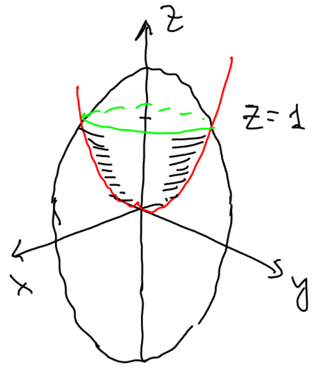
Lösningsförslag, Tentamen TATA44, 2024-10-29

(1) Skärningskurvan mellan paraboloiden $z=x^2+y^2$ och ellipsoiden $2x^2+2y^2+(z+1)^2=6$ fås ur $2z+(z+1)^2=6 \iff z^2+4z-5=0 \iff$ $z_1=-5$ (falsk eftersom $x^2+y^2=-5$, $\emptyset$) eller $z_2=1$, se bilden. M.h.a. cylindriska koordinater ges ytan S av $z=x^2+y^2=\rho^2$, d.v.s.



$$\bar r = \rho\hat\rho + z\hat z = \rho\hat\rho+\rho^2\hat z, \quad 0\le\varphi<2\pi,\ 0\le\rho\le 1.$$

$$\Rightarrow \bar r'_\rho = \hat\rho + \rho\frac{\partial\hat\rho}{\partial\rho} + 2\rho\hat z = \Big/\, d\hat\rho = \hat\varphi\, d\varphi \Rightarrow \frac{\partial\hat\rho}{\partial\rho}=0 \,\Big/ = \hat\rho+2\rho\hat z$$

och
$$\bar r'_\varphi = \rho\hat\varphi + 0 = \rho\hat\varphi, \quad \text{alltså} \quad \bar r'_\rho\times\bar r'_\varphi = (\hat\rho+2\rho\hat z)\times\rho\hat\varphi = \rho\hat z - 2\rho^2\hat\rho$$

$$\Rightarrow |\bar r'_\rho\times\bar r'_\varphi| = \rho\sqrt{1+4\rho^2}, \quad dS=\rho\sqrt{1+4\rho^2}\,d\rho\, d\varphi, \ \text{och således}$$

$$\text{Area}(S) = \int_0^{2\pi}\int_0^1 \rho\sqrt{\underbrace{1+4\rho^2}_{t}}\,d\rho\, d\varphi = 2\pi\cdot\int_1^5 \sqrt t\,\frac{dt}{8} = \frac{\pi}{4}\cdot\left[\frac{2t^{3/2}}{3}\right]_1^5$$

Svar: area $=\frac{\pi}{6}(5\sqrt5-1)$

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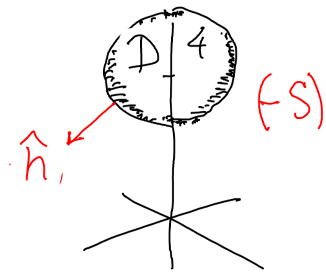
(2) $\operatorname{div}\bar A = 2x+2y+2z = \big/$ i sfäriska koordinater med $z=4+r\cos\theta$ $\big/$
$$= 2r(\cos\varphi\sin\theta+\sin\varphi\sin\theta+\cos\theta)+8,$$
Gauss' sats ger (obs. orientering!):



$$\iint_S \bar A\cdot d\bar S = -\iint_{\underbrace{-S}_{\text{randen till klotet } D}} \bar A\cdot d\bar S = -\iiint_D \operatorname{div}\bar A\cdot dV =$$

$$= -\int_0^{2\pi}\int_0^{\pi}\int_0^{2}\Big(2r(\underline{\cos\varphi}\sin\theta+\underline{\sin\varphi}\sin\theta+\cos\theta)+8\Big)\cdot r^2\sin\theta\cdot dr\,d\theta\,d\varphi =$$

$$= -2\pi\int_0^{\pi}\int_0^{2}(8+2r\cos\theta)\,r^2\sin\theta\, dr\,d\theta = -2\pi\int_0^{\pi}\int_0^{2} 8r^2\sin\theta\,dr\,d\theta = -\frac{256\pi}{3}$$

