Exempel: Beräkna

(a) $\sum_{n=1}^{\infty} \frac{1}{n3^n}$ och (b) $\sum_{n=1}^{\infty} \frac{n}{2^n}$

genom att integrera respektive derivera den geometriska serien

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad |x|<1.$$

(a)

$$\int_0^x \left(\sum_{n=0}^{\infty} t^n\right) dt = \int_0^x \frac{1}{1-t}\, dt$$

$$\sum_{n=0}^{\infty} \int_0^x t^n\, dt = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = -\ln(1-x), \qquad \underline{\underline{|x|<1}}$$

$$\sum_{n=1}^{\infty} \frac{1}{n3^n} = \sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{1}{3}\right)^n = \sum_{n=0}^{\infty} \frac{1}{n+1}\left(\frac{1}{3}\right)^{n+1} = -\ln(1-1/3)$$

$$= -\ln(2/3)$$

$$= \underline{\underline{\ln(3/2)}}.$$

(b) $\left(\sum_{n=0}^{\infty} x^n\right)' = \left(\frac{1}{1-x}\right)'$ $\quad |x|<1$

$$\parallel$$

$$\sum_{n=0}^{\infty} \frac{d}{dx}(x^n) = \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}$$

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = \sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^n = \frac{1}{2}\sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^{n-1} = \frac{1}{2}\,\frac{1}{(1-1/2)^2} = \underline{\underline{2}}.$$