Exempel

Bestäm den allmänna lösningen till

$$y^{(4)} - 2y^{(3)} + y'' + 2y' - 2y = \mathrm{e}^x \cos x.$$

Svaret ska ges på reell form.

Lösning

Lösning

Homogen ekvation:

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$p(r) = r^4 - 2r^3 + r^2 + 2r - 2 = \ldots (r+1)(r-1)(r-(1+i))(r-(1-i)).$

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$$y_h = C_1 \mathrm{e}^{x} + C_2 \mathrm{e}^{-x} + C_3 \mathrm{e}^{x} \cos x + C_4 \mathrm{e}^{x} \sin x.$$

Lösning

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Vi hittar en partikulärlösning till
$p(D)w = \mathrm{e}^{(1+i)x} = \mathrm{e}^x \cos x + i\mathrm{e}^x \sin x.$

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$p(D)w = \mathrm{e}^{(1+i)x} = \mathrm{e}^x \cos x + i\mathrm{e}^x \sin x$.
Då blir $y_p = \mathrm{Re}\, w$ en partikulärlösning till $p(D)y_p = \mathrm{e}^x \cos x$.

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Då blir $y_p = \mathrm{Re}\, w$ en partikulärlösning till $p(D)y_p = \mathrm{e}^x \cos x$.
Vi använder förskjutningsregeln med

$$w = z\mathrm{e}^{(1+i)x}.$$

Lösning

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$$= D^4 + (2+4i)D^3 + (-5+6i)D^2 + (-4-2i)D.$$

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Det fungerar att ta z' konstant, så att $z'' = z^{(3)} = z^{(4)} = 0$:

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Det fungerar att ta z' konstant, så att $z'' = z^{(3)} = z^{(4)} = 0$:

$$(-4-2i)z' = 1 \Leftrightarrow z' = \frac{1}{-4-2i} = \frac{-2+i}{10} \Leftarrow z = \frac{-2x+ix}{10}.$$

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Lösning

$$
\begin{aligned}
y_p &= \mathrm{Re} w = \mathrm{Re}(z\mathrm{e}^{(1+i)x}) \\
&= \mathrm{Re}\left(\frac{-2x+ix}{10}(\mathrm{e}^x\cos x + i\mathrm{e}^x \sin x)\right) \\
&-\frac{x}{5}\mathrm{e}^x\cos x + \frac{ix}{10}i\mathrm{e}^x\sin x \\
&= \frac{-x}{5}\mathrm{e}^x\cos x - \frac{x}{10}\mathrm{e}^x\sin x.
\end{aligned}
$$

Svar: $y = y_h + y_p =$
$C_1\mathrm{e}^x + C_2\mathrm{e}^{-x} + C_3\mathrm{e}^x\cos x + C_4\mathrm{e}^x\sin x + \frac{-x}{5}\mathrm{e}^x\cos x - \frac{x}{10}\mathrm{e}^x\sin x.$