Svar: $-\frac{256}{3}\pi$

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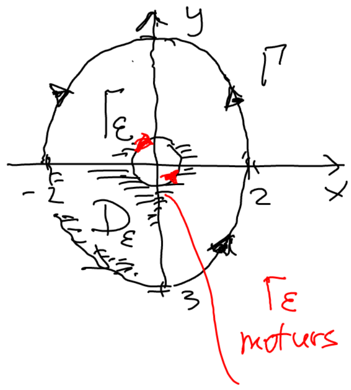
(3) I cylindriska koordinater kan $\bar A$ skrivas som
$$\bar A = \frac{x}{\sqrt{x^2+y^2}}\hat x + \frac{y}{\sqrt{x^2+y^2}}\hat y + (-y\hat x + x\hat y) = (\cos\varphi\,\hat x+\sin\varphi\,\hat y)+\rho(\cos\varphi\,\hat y-\sin\varphi\,\hat x) =$$
$$= \hat\rho + \rho\hat\varphi = (1,\rho,0)_{\text{cyl.}}$$

$$\Rightarrow \operatorname{rot}\bar A = \nabla\times\bar A = \frac{1}{\rho}\begin{vmatrix}\hat\rho & \rho\hat\varphi & \hat z\\ \partial_\rho & \partial_\varphi & \partial_z \\ 1 & \rho^2 & 0\end{vmatrix} = \left(0;0;\frac{2\rho}{\rho}\right) = (0,0,2) = 2\hat z$$

(alternativt kan man räkna divergensen direkt i kartesiska koordinater)

Observera $\bar{A}$ är singulärt för $x=y=0$ (z-axeln).

Väljer $D_\varepsilon = \{(x,y,z)\in\mathbb{R}^3 ;\ 9x^2+4y^2 \le 36,\ x^2+y^2 \ge \varepsilon^2,\ z=0\}$ med orientering $\hat{n}=\hat{z}$, så att Γ är positivt orienterade (m.a.p. D_ε), alltså ger Stokes sats:



$$\iint_{D_\varepsilon} (\nabla\times\bar{A})\cdot d\bar{S} = \int_\Gamma \bar{A}\cdot d\bar{r} - \int_{\Gamma_\varepsilon} \bar{A}\cdot d\bar{r}$$

- $d\bar{S} = \hat{z}\,dxdy \Rightarrow \iint_{D_\varepsilon}(\nabla\times\bar{A})\cdot d\bar{S} = \iint_{D_\varepsilon} 2\hat{z}\cdot\hat{z}\,dxdy = 2\,\text{Area}(D_\varepsilon) =$
$$= 2\left[\pi\cdot 2\cdot 3 - \pi\varepsilon^2\right] = 12\pi - 2\pi\varepsilon^2.$$

- $$\int_{\Gamma_\varepsilon} \bar{A}\cdot d\bar{r} = \left| \begin{array}{l} \bar{r} = \rho\hat{\rho} + z\hat{z} = \varepsilon\hat{\rho} \\ d\bar{r} = \varepsilon\hat{\varphi}\,d\varphi \end{array}\right| = \int_{\Gamma_\varepsilon} (\hat{\rho} + \underbrace{\rho}_{\varepsilon}\hat{\varphi})\cdot\varepsilon\hat{\varphi}\,d\varphi = \int_0^{2\pi} \varepsilon^2\,d\varphi = 2\pi\varepsilon^2$$

Således:
$$\int_\Gamma \bar{A}\cdot d\bar{r} = 12\pi - 2\pi\varepsilon^2 + 2\pi\varepsilon^2 = \underline{12\pi}$$

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(4) Gradient i sfäriska koordinater ges av $\nabla\Phi = \frac{\partial\Phi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\cdot\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\varphi}\cdot\hat{\varphi}$, vilket ger i vårt fal: $\nabla\Phi = \bar{F} \Leftrightarrow$

$$\begin{cases} (a)\ \frac{\partial\Phi}{\partial r} = 2r\cos\varphi\sin 2\theta \\ (b)\ \frac{1}{r}\frac{\partial\Phi}{\partial\theta} = 2r\cos\varphi\cos 2\theta \\ (c)\ \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\varphi} = -2r\sin\varphi\cos\theta \end{cases}$$

