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Hand-in exercises for TATA34 Real Analysis, Honours Course, 2025

I will send out information by personal email during the course, so please send me an email to anders.bjorn@liu.se with your name, preferred email address, programme and year. Let me know if you decide to drop the course, and I won't bother you with further emails.

The examination of this course consists of 6 rounds of hand-in exercises as given below.

The exercises should be **solved individually** and you have to be prepared to demonstrate your solutions at the board during the exercise sessions held after each round (but for the last one).

Each exercise can give a maximum of one point, but often fractions of a point will be given.

For grade 3 it is enough to get 5 points in each round. For grades 4 and 5 it is enough with 6 resp. 7 points in each round, provided that among them there are at least 6 resp. 12 *-marked points in total (i.e. from all the rounds together). There is no *-marked requirement for grade 3.

If you are very close to fulfilling these requirements I will take an overall look after all 6 rounds, and perhaps give you (individually) some extra exercise(s) to solve in order for you to obtain a certain grade. This mainly applies for passing the course.

In addition, to pass the course you are **also required** to be present and demonstrate solutions at the board during at least 3 exercise sessions. You will be asked to demonstrate some exercise (that I choose), or part of an exercise, that you've solved.

Instructions for the exercises

- For deadlines see the **lecture plan**. If for some reason you are not able to hand in the solutions by the deadline, you should notify **the examiner** as early as possible.
- Don't be afraid of the *-marked exercises, some of them are not so difficult.
- Read the exercises carefully. Sometimes there are extra twists.
- If you need to do a maxmin investigation in some problem, then please provide the details as required in the first year single-variable analysis course TATA41.
- You may use any result that is **proved** in the course book by Abbott, the course book by Forsling–Neymark for the 1st year single-variable analysis courses TATA41 and TATA42, or during the lectures.
- For some exercises there are hints. These are just suggestions of how you may solve the exercise. You're free to choose other ways if you prefer.
- In all exercises (with several parts) it is ok to use the result in one part to show a later part, even if you haven't solved the first part, unless said otherwise.
- You are allowed to discuss the problems with the other students taking the course, but you are **not allowed** to copy anyone else's solution. **You must formulate your own solutions, and understand them.** You are **not allowed** to use ChatGPT or similar tools.

Continued on the next page.

- Swedish-speaking students are **encouraged** to write their solutions in Swedish.
- A good idea is to first solve a problem on **scrap paper**, and later write it down more carefully on another piece of paper.
- You should put an effort into writing **clearly** and **legibly**, and to carefully **justify** your arguments.
- You may of course write your solutions using **L^AT_EX** (or some similar program), but this is not at all necessary.
- The exercises should be **numbered** according to the list below, and **sorted** according to the **numbering**.
- There should be **ample space for my comments** on each page. Please use wide enough margins.
- You may solve several exercises on the same page, and use both sides of the paper, but there should always be space between the solutions and in the margins so that I have ample space for my comments.
- Each round should have a **cover page** on which you should write your name, programme and year. In addition you should tick off which exercises you have solved in a table of the following form: Mark solved exercises with X and unsolved with –. (The third line is meant for me to mark how many points you get on each exercise.) Don't make the squares too small.

1	2	3	4	5	6	7	8	9	10	11
X	X	–	X	X	X	–	–	X	X	X

- **Staple** the cover page together with your solutions in the upper left corner. (And make sure that no text is hidden on later pages, not even any exercise numbers.)
- **Do not** put the solutions in a folder (plastficka).
- The solutions should be **handed** into the course's mailbox, B-house, top floor, near entrance 21.

Course evaluations: This course has generally been given very positive course evaluations by the students. There has not been any major specific adjustments suggested. Nevertheless, there are always some minor adjustments each year, as part of the continuing course development. I **encourage** you to fill in the course evaluation after the end of the course.

Round 1

In this round all the involved numbers and sets are within \mathbf{R} , unless explicitly said otherwise.

In all exercises it is ok to use the result in one part to show a later part, even if you haven't solved the first part, unless said otherwise. This applies to all later rounds as well.

