Celestial Mech Dyn Astr (2006) 96:239–251
DOI 10.1007/s10569-006-9037-6

ORIGINAL ARTICLE

Quaternions and the rotation of a rigid body

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Received: 10 December 2005 / Revised: 21 May 2006 /
Accepted: 29 May 2006 / Published online: 27 October 2006

Abstract The orientation of an arbitrary rigid body is specified in terms of a quaternion based upon a set of four Euler parameters. A corresponding set of four generalized angular momentum variables is derived (another quaternion) and then used to replace the usual three-component angular velocity vector to specify the rate by which the orientation of the body with respect to an inertial frame changes. The use of these two quaternions, coordinates and conjugate moments, naturally leads to a formulation of rigid-body rotational dynamics in terms of a system of eight coupled first-order differential equations involving the four Euler parameters and the four conjugate momenta. The equations are formally simple, easy to handle and free of singularities. Furthermore, integration is fast, since only arithmetic operations are involved.

Keywords Attitude dynamics · Rigid body motion · Rotations · Quaternions

1 Introduction

The rotational dynamics of a rigid body is an initial value problem. For a given orientation and its change rate with respect to an inertial frame at an initial time and the force acting on it, find its attitude at any instant.

Usually, attitude representation is given by means of three angles and their derivatives (or conjugate moments), although there are many possibilities to represent the attitude (see for instance the complete survey by Shuster 1993).

It is well known that three angles cannot afford a regular representation of the rotation group SO(3), since there are singularities. Euler proposed a solution to circumvent this problem by introducing a set of four quantities, the so-called *Euler parameters*, based on relations among the above-mentioned three angles. Later on,

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Hamilton (1844) invented the *quaternions*, an extension of the complex numbers $\mathbb{C}$, and soon after, it was discovered that rotations may be represented by quaternions. Names like Rodrigues, Cayley, Klein, Whittaker, etc. are attached to quaternion representation (see again Shuster 1993 for a historical description of the topic).

In Hamiltonian formalism, the increase of the dimension is quite exceptional. Perhaps, the first case found when increasing the degrees of freedom by one is the K–S transformation for orbital problems. The extension was achieved in matricial form (Stiefel and Scheifele 1971). Soon after, other extensions for orbital problems were made, for instance, Burdet (1968) and Ferrándiz (1988), among others. Deprit et al. (1994) obtained the K–S transformation by means of quaternions, albeit Stiefel and Scheifele (1971) (Sect. 44, p. 286) claimed that "a transfer from matrices to quaternions would lead to failure or at least to a very unwieldy formalism." This challenge has recently been taken on by Waldvogel (2006).

With respect to the rotation of rigid bodies, Euler parameters have been the favorite variables to increase the dimension. Thus, Maciejewski (1985), Morton (1993), and Junkins and Singla (2004) obtained the Hamiltonian in terms of Euler parameters by using matrices, in a similar way to the one used by Stiefel and Scheifele (1971) to obtain the K–S transformation. A different approach was given in Cid and Sansaturio (1988) and Abad et al. (1989), where the conjugate momenta were obtained by constructing a *weakly* canonical transformation ad hoc.

In this paper, we formulate the problem in terms of quaternions from scratch. By making use of some properties of the algebra of quaternions, we show the equivalence between unit quaternions and rotations (Sect. 2); then, we build up the quaternion conjugate moment of the unit quaternion coordinate, and both the Hamiltonian function and Hamilton's equations are derived in Sect. 3. Since another set of conjugate momenta was obtained in previous papers (Cid and Sansaturio 1988; Abad et al. 1989), we reproduce (Sect. 5) the construction of these canonical variables in terms of quaternions by means of a weakly canonical transformation, showing that both sets of variables are the same. Finally (Sect. 6), some academic applications will be made. The corresponding Hamilton equations are formally simple, singularity-free, and fast to manipulate, since addition and multiplication are the only arithmetic operations involved. Applications to more realistic cases like the rotation of natural celestial bodies (Henrard 2005) or binary asteroids (Scheeres 2004) are in progress.