$\overset{(a)}{\Longmapsto}$ $\Phi(r,\theta,\varphi) = r^2\cos\varphi\sin 2\theta + h(\theta,\varphi) \Longmapsto$

$\Longmapsto \Phi'_\theta = 2r^2\cos\varphi\cos 2\theta + h'_\theta(\theta,\varphi) = /(b)/ = 2r^2\cos\varphi\cos 2\theta$

$\Longmapsto h'_\theta(\theta,\varphi) = 0 \Rightarrow h(\theta,\varphi) = f(\varphi)$, alltså

$\Phi(r,\theta,\varphi) = r^2\cos\varphi\sin 2\theta + h(\varphi) \Longmapsto$

$\Phi'_\varphi = -r^2\sin\varphi\underbrace{\sin 2\theta} + h'(\varphi) = /(c)/ = -2r^2\sin\varphi\underbrace{\sin\theta\cos\theta}$

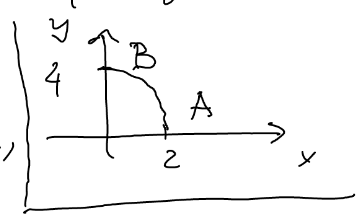
$\Longmapsto h'(\varphi) = 0,\ h(\varphi) = C \in \mathbb{R}$, slutligen blir potentialen

$$\Phi(r,\theta,\varphi) = r^2\cos\varphi\sin 2\theta + C = 2\underbrace{r\cos\varphi\sin\theta}_{x}\cdot\underbrace{r\cos\theta}_{z} + C = 2xz + C$$

Kurvan Γ: $z=0 \cap 4x^2+y^2+2z^2=16$, $x,y\ge 0$ är en ellipsbåge:

$4x^2+y^2=16,\ x\ge 0,\ y\ge 0$ i planet $z=0$, d.v.s

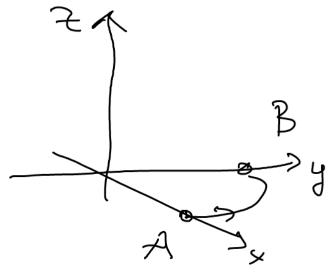
met ändpunkter $A(2,0,0)$ och $B(0,4,0)$. Enligt uppgiften, $\Gamma: A \to B$, alltså kurvintegralen blir



$$\int_\Gamma \bar F\cdot d\bar r = \int_\Gamma \nabla\Phi\cdot d\bar r = \Phi(B)-\Phi(A) = \Big/\text{m.h.a. } \Phi = 2xz\Big/ =$$

$$= \Phi(0,4,0)-\Phi(2,0,0) = 0-0 = 0.$$

Alternativt, i sfäriska koordinater



A ges av $r=2;\ \varphi=0;\ \theta=\frac{\pi}{2}$

B ges av $r=4,\ \varphi=\frac{\pi}{2},\ \theta=\frac{\pi}{2}.$

$$\Phi(A) = \left(r^2\cos\varphi\sin 2\theta + C\right)\Big|_A = 4\cdot 1\cdot 0 + C = C$$

$$\Phi(B) = \left(r^2\cos\varphi\sin 2\theta + C\right)\Big|_B = 16\cdot 0\cdot 0 + C = C$$

$$\Rightarrow \quad \Phi(B)-\Phi(A) = 0$$

Svar: $\Phi = r^2\cos\varphi\sin 2\theta + C = 2xz + C, \quad \int_\Gamma \bar F\cdot d\bar r = 0.$

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5 a) $\bar A = (x+y)\hat x + (y-x)\hat y + z\hat z =$

$$= \underbrace{(x\hat x + y\hat y + z\hat z)}_{\bar r} + (y\hat x - x\hat y) = \bar r + r\sin\theta\,\underbrace{(\sin\varphi\cdot\hat x - \cos\varphi\cdot\hat y)}_{-\hat\varphi} =$$

$$= r\hat r - r\sin\theta\cdot\hat\varphi = (r;0;-r\sin\theta)_{\text{sfär.}}$$

b) $$\operatorname{div} r^m\bar A = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^{m+1}\cdot r^2\sin\theta\right) + 0 + \frac{1}{r\sin\theta}\cdot\underbrace{\frac{\partial}{\partial\varphi}\left(-r\sin\theta\cdot r^{m+1}\sin\theta\right)}_{0}\right] =$$

$$= (m+3)\frac{r^{m+2}}{r^2} = (m+3)r^m$$

$$\operatorname{dim} r^m\bar A = 0 \iff m+3 = 0$$

Svar a) $\bar A(r,\theta,\varphi) = r\hat r - r\sin\theta\cdot\hat\varphi$, b) $m=-3$