1. Let $A_1 \supset A_2 \supset A_3 \supset \dots$. Decide if each of the following statements is true. If not give a counterexample. No proof needed if a statement is true. (Note that, in contrast to Abbott, when I write $A \subset B$ it is allowed that $A = B$.)
 - (a) If each A_n is infinite (i.e. contains an infinite number of points), then $\bigcap_{n=1}^{\infty} A_n$ is infinite.
 - (b) If each A_n is finite and nonempty, then $\bigcap_{n=1}^{\infty} A_n$ is nonempty.
2. Let $|x| = (\sum_{j=1}^n x_j^2)^{1/2}$ for $x = (x_1, \dots, x_n) \in \mathbf{R}^n$. Prove that $||a| - |b|| \leq |a - b|$ for $a, b \in \mathbf{R}^n$. *Hint:* Use the triangle inequality from linear algebra (no need to prove that). You can get half credit if you only prove it for $n = 1$.
3. Let $x_1 = 1$ and define $x_{n+1} = (2x_n + 5)/3$ for $n = 1, 2, \dots$.
 - (a) Use induction to prove that $x_n < 5$ for each n .
 - (b) Show that the sequence is strictly increasing.
 - (c) It follows from the monotone convergence theorem (MCT, SOMK) that the sequence has a limit $x = \lim_{n \rightarrow \infty} x_n$. Determine this limit.
4. Compute, without proofs, the suprema and infima of

$$A = \left\{ \frac{n}{5n+2} : n = 1, 2, \dots \right\} \quad \text{and} \quad B = \left\{ \frac{m}{2m+n} : m, n = 1, 2, \dots \right\}.$$

Is $\inf A \in A$?, $\sup A \in A$?, $\inf B \in B$?, $\sup B \in B$?

5. Let $A, B \subset \mathbf{R}$ be nonempty bounded sets. Decide if each of the following statements is true. If so give a short proof and otherwise a counterexample.
 - (a) If $\sup A < \inf B$, then there exists $c \in \mathbf{R}$ such that $a < c < b$ for all $a \in A$ and $b \in B$.
 - (b) If $c \in \mathbf{R}$ is such that $a < c < b$ for all $a \in A$ and $b \in B$, then $\sup A < \inf B$.
- *6. Let $\mathbf{I} = \mathbf{R} \setminus \mathbf{Q}$ be the set of irrational numbers.
 - (a) Show that if $a, b \in \mathbf{Q}$, then $a + b, ab \in \mathbf{Q}$.
 - (b) Show that if $a \in \mathbf{Q} \setminus \{0\}$ and $b \in \mathbf{I}$, then $a + b, ab \in \mathbf{I}$.
 - (c) What happens if $a, b \in \mathbf{I}$? Is it possible to have $a + b \in \mathbf{Q}$? How about $a + b \in \mathbf{I}$? How about $ab \in \mathbf{Q}$? How about $ab \in \mathbf{I}$? (For each question, give an example or prove none exists.)
7. Prove that the following numbers are irrational.

(a) $\sqrt{7}$,	(b) $\sqrt{10}$,	(c) $\sqrt{2} + \sqrt{5}$.
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8. (a) Show that $\sqrt[3]{5}$ and $\sqrt{2} + \sqrt{5}$ are algebraic.
 - (b) Show that if $a \in \mathbf{A}$, $a \neq 0$, (i.e. a is algebraic) then $-a, 1/a \in \mathbf{A}$. (You are not allowed to use that \mathbf{A} is a field.)

There are more questions on the next page.

- *9. (a) Show that $(0, 1) \sim \mathbf{R}$ (where $(0, 1) = \{x : 0 < x < 1\}$).
 (b) Show that $[0, 1] \sim \mathbf{R}$ (where $[0, 1] = \{x : 0 \leq x \leq 1\}$).
 (c) Show that $(0, 1) \times (0, 1) \sim \mathbf{R}^2$.
 (d) Show that $(0, 1) \preceq (0, 1) \times (0, 1)$.
 (e) Show that $(0, 1) \times (0, 1) \preceq (0, 1)$. *Hint:* Use a mapping which maps $(x, y) \in (0, 1) \times (0, 1)$, where $x = 0.x_1x_2x_3\dots$ and $y = 0.y_1y_2y_3\dots$, to $z = 0.x_1y_1x_2y_2x_3y_3\dots$. Make sure your mapping is well-defined.
 (f) Show that $(0, 1) \times (0, 1) \sim (0, 1)$.
 (g) Show that $\mathbf{R}^2 \sim \mathbf{R}$.
- *10. Show that the set consisting of all finite subsets of \mathbf{N} is countable.
- *11. Let $a_n = 1/n$. Then we know that $a_n \rightarrow 0$, but is this completely obvious. In this exercise we'll look at how this follows from our axioms. (You don't have to show that the sequence is decreasing.)
 (a) Let $i = \inf\{a_1, a_2, \dots\}$, which exists by the supremum axiom (SA, Axiom of Completeness). As 0 is a lower bound for the set, we see that $i \geq 0$. Show that the archimedean property (AP, AE) $\Rightarrow i = 0$. (Hence we know that SA \Rightarrow AP $\Rightarrow i = 0$.) *Hint:* Consider the contrapositive implication.
 (b) Let $a = \lim_{n \rightarrow \infty} a_n$, which exists by the monotone convergence theorem (MCT, SOMK). By the order limit theorem, $a \geq 0$. Show that $a = 0$ without using the archimedean property (AP, AE). *Hint:* Assume $a > 0$, then one may use $\varepsilon = a$ in the definition of limits.