2 Representation of a rotation

We can think of a rotation about a fixed point O as an operator R in the space $\mathbf{E}$ of three-dimensional vectors; it is determined by its axis, defined by a unit vector $\boldsymbol{a}$, and by its amplitude ω. When it is necessary to specify these elements, the rotation is denoted as the operator $R(\omega, \boldsymbol{a})$.

Any vector $\boldsymbol{x}$ can be decomposed into the sum

$$\boldsymbol{x} = (\boldsymbol{x} \cdot \boldsymbol{a})\boldsymbol{a} + (\boldsymbol{a} \times \boldsymbol{x}) \times \boldsymbol{a}, \tag{1}$$

and since the vectors $\boldsymbol{a}$, $\boldsymbol{a} \times \boldsymbol{x}$, and $(\boldsymbol{a} \times \boldsymbol{x}) \times \boldsymbol{a}$ are mutually orthogonal, the rotation of a vector $\boldsymbol{x}$ is

$$\begin{aligned} R(\boldsymbol{x}) &= (\boldsymbol{x} \cdot \boldsymbol{a})\boldsymbol{a} + R[(\boldsymbol{a} \times \boldsymbol{x}) \times \boldsymbol{a}] \\ &= (\boldsymbol{x} \cdot \boldsymbol{a})\boldsymbol{a} + (\boldsymbol{a} \times \boldsymbol{x}) \sin\omega + [(\boldsymbol{a} \times \boldsymbol{x}) \times \boldsymbol{a}] \cos\omega \end{aligned} \tag{2}$$

equation that, by using the radial and transversal decomposition (1), can be written as the well known Euler–Rodrigues formula

$$R(\boldsymbol{x}) = \boldsymbol{x} + (\boldsymbol{a} \times \boldsymbol{x})\ \sin\omega + [\boldsymbol{a} \times (\boldsymbol{a} \times \boldsymbol{x})]\,(1 - \cos\omega). \tag{3}$$

Let us now consider the set $\mathbb{R} \times \mathbb{R}^3$ of pairs $\{a_0, \boldsymbol{a}\}$ made of a scalar a_0 and a vector $\boldsymbol{a}$. It is a real vector space with the usual operations of addition and multiplication by a real number. For any $q = \{q_0, \boldsymbol{q}\}$ and $p = \{p_0, \boldsymbol{p}\}$, the law of composition

$$(q,p) \to q\,p = \{q_0\,p_0 - \boldsymbol{q} \cdot \boldsymbol{p}, q_0\,\boldsymbol{p} + p_0\,\boldsymbol{q} + \boldsymbol{q} \times \boldsymbol{p}\}$$

is bilinear; hence it endows $\mathbb{R} \times \mathbb{R}^3$ with a structure of *algebra* (see Deprit et al. 1994 for details). The set $\mathbb{R} \times \mathbb{R}^3$ endowed with that structure is designated as $\mathcal{Q}$, its elements are called *quaternions*, and the mapping $(q,p) \to q\,p$ is called the product of q to the right by p (or the product of p to the left by q). The product of quaternions is manifestly associative; it is not commutative, however, since it involves a cross product.

The components q_0 and $\boldsymbol{q}$ of a quaternion $q = (q_0, \boldsymbol{q})$ are referred to as its *real* part $\Re(q)$ and its *imaginary* part $\Im(q)$, respectively.

The quaternion $(q_0, -\boldsymbol{q})$ is called the *conjugate* of $q = (q_0, \boldsymbol{q})$; it is denoted as $\tilde{q}$. The mapping $q \to \tilde{q}$ is an automorphism of the vector space $\mathcal{Q}$, but an antiautomorphism of the algebra structure since $\widetilde{p\,q} = \tilde{q}\,\tilde{p}$ for any p and q.

The application $(p, q) \to \Re(p\,q) : \mathcal{Q} \times \mathcal{Q} \to \mathbb{R}$ is a bilinear form; it is called the *dot product* of quaternions p and q and is denoted as $p \cdot q$. It is not degenerate, i.e., if $p \cdot q = 0$ for any $q \in \mathcal{Q}$, then $p = 0$.