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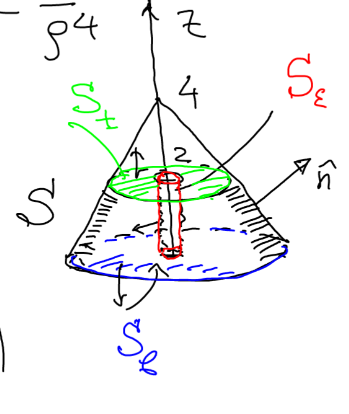
6 $$\bar A = \frac{1}{(x^2+y^2)^2}(x,y,z) = \frac{1}{(x^2+y^2)^2}\bar r = \Big/\begin{matrix}\text{cylindriska}\\ \text{koordinater}\end{matrix}\Big/ = \frac{\rho\hat\rho + z\hat z}{\rho^4} = \frac{1}{\rho^3}\hat\rho + \frac{z}{\rho^4}\hat z,$$

$$\operatorname{div}\bar A = \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}\left(\frac{1}{\rho^3}\cdot\rho\right) + 0 + \frac{\partial}{\partial z}\left(\frac{z}{\rho^4}\cdot\rho\right)\right] = -\frac{2}{\rho^4} + \frac{1}{\rho^4} = -\frac{1}{\rho^4}$$



Kroppen $D_\varepsilon = \left\{\bar r:\ x^2+y^2 \geq \varepsilon,\ z \leq 4-\sqrt{x^2+y^2},\ 0\leq z\leq 2\right\}$

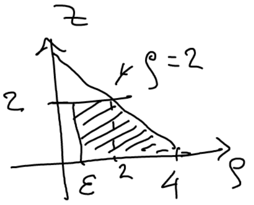
$\partial D_\varepsilon = S + S_t - S_\varepsilon + S_b$, där S_t orienterad med $\hat n = \hat z$

S_ε orienterad "utåt"

S_b orienterad med $\hat n = -\hat z$

$$\left(\iint_S + \iint_{S_t} - \iint_{S_\varepsilon} + \iint_{S_b}\right)(\bar{A}\bullet d\bar{S}) = \iiint_{D_\varepsilon} \operatorname{div}\bar{A}\cdot dV = -\iiint_{D_\varepsilon} \frac{1}{\rho^4}\,dxdydz = I_4$$

$$\iint_{S_t} = I_1,\quad \iint_{S_\varepsilon} = I_2,\quad \iint_{S_b} = I_3$$



- D_ε i cylinderkoordinater ges av: $0\le z\le 2,\ 0\le\varphi\le 2\pi$, $\varepsilon\le\rho\le 4-z$

$$\Rightarrow I_4 = -\int_0^{2\pi}\int_0^2\int_\varepsilon^{4-z}\frac{1}{\rho^4}\cdot\rho\,d\rho\,dz\,d\varphi = -2\pi\int_0^2\left[\frac{1}{-2\rho^2}\right]_\varepsilon^{4-z}dz = \pi\int_0^2\left[\frac{1}{(4-z)^2}-\frac{1}{\varepsilon^2}\right]dz =$$

$$= \pi\left[\frac{1}{4-z}\right]_0^2 - \frac{2\pi}{\varepsilon^2} = \frac{\pi}{4} - \frac{2\pi}{\varepsilon^2}$$