Round 2

In this round all the involved numbers and sets are within \mathbf{R} , unless explicitly said otherwise.

1. Verify, using the (ε, N) -definition of limit, that $\lim_{n \rightarrow \infty} \frac{3n+2}{5n+3} = \frac{3}{5}$.
2. Let $x_n \geq 0$ and $x \geq 0$.
 - (a) Show that if $x_n \rightarrow 0$, then $\sqrt{x_n} \rightarrow 0$.
 - (b) Show that if $x_n \rightarrow x$, then $\sqrt{x_n} \rightarrow \sqrt{x}$.
(In this exercise you're not allowed to use that \sqrt{x} is a continuous function.)
3. Let $(x_j)_{j=1}^\infty$ be a sequence of points in \mathbf{R}^n such that $x_j \rightarrow a$ and $x_j \rightarrow b$, as $j \rightarrow \infty$. Show that $a = b$. You can get half credit if you only prove it for $n = 1$.
4. What happens if we reverse the order of the (first two) quantifiers in the definition of limits? We get the following (nonstandard) definition.
Definition: A sequence $(x_n)_{n=1}^\infty$ *conges* to x if $\exists \varepsilon > 0 \forall N \in \mathbf{N} \forall n \geq N |x_n - x| < \varepsilon$.
 - (a) Give an example of a congent sequence (and state what it conges to).
 - (b) Does there exist a congent sequence which is divergent?
 - (c) Can a sequence cong to two different values?
 - (d) Exactly what is described by this weird definition? (It is enough to give a description.)
5. Let

$$x_1 = 5 \quad \text{and} \quad x_{n+1} = \frac{1}{2} \left(x_n + \frac{3}{x_n} \right), \quad n = 1, 2, \dots$$

Show that $x_n \rightarrow \sqrt{3}$. *Hint:* Prove that $x_n > \sqrt{3}$ for each n , and then that $(x_n)_{n=1}^\infty$ is decreasing.

- *6. Let $(a_n)_{n=1}^\infty$ be a bounded sequence and define the *limit superior*

$$\limsup_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} x_n, \quad \text{where } x_n = \sup\{a_k : k \geq n\}.$$

- (a) Show that $(x_n)_{n=1}^\infty$ converges.
 - (b) Give a reasonable definition for $\liminf_{n \rightarrow \infty} a_n$.
 - (c) Prove that $\liminf_{n \rightarrow \infty} a_n \leq \limsup_{n \rightarrow \infty} a_n$, and show with an example that the inequality can be strict.
 - (d) Show that $\liminf_{n \rightarrow \infty} a_n = \limsup_{n \rightarrow \infty} a_n = a$ if and only if $\lim_{n \rightarrow \infty} a_n = a$.
 - (e) Give a reasonable definition for $\limsup_{x \rightarrow \infty} f(x)$, where $f : \mathbf{R} \rightarrow \mathbf{R}$ is a bounded function.
7. Let (a_n) and (b_n) be Cauchy sequences, and let $c_n = |a_n - b_n|$. Is (c_n) always a Cauchy sequence? Show this or give a counterexample.
8. (a) Show that if $\sum_{n=1}^\infty a_n$ is absolutely convergent then also $\sum_{n=1}^\infty a_n^2$ is absolutely convergent.
 - (b) Is it true that if $\sum_{n=1}^\infty a_n$ is convergent, then also $\sum_{n=1}^\infty a_n^2$ is convergent? Give a proof or a counterexample.
 - (c) Assume that both $\sum_{n=1}^\infty x_n$ and $\sum_{n=1}^\infty y_n$ diverge. Can it happen that $\sum_{n=1}^\infty x_n y_n$ converges? Give an example or prove that none exist.

There are more questions on the next page.