Because $q\,\tilde{q} = \tilde{q}\,q = (\|\Re(q)\|^2 + \|\Im(q)\|^2, 0) = \|\Re(q)\|^2 + \|\Im(q)\|^2$ is the sum of two positive numbers, it makes sense to define the *norm* of a quaternion as the scalar $\|q\| = \sqrt{q\,\tilde{q}}$. Evidently, $q = 0$ if and only if $\|q\| = 0$; furthermore, for any p and $q \in \mathcal{Q}$,

$$\|p\,q\|^2 = (p\,q)\,\widetilde{(p\,q)} = (p\,q)\,(\tilde{q}\,\tilde{p}) = p\,(q\,\tilde{q})\,\tilde{p} = (p\,\tilde{p})\,\|q\|^2 = \|p\|^2\,\|q\|^2,$$

which means that the norm $q \to \|q\| : \mathcal{Q} \to \mathbb{R}^+$ makes of $\mathcal{Q}$ a normed algebra.

Of a quaternion q such that $\|q\| = 1$, it is said that it is a *unit* quaternion. Let $\mathcal{S}^3(\mathbb{R})$ denote the set of all unit quaternions. It is not empty: $1 \in \mathcal{S}^3(\mathbb{R})$. If $q \in \mathcal{S}^3(\mathbb{R})$, so does $\tilde{q}$; in fact, $\tilde{q}$ is the inverse of q. If p and $q \in \mathcal{S}^3(\mathbb{R})$, so does their product $p\,q$. In sum, the product of quaternions endows $\mathcal{S}^3(\mathbb{R})$ with the structure of a group. It should be noted that the group $\mathcal{S}^3(\mathbb{R})$ is not commutative.

Theorem 1 *Let q be the quaternion $(\cos\omega/2, \boldsymbol{a}\ \sin\ \omega/2)$ where $\boldsymbol{a}$ is a unit vector; for any $\boldsymbol{x} \in \mathbb{R}^3$, the product $q\boldsymbol{x}\tilde{q}$ belongs to $\mathbb{R}^3$. Moreover, the application $\boldsymbol{x} \to q\boldsymbol{x}\tilde{q} : \mathbb{R}^3 \to \mathbb{R}^3$ is identical to the rotation $\mathcal{R}(\omega, \boldsymbol{a})$.*

The theorem is proved by calculating that

$$q\boldsymbol{x} = \Big((-\boldsymbol{a} \cdot \boldsymbol{x})\ \sin\tfrac{1}{2}\,\omega, \boldsymbol{x}\ \cos\tfrac{1}{2}\,\omega + (\boldsymbol{a} \times \boldsymbol{x})\ \sin\tfrac{1}{2}\,\omega\Big),$$
$$q\boldsymbol{x}\tilde{q} = \Big(0, (\boldsymbol{a} \cdot \boldsymbol{x})\,\boldsymbol{a} + (\boldsymbol{a} \times \boldsymbol{x})\ \sin\omega + (\boldsymbol{a} \times \boldsymbol{x}) \times \boldsymbol{a}\ \cos\omega\Big)$$

that coincides with the expression (2) of a rotation, and that is the equivalent of Euler's formula in quaternion form.

Proposition 1 *If R is a time depending rotation, then the instantaneous angular velocity is the vector*

$$\boldsymbol{\Omega} = 2\dot{q}\tilde{q}.$$

Proof Let q the quaternion equivalent to the rotation R, that is, $\boldsymbol{y} = q\boldsymbol{x}\tilde{q}$, hence, $\boldsymbol{x} = \tilde{q}\boldsymbol{y}q$. The time derivative of $\boldsymbol{y}$ is

$$\dot{\boldsymbol{y}} = \dot{q}\boldsymbol{x}\tilde{q} + q\boldsymbol{x}\dot{\tilde{q}} = \dot{q}(\tilde{q}\boldsymbol{y}q)\tilde{q} + q(\tilde{q}\boldsymbol{y}q)\dot{\tilde{q}} = \dot{q}\tilde{q}\boldsymbol{y} + \boldsymbol{y}q\dot{\tilde{q}} = (\dot{q}\tilde{q})\boldsymbol{y} - \widetilde{(\dot{q}\tilde{q})}\boldsymbol{y},$$

