous at every $x \in \mathbf{Q}$.
   (d) Show that f is lower semicontinuous (see the previous exercise).

Round 6 *You need to justify all your answers.*

1. The series

$$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$$

converges for every $x \in [-1, 1)$ but not for $x = 1$. For a fixed $x_0 \in (-1, 1)$ explain how it is still possible to use the Weierstrass M-test to show that f is continuous at x_0.

2. Determine coefficients a_n so that the power series $\sum_{n=1}^{\infty} a_n x^n$ has the following properties, if it is possible. Explain why the property is satisfied, or why it is impossible to satisfy.
   (a) Converges absolutely for all $x \in [-1, 1]$ and diverges elsewhere.
   (b) Converges conditionally at $x = 1$ and diverges at $x = -1$.
   (c) Converges conditionally both at $x = -1$ and $x = 1$.
   (d) Converges conditionally at $x = -1$ and absolutely at $x = 1$.

3. Define

$$g(x) = x - \frac{(2x)^4}{4} + \frac{(2x)^7}{7} - \frac{(2x)^{10}}{10} + \frac{(2x)^{13}}{13} - \dots .$$

   (a) Where is g defined?
   (b) Where is g continuous?
   (c) Where is g differentiable?
   (d) Give an (as simple as possible) expression for g' where it exists.

*4. Assume that $\sum_{k=1}^{\infty} a_k = A \in \mathbf{R}$ and $\sum_{k=1}^{\infty} b_k = B \in \mathbf{R}$, with partial sums $r_n = \sum_{k=1}^{n} a_k$ and $s_n = \sum_{k=1}^{n} b_k$. If one wants to sum the double series $\sum_{k=1}^{\infty} \sum_{l=1}^{\infty} a_k b_l$ the sum may depend on the order of the summation. If we sum over squares we obtain

$$\lim_{n\to\infty} \sum_{k=1}^{n} \sum_{l=1}^{n} a_k b_l = \lim_{n\to\infty} r_n s_n = AB.$$

If the series $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ are absolutely convergent, then all orders of summation give the same result, i.e. AB. (You don't need to prove the facts above.) If we sum over triangles we obtain

$$\lim_{n\to\infty} \sum_{k=1}^{n} c_k, \quad \text{where } c_k = \sum_{l=1}^{k-1} a_l b_{k-l}.$$

In an earlier exercise we saw that $\sum_{k=1}^{\infty} c_k$ may diverge. However, show that if $\sum_{k=1}^{\infty} c_k$ does converge then it must converge to AB. *Hint*: Consider $f(x) = \sum_{k=1}^{\infty} a_k x^k$, $g(x) = \sum_{k=1}^{\infty} b_k x^k$, $h(x) = \sum_{k=1}^{\infty} c_k x^k$ and use Abel's theorem.

5. For each of the functions f below find a formula for $F(x) = \int_{-1}^{x} f\,dx$ (not involving any integrals). Where is F continuous? Where is F differentiable? Where is $F'(x) = f(x)$?

(a) $f(x) = |x|$, (b) $f(x) = \begin{cases} 1, & \text{if } x < 0, \\ 2, & \text{if } x \geq 0, \end{cases}$ (c) $f(x) = \begin{cases} x, & \text{if } x < 0, \\ 1, & \text{if } x = 0, \\ 2x, & \text{if } x > 0. \end{cases}$

There are further questions on the next page.

In all the exercises below, integrable means Riemann integrable.

6. Dirichlet's function $\chi_{\mathbf{Q}}$ is not (Riemann) integrable on $[0,1]$ (you don't need to show this). Construct a sequence (f_n) of integrable functions such that $f_n \to \chi_{\mathbf{Q}}$ pointwise on $[0,1]$.

*7. (a) Show that if $f_n \to f$ uniformly on $[0,1]$ and f_n are integrable functions, then f is integrable.

(b) Show that

$$\int_0^{\pi} \sqrt[n]{\sin x}\, dx \to \pi, \quad \text{as } n \to \infty,$$

without using the dominated and monotone convergence theorems (which we didn't prove in this course).

(c) Show the result in (b) using the dominated and/or monotone convergence theorems.

8. Find sequences (f_n) of integrable functions on $[0,1]$ such that $f_n \to 0$ pointwise and

(a) $\lim_{n\to\infty} \int_0^1 f_n\, dx = \infty$,

(b) $\lim_{n\to\infty} \int_0^1 f_n\, dx$ doesn't exist (neither finite nor infinite).

9. Let f be an integrable function on $I = [0,1]$ and define

$$m = \inf\{f(x) : x \in I\}, \qquad M = \sup\{f(x) : x \in I\},$$
$$m' = \inf\{|f(x)| : x \in I\}, \quad M' = \sup\{|f(x)| : x \in I\}.$$

(a) Show that $M' - m' \le M - m$.

(b) Show that $|f|$ is integrable on I, without using Lebesgue's theorem.

(c) Show that $|f|$ is integrable on I, using Lebesgue's theorem.

(d) Show that

$$\left| \int_0^1 f\, dx \right| \le \int_0^1 |f|\, dx.$$

*10. Let

$$f(x) = \begin{cases} 1, & \text{if } x = 1/n \text{ for some } n \in \mathbf{N}, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Show that f is integrable on $[0,1]$ without using Lebesgue's theorem.

(b) Use Lebesgue's theorem to show that f is integrable on $[0,1]$.

(c) Calculate $\int_0^1 f\, dx$.

11. Assume that $f : [0,1] \to \mathbf{R}$ is bounded. Define $f^2(x) = f(x)^2$ and $f^3(x) = f(x)^3$.

(a) Is it true that if f^2 is integrable, then f is integrable?
(Give proof or counterexample.)

(b) Is it true that if f^3 is integrable, then f is integrable?
(Give proof or counterexample.)