*9. The *Leibniz' criterium* says that if $(a_n)_{n=1}^{\infty}$ is a decreasing sequence tending to 0, then

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n$$

converges. Prove this in the following three ways:

(a) Prove that the partial sums

$$s_n = \sum_{k=1}^n (-1)^{k-1} a_k$$

form a Cauchy sequence, and use this to prove Leibniz' criterium.

(b) Use Cantor's encapsulation theorem for intervals to prove Leibniz' criterium

(c) Show that the subsequences $(s_{2n})_{n=1}^{\infty}$ and $(s_{2n+1})_{n=1}^{\infty}$ are monotone, and use this to prove Leibniz' criterium.

10. Assume that $\sum_{k=1}^{\infty} a_k = A \in \mathbf{R}$ and $\sum_{k=1}^{\infty} b_k = B \in \mathbf{R}$. If one wants to sum the double series $\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} a_k b_j$ it may depend on the order of summation. If we sum over squares this leads to $\lim_{n \rightarrow \infty} \sum_{k=1}^n \sum_{j=1}^n a_k b_j$, while if we sum over triangles we obtain

$$\lim_{n \rightarrow \infty} \sum_{k=2}^n c_k, \quad \text{where } c_k = \sum_{j=1}^{k-1} a_j b_{k-j}.$$

(a) Show that if we sum over squares we always get

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \sum_{j=1}^n a_k b_j = AB.$$

(b) Show, by giving an example, that it can happen that the sum over triangles $\sum_{k=1}^{\infty} c_k$ diverges. *Hint:* Consider $a_k = b_k = (-1)^k / \sqrt{k}$. If you make other choices you need to show that $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ are convergent.

*11. (See above for the definition of \limsup .)

(a) Show that $\sum_{k=1}^{\infty} a_k$ is absolutely convergent if $\limsup_{k \rightarrow \infty} |a_k|^{1/k} < 1$.

(b) Show that $\sum_{k=1}^{\infty} a_k$ is divergent if $\limsup_{k \rightarrow \infty} |a_k|^{1/k} > 1$.

(c) Give an example where $\limsup_{k \rightarrow \infty} |a_k|^{1/k} = 1$ and $\sum_{k=1}^{\infty} a_k$ is absolutely convergent.

(d) Give an example where $\limsup_{k \rightarrow \infty} |a_k|^{1/k} = 1$ and $\sum_{k=1}^{\infty} a_k$ is conditionally convergent.

(e) Give an example where $\limsup_{k \rightarrow \infty} |a_k|^{1/k} = 1$ and $\sum_{k=1}^{\infty} a_k$ is divergent.

Round 3

1. Let

$$a_n = (-1)^n + \frac{2}{n}, \quad n = 1, 2, 3, \dots, \quad \text{and} \quad A = \{a_n : n = 1, 2, 3, \dots\}.$$

- (a) Calculate $\limsup_{n \rightarrow \infty} a_n$ and $\liminf_{n \rightarrow \infty} a_n$. *Only answers required in (a), (b), (e) and (f).*
 - (b) Determine the limit points of A .
 - (c) Is A open? Why?
 - (d) Is A closed? Why?
 - (e) Has A any isolated points? If so which?
 - (f) Determine \bar{A} . (In this course, \bar{A} denotes the *closure* of A , not the complement.)
2. For each of the following sets determine if it is open, closed, both or neither (with respect to \mathbf{R}). If a set isn't open, find a point in it without a neighbourhood within the set. If a set isn't closed, find a limit point which isn't in the set. No further justification has to be given.
- (a) \mathbf{Q} , (b) \mathbf{N} , (c) $\{x \in \mathbf{R} : x \neq 0\}$,
 - (d) $\{1 + 1/2 + 1/3 + \dots + 1/n : n = 1, 2, \dots\}$, (e) $\{1 + 1/4 + 1/9 + \dots + 1/n^2 : n = 1, 2, \dots\}$.
3. (a) Show that $\overline{A \cup B} = \bar{A} \cup \bar{B}$. *In both parts you may assume that the sets are subsets of \mathbf{R} .*
- (b) Is the corresponding identity true for infinite unions? Give a proof or a counterexample.
4. Decide which of the following statements are true. For each one give a proof or a counterexample.
- (a) If $A \subset \mathbf{R}$ is open and $\mathbf{Q} \subset A$, then $A = \mathbf{R}$.
 - (b) Every nonempty open set $A \subset \mathbf{R}$ contains a rational number.
 - (c) Every infinite closed set $A \subset [0, 1]$ contains a rational number.
 - (d) Every infinite countable set $A \subset [0, 1]$ has a limit point.
 - (e) Every infinite countable set $A \subset [0, 1]$ has an isolated point.
5. Which of the following sets are compact? For the noncompact ones, show why it isn't sequentially compact, i.e. give an example of a sequence in the set without convergent subsequence with limit in the set. (For compact sets it is enough to say so.)
- (a) \mathbf{N} , (b) $\mathbf{Q} \cap [0, 1]$, (c) $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$, (d) $\{1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\}$,
 - (e) $\{1 + 1/4 + 1/9 + \dots + 1/n^2 : n = 1, 2, \dots\}$.
6. For the sets which were noncompact in the previous exercise, find an open cover without finite subcover.
- *7. Let X be a metric space. Let $a \in X$, and let $(a_n)_{n=1}^\infty$ be a sequence in X such that every subsequence has a subsubsequence converging to a . Show that $\lim_{n \rightarrow \infty} a_n = a$. *Hint: Assume that $a_n \not\rightarrow a$. You can get 0.7p if you only consider this for $X = \mathbf{R}$.*