since q is a unit quaternion. Moreover, by this reason, $\dot{q}\tilde{q} + q\dot{\tilde{q}} = 0$, hence $\Re(\dot{q}\tilde{q}) = 0$, that is, it is a vector. On the other hand, for any two quaternions a and b, there results that

$$ab - \tilde{b}\tilde{a} = (0, 2a_0\boldsymbol{b} + 2b_0\boldsymbol{a} + 2\boldsymbol{a} \times \boldsymbol{b}),$$

but if both quaternions are vectors, then,

$$ab - \tilde{b}\tilde{a} = (0, 2\boldsymbol{a} \times \boldsymbol{b}),$$

hence, $\dot{\boldsymbol{y}} = 2(\dot{q}\tilde{q})\,\boldsymbol{y} = 2(\dot{q}\tilde{q}) \times \boldsymbol{y} = \boldsymbol{\Omega} \times \boldsymbol{y}$. □

Incidentally, let us point out that the relation between the traditional way of representing a rotation by its orthogonal matrix $R = (r_{ij}) \in \mathrm{SO}(3)$ and its quaternion $p = (p_0, \boldsymbol{p})$ is simple. Indeed, we only need to remember that the ith row of R is made of the components of $R(\boldsymbol{e}_i)$ in the orthonormal canonical basis $\mathcal{E} = (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$. Thus, for instance,

$$R(\boldsymbol{e}_1) = p\boldsymbol{e}_1\tilde{p} = \boldsymbol{e}_1 + 2p_0(\boldsymbol{p} \times \boldsymbol{e}_1) + 2\boldsymbol{p} \times (\boldsymbol{p} \times \boldsymbol{e}_1).$$

Circular permutations of the indices $\{1, 2, 3\}$ in the preceding formula yields the last two rows in the matrix below

$$R = \begin{pmatrix} p_1^2 - p_2^2 - p_3^2 + p_0^2 & 2(p_1p_2 + p_0p_3) & 2(p_1p_3 - p_0p_2) \\ 2(p_2p_1 - p_3p_0) & p_2^2 - p_3^2 - p_1^2 + p_0^2 & 2(p_2p_3 + p_1p_0) \\ 2(p_3p_1 + p_2p_0) & 2(p_3p_2 - p_1p_0) & p_3^2 - p_1^2 - p_2^2 + p_0^2 \end{pmatrix}.$$

The inverse procedure, i.e., the obtaining of the quaternion for a rotation matrix given, in principle, is easy to obtain. Indeed, from the above matrix one deduces that

$$4p_0^2 = r_{11} + r_{22} + r_{33} + 1, \qquad 4p_1^2 = r_{11} - r_{22} - r_{33} + 1,$$
$$4p_2^2 = r_{22} - r_{33} - r_{11} + 1, \qquad 4p_3^2 = r_{33} - r_{11} - r_{22} + 1$$

an thus, it is possible to calculate the axis and amplitude of the rotation. Other ways to solve the problem of extracting the quaternion from the direction-cosine matrix are given in Shuster and Natanson (1993).

3 Hamiltonian in canonical quaternion variables

To study the rotation of a rigid body about a point O, we use two frames: one inertial or fixed in space $\mathcal{S} \equiv O\boldsymbol{s}_1\boldsymbol{s}_2\boldsymbol{s}_3$, and another fixed in the body $\mathcal{B} \equiv O\boldsymbol{b}_1\boldsymbol{b}_2\boldsymbol{b}_3$. In most of the cases, it is quite convenient to choose this body frame to be the principal axes of inertia. The attitude of $\mathcal{B}$ in $\mathcal{S}$ results from one rotation, which usually is decomposed into three rotations of the basis of the group of rotations SO(3), as we shall see in the next section.

Let us assume $R(\omega, \boldsymbol{a})$ is such rotation. Then, the quaternion $q = (q_0, \boldsymbol{q})$ defined in Theorem 1 is a unit quaternion that represents the rotation. Its components

$$q_0 = \cos(\omega/2), \qquad \boldsymbol{q} = \boldsymbol{a}\,\sin(\omega/2)$$

are referred to as *Euler parameters*.