- $$I_2 = \iint_{S_\varepsilon}\bar{A}\bullet d\bar{S} = \iint_{S_\varepsilon}\left(\frac{\hat{\rho}}{\varepsilon^3}+\frac{z}{\varepsilon^4}\hat{z}\right)\bullet\hat{\rho}\cdot\varepsilon\,dz\,d\varphi = \int_0^{2\pi}\int_0^2\frac{1}{\varepsilon^2}\,dz\,d\varphi = \frac{4\pi}{\varepsilon^2}$$

$\uparrow$ $\rho=\varepsilon$-nivåyta: $d\bar{S} = \hat{\rho}\cdot\rho\,dz\,d\varphi$

- $$I_1 = \iint_{S_t}\bar{A}\bullet d\bar{S} = \left/\begin{matrix} z=2\text{-nivåyta} \\ d\bar{S} = \hat{z}\cdot\rho\,d\rho\,d\varphi\end{matrix}\right| = \iint_{S_t}\left(\frac{\hat{\rho}}{\rho^3}+\frac{\overset{=2}{z}}{\rho^4}\hat{z}\right)\bullet\hat{z}\,\rho\,d\rho\,d\varphi = \int_0^{2\pi}\int_\varepsilon^2\frac{2}{\rho^3}\,d\rho\,d\varphi =$$

$$= 4\pi\left[-\frac{1}{2\rho^2}\right]_\varepsilon^2 = -\frac{4\pi}{8} + \frac{2\pi}{\varepsilon^2}$$

Analogt för S_b:

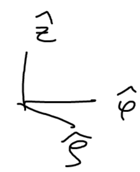
$$I_3 = \iint_{S_b}\bar{A}\bullet d\bar{S} = \left/\begin{matrix} z=0\text{- nivåytan} \\ d\bar{S} = -\hat{z}\,\rho\,d\rho\,d\varphi\end{matrix}\right| = \iint_{S_b}\left(\frac{\hat{\rho}}{\rho^3}+\frac{\overset{=0}{z}}{\rho^4}\hat{z}\right)\bullet\hat{z}\rho\,d\rho\,d\varphi = 0$$

$$I = I_4 - I_1 + I_2 - I_3 = \frac{\pi}{4} - \frac{2\pi}{\varepsilon^2} + \frac{4\pi}{8} - \frac{2\pi}{\varepsilon^2} + \frac{4\pi}{\varepsilon^2} - 0 = \frac{3\pi}{4}.$$

Alternativ II (m.h.a. definition): S ges av $\begin{cases} z = 4-\rho, & 2\le\rho\le 4 \\ \varphi=\varphi, & 0\le\varphi\le 2\pi\end{cases}$

$\bar{r} = \rho\hat{\rho} + z\hat{z} = \rho\hat{\rho} + (4-\rho)\hat{z}$, $\quad d\hat{\rho} = \hat{\varphi}\,d\varphi$, $d\hat{z} = 0$ $\quad$ ger $d\bar{r} = \underline{\underline{\hat{\rho}}}\,d\rho + \rho\hat{\varphi}\,d\varphi - \underline{\hat{z}\,d\rho}$

$$\Rightarrow \bar{n} = \bar{r}'_\rho\times\bar{r}'_\varphi = (\hat{\rho}-\hat{z})\times(\rho\hat{\varphi}) = \rho\cdot\hat{\rho}\times\hat{\varphi} - \rho\,\hat{z}\times\hat{\varphi} = \rho\hat{z} + \rho\hat{\rho}$$



Höger bas

Obs. att $\bar{n}\bullet\hat{z} = \rho > 0 \Rightarrow$ korrekt orientering!

Således

$$\iint_S\bar{A}\bullet d\bar{S} = \iint_D\left(\frac{1}{\rho^3}\hat{\rho} + \frac{\overset{=4-\rho}{z}}{\rho^4}\hat{z}\right)\bullet(\rho\hat{\rho}+\rho\hat{z})\,d\rho\,d\varphi = \int_0^{2\pi}\int_2^4\left(\frac{1}{\rho^2}+\frac{4-\rho}{\rho^3}\right)d\rho\,d\varphi =$$

$$= 2\pi\int_2^4\frac{4}{\rho^3}\,d\rho = 8\pi\left[-\frac{1}{2\rho^2}\right]_2^4 = \frac{3\pi}{4}$$

Svar: $\frac{3\pi}{4}$