There are more questions on the next page.

8. Decide which of the following statements are true. For each one either give a proof or a counterexample. For full credit you should consider if the statement is true in arbitrary metric spaces, but partial credit will be given if you only consider subsets of \mathbf{R} .
- (a) An arbitrary intersection of compact sets is compact.
 - (b) An arbitrary union of compact sets is compact.
 - (c) If A is arbitrary and K compact, then $A \cap K$ is compact.
 - (d) If $F_1 \supset F_2 \supset F_3 \supset \dots$ is a decreasing sequence of nonempty closed sets, then $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.
 - (e) A finite set is always compact.
 - (f) A countable set is always compact.
- *9. Let (X, d_X) and (Y, d_Y) be nonempty metric spaces. Let

$$d_{X \times Y}((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2).$$

- (a) Show that $(X \times Y, d_{X \times Y})$ is a metric space, i.e. that $d_{X \times Y}$ is a metric on $X \times Y$.
 - (b) Show that $X \times Y$ is complete if and only if both X and Y are complete.
10. A set is *clact* if every cover by closed sets has a finite subcover. (This is a nonstandard definition.) Describe all clact subsets of \mathbf{R} . Prove that your description is correct.
- *11. Repeat the Cantor construction starting with the interval $A_0 = B_0 = [0, 1]$. This time, however, remove the open middle quarter of each interval in A_{n-1} when constructing A_n , and the open middle interval of length 4^{-n} of each interval in B_{n-1} when constructing B_n . The set $A = \bigcap_{n=0}^{\infty} A_n$ is a selfsimilar Cantor set, whereas $B = \bigcap_{n=0}^{\infty} B_n$ is a nonselfsimilar Cantor set.
- (a) Draw A_1, A_2, B_1 and B_2 , with explicit coordinates using fractions. $A_1 = B_1$ but how can you compare A_2 and B_2 ? (Is $A_2 = B_2$, $A_2 \subset B_2$, $A_2 \supset B_2$ or none of these?)
 - (b) Draw the first two intervals (closest to 0) of A_3 and B_3 , with explicit coordinates using fractions. How can you compare A_3 and B_3 ?
 - (c) Draw the first two intervals (closest to 0) of A_4 and B_4 , with explicit coordinates using fractions. How can you compare A_4 and B_4 ?
 - (d) Are A and B compact? Perfect? Do they have any interior points?
 - (e) What is the total length of the removed intervals when constructing A resp. B ?
 - (f) What is the length (i.e. outer measure) of A resp. B ? What are their dimensions? (We haven't all the tools available for this, but give your best guesses, and try to explain them.)

Round 4

1. Use the (ε, δ) -definition of limit to show:
 - (a) $\lim_{x \rightarrow 1} x^4 - 1 = 0$,
 - (b) $\lim_{x \rightarrow 2} x^2 + x + 1 = 7$.
2. Prove or disprove that the following functions are uniformly continuous:
 - (a) e^x on $(0, 1)$,
 - (b) e^x on $(1, \infty)$,
 - (c) $\ln x$ on $(0, 1)$,
 - (d) $\ln x$ on $(1, \infty)$.
3. Define for each of the following sets A a function $f : \mathbf{R} \rightarrow \mathbf{R}$ which is discontinuous at every point in A but continuous at every point in A^c : (It is enough if you define them.)
 - (a) $A = \mathbf{Z}$,
 - (b) $A = [0, 1]$,
 - (c) $A = (0, 1)$,
 - (d) $A = \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$.
- *4. (a) Prove that Dirichlet's function $\chi_{\mathbf{Q}}$ isn't continuous at any point. Here

$$\chi_{\mathbf{Q}}(x) = \begin{cases} 1, & \text{if } x \in \mathbf{Q}, \\ 0, & \text{otherwise.} \end{cases}$$