Proposition 2 *The instantaneous angular velocity, expressed in the body frame* $\mathcal{B}$ *is the vector*

$$\boldsymbol{\omega} = 2\tilde{q}\dot{q}. \tag{4}$$

Proof By proposition 1, the instantaneous angular velocity in the fixed frame is $\boldsymbol{\Omega} = 2\dot{q}\tilde{q}$. This vector, expressed in the body frame is $\boldsymbol{\omega} = \tilde{q}\,\boldsymbol{\Omega}\,q = 2\tilde{q}\dot{q}$. □

The kinetic energy of a rotating rigid body is

$$T = \frac{1}{2}\boldsymbol{\omega}.\mathbb{I}\,\boldsymbol{\omega},$$

where $\mathbb{I}$ is the tensor of principal moments of inertia. This expression may be considered as an expression involving quaternions. Indeed, let us consider the application $(\lambda, \mu) \to \lambda \otimes \mu : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}^4$ defined by

$$\lambda \otimes \mu = (\lambda_0\mu_0, \lambda_1\mu_1, \lambda_2\mu_2, \lambda_3\mu_3);$$

this operation is obviously commutative, and the inverse λ^{-1} of λ is $\lambda^{-1} = (1/\lambda_0, 1/\lambda_1, 1/\lambda_2, 1/\lambda_3)$.

Let us now define $\mathbf{I} = (I_0, I_1, I_2, I_3) \in \mathbb{R}^4$. Note that we introduced a new quantity I_0, which has dimensions of a moment of inertia, although its value is irrelevant (Morton 1993) since it is multiplying a vector, that is, a quaternion with real part null.

In terms of quaternions, and taking into account (4), the kinetic energy is

$$T = 2(\tilde{q}\dot{q}) \cdot (\mathbf{I} \otimes \tilde{q}\dot{q}). \tag{5}$$

Straight computations lead us to the following

Proposition 3 *Let a and b two quaternions, and let F be the function $F(a, b) = (ab) \cdot (\mathbf{I} \otimes ab)$, then*

$$\frac{\partial F}{\partial b} = 2\tilde{a}(\mathbf{I} \otimes ab).$$

With this result, Q, the conjugate moment of the quaternion q is

$$Q = \frac{\partial T}{\partial \dot{q}} = 4q(\mathbf{I} \otimes \tilde{q}\dot{q}).$$

This expression is easily inverted, resulting in

$$\dot{q} = \frac{1}{4}q\left(\mathbf{I}^{-1} \otimes \tilde{q}Q\right),$$

hence the kinetic energy is

$$T = \frac{1}{8}(\mathbf{I}^{-1} \otimes \tilde{q}Q) \cdot (\tilde{q}Q), \qquad \text{or} \qquad T = \frac{1}{8}(\mathbf{I}^{-1} \otimes \tilde{Q}q) \cdot (\tilde{Q}q). \tag{6}$$

If the rigid body is under the influence of a potential field $U(q)$, the Hamiltonian is $H = T + U$, and the equations of motion are

$$\begin{aligned} \dot{Q} &= -\frac{\partial H}{\partial q} = -\frac{1}{4} Q(\mathbf{I}^{-1} \otimes \tilde{Q} q) - \frac{\partial U}{\partial q}, \\ \dot{q} &= \frac{\partial H}{\partial Q} = \frac{1}{4} q(\mathbf{I}^{-1} \otimes \tilde{q} Q). \end{aligned} \tag{7}$$

An equivalent matricial formulation was derived by Morton (1993).

4 Hamiltonian in canonical Euler variables

The most usual way to represent rotations about a fixed point in astrodynamics and mechanics is by means of the so-called *Euler angles* (ϕ, ϑ, ψ), because the rotation matrix is defined in terms of three consecutive rotations. There are 12 possible sets of Euler angles, six symmetric sets

$$\begin{array}{ccc} 1-2-1 & 1-3-1 & 2-3-2, \\ 2-1-2 & 3-1-3 & 3-2-3 \end{array}$$

and six asymmetric sets

$$\begin{array}{ccc} 1-2-3 & 1-3-2 & 2-3-1, \\ 2-1-3 & 3-1-2 & 3-2-1. \end{array}$$

All of them have the disadvantage of presenting a polar-like singularity, but are otherwise simple and convenient. In this paper, we will use the sequence $3-1-3$.