- (b) Show that Thomae's function t , see Section 4.1, isn't continuous at any rational point.
- (c) Show that t is continuous at every irrational point. *Hint*: Study the set $\{x \in \mathbf{R} : t(x) \geq \varepsilon\}$.
5. Let X and Y be metric spaces and $f : X \rightarrow Y$ be continuous. Show that $f^{-1}(F)$ is closed if F is closed. You can get half credit if you only consider continuous functions $f : \mathbf{R} \rightarrow \mathbf{R}$.
6. Give for each part an example of such an f and a Cauchy sequence (x_n) , with $x_n \in D_f$ (f 's domain of definition), such that $(f(x_n))$ isn't a Cauchy sequence, or explain why this is impossible to fulfill.
 - (a) With a continuous $f : (0, 1) \rightarrow \mathbf{R}$.
 - (b) With a continuous $f : [0, 1] \rightarrow \mathbf{R}$.
 - (c) With a continuous $f : [0, \infty) \rightarrow \mathbf{R}$.
 - (d) With a uniformly continuous $f : (0, 1) \rightarrow \mathbf{R}$.
 - (e) With a uniformly continuous $f : [0, 1] \rightarrow \mathbf{R}$.
 - (f) With a uniformly continuous $f : [0, \infty) \rightarrow \mathbf{R}$.
- *7. A function $f : A \rightarrow \mathbf{R}$ is *Lipschitz* with Lipschitz constant M if $|f(x) - f(y)| \leq M|x - y|$ for all $x, y \in A$.
 - (a) Show that if f is Lipschitz on $(0, 1]$, then $\lim_{x \rightarrow 0+} f(x)$ exists and is finite.
 - (b) Show that if f is Lipschitz on $(0, 1]$, then f has a continuous extension (utvidgning) $g : [0, 1] \rightarrow \mathbf{R}$, i.e. such that $g(x) = f(x)$ for $x \in (0, 1]$.
 - (c) Is g in (c) always Lipschitz on $[0, 1]$? (Give proof or counterexample.)
8. Let I be an interval, $f : I \rightarrow \mathbf{R}$ be differentiable and $M = \sup_{x \in I} |f'(x)|$. Here $f'(a)$ is defined using a one-sided limit if $a \in I$ is an end point of I .

Definition: $f \in C^1(I)$ if f' (exists and) is continuous on I .

 - (a) Show that if $f \in C^1(I)$ with $I = [0, 1]$, then $M < \infty$.
 - (b) Given an example of $f \in C^1(I)$ with $I = (0, 1)$ such that $M = \infty$.
 - (c) Given an example of $f \in C^1(I)$ with $I = [0, \infty)$ such that $M = \infty$.
 - (d) Given an example of a differentiable f on $I = [0, 1]$ such that $M = \infty$.
 - (e) Show that if $M < \infty$, then f is Lipschitz on I .
 - (f) Show that if $M = \infty$, then f is not Lipschitz on I .

There are more questions on the next page.

9. Show that a metric space X is connected if and only if X and \emptyset are the only sets which are both open and closed.
- *10. Let (X, d_X) be a nonempty metric space, and $I = [0, 1]$. Let

$$d_{X \times I}((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + |y_1 - y_2|.$$

- (a) Show that if X is pathconnected then $X \times I$ is pathconnected.
- (b) Show that if $X \times I$ is pathconnected then X is pathconnected.
- (c) Show that if X is connected then $X \times I$ is connected.
- (d) Show that if $X \times I$ is connected then X is connected.

Hint: The projections $P_X : X \times I \rightarrow X$, $P_X(x, t) = x$, and $P_I : X \times I \rightarrow I$, $P_I(x, t) = t$, which are clearly Lipschitz (you don't need to prove that) and the fibers $X \times \{t\} = \{(x, t) : x \in X\}$, $t \in I$, and $\{x\} \times I$, $x \in X$, may be of some help.

- *11. Let X be a metric space and $E \subset X$ be nonempty. Show that if E is connected, then so is \bar{E} ? You can get 0.7p if you only consider this for $X = \mathbf{R}^n$, and 0.3p if you only consider $X = \mathbf{R}$.

Later rounds will be given later.