Let us now to proceed to establish the relation between the inertial and mobile frames through the Euler angles (ϕ, ϑ, ψ). We define the inclination angle ϑ of the plane $\boldsymbol{b}_1\boldsymbol{b}_2$ and the plane $\boldsymbol{s}_1\boldsymbol{s}_2$, such that:

$$\boldsymbol{s}_3 \cdot \boldsymbol{b}_3 = \cos\vartheta, \quad 0 \le \vartheta \le \pi.$$

The node vector $\boldsymbol{\ell}$ of plane $\boldsymbol{b}_1\boldsymbol{b}_2$ on plane $\boldsymbol{s}_1\boldsymbol{s}_2$ is obtained by the relation:

$$\boldsymbol{s}_3 \times \boldsymbol{b}_3 = \boldsymbol{\ell} \sin\vartheta$$

(defined only if $\vartheta \neq 0$) and its longitude in $\boldsymbol{s}_1\boldsymbol{s}_2$ is given by the angle ϕ, reckoned from $\boldsymbol{s}_1$,

$$\boldsymbol{\ell} = \boldsymbol{s}_1 \cos\phi + \boldsymbol{s}_2 \sin\phi, \quad 0 \le \phi < 2\pi.$$

Finally, the angle of proper rotation is defined by the relation

$$\boldsymbol{b}_1 = \boldsymbol{\ell} \cos\psi + (\boldsymbol{s}_3 \times \boldsymbol{\ell}) \sin\psi, \quad 0 \le \psi < 2\pi.$$

Thus, the relation between the inertial and body frames is a composition of three rotations

$$R(\psi, \boldsymbol{b}_3)\, R(\vartheta, \boldsymbol{\ell})\, R(\phi, \boldsymbol{s}_3).$$

Note that when $\boldsymbol{b}_3 = \boldsymbol{s}_3$, then $\vartheta = 0$; when $\boldsymbol{b}_3 = -\boldsymbol{s}_3$, then $\vartheta = \pi$. In both cases, the node $\boldsymbol{\ell}$, is an arbitrary unit vector in the plane spanned by $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$. We set aside these singular cases by assuming the three vectors $\boldsymbol{s}_3$, $\boldsymbol{\ell}$, and $\boldsymbol{b}_3$ are linearly independent, and thus, can be considered as a non-orthogonal base. It can be shown (Deprit and

Elipe 1993) that the conjugate moments (Φ, Θ, Ψ) of the Euler angles (ϕ, ϑ, ψ) are precisely the components of the angular momentum vector $\boldsymbol{G}$ on this base, that is,

$$\Phi = \boldsymbol{G} \cdot \boldsymbol{s}_3, \qquad \Theta = \boldsymbol{G} \cdot \boldsymbol{\ell}, \qquad \Psi = \boldsymbol{G} \cdot \boldsymbol{b}_3.$$

With this, it is a matter of some algebra to find that the components of the angular momentum in the body frame are

$$\begin{aligned} g_1 &= \left(\frac{\Phi - \Psi \cos\vartheta}{\sin\vartheta}\right) \sin\psi + \Theta \cos\psi, \\ g_2 &= \left(\frac{\Phi - \Psi \cos\vartheta}{\sin\vartheta}\right) \cos\psi - \Theta \sin\psi, \\ g_3 &= \Psi \end{aligned} \tag{8}$$

and since the rotational kinetic energy is

$$T = \frac{1}{2}(I_1^{-1} g_1^2 + I_2^{-1} g_2^2 + I_3^{-1} g_3^2) \tag{9}$$

the Hamiltonian is

$$\begin{aligned} \mathcal{H} = {} & \frac{1}{2}(I_1^{-1} \sin^2\psi + I_2^{-1} \cos^2\psi) \left(\frac{\Phi - \Psi \cos\vartheta}{\sin\vartheta}\right)^2 \\ & + \frac{1}{2}\Psi^2 + \frac{1}{2}(I_1^{-1} \cos^2\psi + I_2^{-1} \sin^2\psi)\Theta^2 \\ & + (I_1^{-1} - I_2^{-1}) \left(\frac{\Phi - \Psi \cos\vartheta}{\sin\vartheta}\right) \Theta \sin\psi \cos\psi + V(\phi, \vartheta, \psi). \end{aligned} \tag{10}$$

5 Weakly canonical transformation

The use of quaternions in Hamiltonian formalism presents an additional difficulty, since quaternions extend the dimension of the Hamiltonian system. In order to do so, several solutions have been proposed, for instance, the ones given in Maciejewski (1985), Cid and Sansaturio (1988), Abad et al. (1989), and Morton (1993).

Lowering the dimension of a Hamiltonian system is a step most frequent in mechanics. Indeed, for a Hamiltonian $\mathcal{H}(\boldsymbol{p}, \boldsymbol{P})$ given in n coordinates $\boldsymbol{p}$ and n momenta $\boldsymbol{P}$, one builds a canonical transformation

$$\lambda\colon (\boldsymbol{q}, \boldsymbol{Q}) \longrightarrow (\boldsymbol{p}, \boldsymbol{P})\colon \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n \times \mathbb{R}^n$$

that will render $m < n$ coordinates ignorable, $(q_i)_{1 \le i \le m}$ in the pullback of the Hamiltonian $\mathcal{H}$. Thus, there are m integrals, namely the conjugate momenta $(Q_i)_{1 \le i \le m}$, and the problem is reduced to another one which dimension is $2m$ lower.

The reverse problem, that of increasing the dimension of a Hamiltonian system appear in orbital dynamics (Stiefel and Scheifele 1971) in connection with the K–S transformation. Assume that a transformation λ maps a domain of dimension $2(n+m)$ in the phase space $(\boldsymbol{q}, \boldsymbol{Q})$ into a domain of dimension $2n$ in the phase space $(\boldsymbol{p}, \boldsymbol{P})$. Provided that the Poisson brackets

$$\{p_i; p_j\} = \{P_i; P_j\} = 0 \quad \text{for} \quad 1 \le i \le j \le n$$

and that exists a so-called multiplier μ independent of $\boldsymbol{q}$ and $\boldsymbol{Q}$ such that

$$\{p_i; P_j\} = \mu \delta_{ij} \quad \text{for} \quad 1 \leq i \leq j \leq n$$

with δ_{ij} the Kronecker symbol, then

$$\lambda^* \boldsymbol{p}(t) = \boldsymbol{p}(\boldsymbol{q}(t), \boldsymbol{Q}(t)) \qquad \text{and} \qquad \lambda^* \boldsymbol{P}(t) = \boldsymbol{P}(\boldsymbol{q}(t), \boldsymbol{Q}(t))$$

is solution of the Hamilton equations

$$\dot{\boldsymbol{p}} = \nabla_{\boldsymbol{P}} \mathcal{H}, \qquad \dot{\boldsymbol{P}} = -\nabla_{\boldsymbol{p}} \mathcal{H},$$

whenever $(\boldsymbol{q}(t), \boldsymbol{Q}(t))$ is a solution of the system

$$\dot{\boldsymbol{q}} = \nabla_{\boldsymbol{Q}} (\lambda^* (\mu^{-1} \mathcal{H})), \qquad \dot{\boldsymbol{Q}} = -\nabla_{\boldsymbol{q}} (\lambda^* (\mu^{-1} \mathcal{H})).$$

Such a transformation is called *weakly* canonical (Deprit et al. 1994).

This result gives us a way to check whether a transformation is weakly canonical or not, but does not supply a technique for building canonical transformations capable to raise the dimension of the Hamiltonian system. This was accomplished in Cid and Sansaturio (1988), where it is stated that a given transformation $\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{p})$ raising the dimension by 1, its extension with the moments obtained by

$$Q_i = \left(\frac{\partial \boldsymbol{p}}{\partial q_i} \right) \cdot \boldsymbol{P} \qquad (1 \leq i \leq n+1) \tag{11}$$

is weakly canonical.

Coming back to the Euler sequence 3 – 1 – 3, one defines the so-called Euler parameters by the relations

$$\begin{aligned} p_0 &= \cos \frac{\vartheta}{2} \cos \frac{\phi + \psi}{2}, \qquad & p_1 &= \sin \frac{\vartheta}{2} \cos \frac{\phi - \psi}{2}, \\ p_2 &= \sin \frac{\vartheta}{2} \sin \frac{\phi - \psi}{2}, \qquad & p_3 &= \cos \frac{\vartheta}{2} \sin \frac{\phi + \psi}{2}. \end{aligned} \tag{12}$$

After some algebra, one finds that inverse transformation is

$$\begin{aligned} \tan \phi &= \frac{p_1 p_3 + p_0 p_2}{p_0 p_1 - p_2 p_3}, \\ \tan \frac{\vartheta}{2} &= \frac{\sqrt{p_1^2 + p_2^2}}{\sqrt{p_0^2 + p_3^2}}, \\ \tan \psi &= \frac{p_1 p_3 - p_0 p_2}{p_0 p_1 + p_2 p_3}. \end{aligned} \tag{13}$$

Then, we can use these relations to compute the partial derivatives of the Euler angles with respect to the Euler parameters and thus to obtain the conjugate moments to the Euler parameters by the relations (11) as

$$\begin{aligned} P_0 &= -(\Phi + \Psi) \frac{\sin{(\phi + \psi)/2}}{\cos \vartheta/2} - 2\Theta \sin \frac{\vartheta}{2} \cos \frac{\phi + \psi}{2}, \\ P_1 &= -(\Phi - \Psi) \frac{\sin{(\phi - \psi)/2}}{\sin \vartheta/2} + 2\Theta \cos \frac{\vartheta}{2} \cos \frac{\phi - \psi}{2}, \\ P_2 &= \quad (\Phi - \Psi) \frac{\cos{(\phi - \psi)/2}}{\sin \vartheta/2} + 2\Theta \cos \frac{\vartheta}{2} \sin \frac{\phi - \psi}{2}, \\ P_3 &= \quad (\Phi + \Psi) \frac{\cos{(\phi + \psi)/2}}{\cos \vartheta/2} - 2\Theta \sin \frac{\vartheta}{2} \sin \frac{\phi + \psi}{2}. \end{aligned} \tag{14}$$

With the help of a symbolic processor, one easily checks that the Poisson brackets satisfy the conditions for the transformation be weakly canonical.

Appropriate combinations of formulas (12) and (14) lead to

$$\cos\phi = \frac{p_0p_1 - p_2p_3}{\sqrt{(p_1^2+p_2^2)(p_0^2+p_3^2)}}, \qquad \sin\phi = \frac{p_1p_3 + p_0p_2}{\sqrt{(p_1^2+p_2^2)(p_0^2+p_3^2)}},$$
$$\cos\vartheta = (p_0^2+p_3^2) - (p_1^2+p_2^2), \qquad \sin\vartheta = 2\sqrt{(p_1^2+p_2^2)(p_0^2+p_3^2)},$$
$$\cos\psi = \frac{p_0p_1 + p_2p_3}{\sqrt{(p_1^2+p_2^2)(p_0^2+p_3^2)}}, \qquad \sin\psi = \frac{p_1p_3 - p_0p_2}{\sqrt{(p_1^2+p_2^2)(p_0^2+p_3^2)}},$$
$$\Phi + \Psi = p_0P_3 - p_3P_0, \qquad \Phi - \Psi = p_1P_2 - p_2P_1,$$
$$\Theta\sin\vartheta = p_1P_1 + p_2P_2$$

and from these expressions, one readily finds that

$$\begin{aligned}
\Phi &= \frac{1}{2}(-p_3P_0 - p_2P_1 + p_1P_2 + p_0P_3),\\
\Theta &= \frac{(-p_0P_0 + p_1P_1 + p_2P_2 - p_3P_3)}{4\sqrt{p_1^2+p_2^2}\sqrt{p_0^2+p_3^2}},\\
\Psi &= \frac{1}{2}(-p_3P_0 + p_2P_1 - p_1P_2 + p_0P_3).
\end{aligned} \tag{15}$$

By replacing Eqs. 13 and 15 into the expressions of the components of the angular momentum (8), and after some algebraic simplifications, there results that

$$\begin{aligned}
g_1 &= \frac{1}{2}(-p_1P_0 + p_0P_1 + p_3P_2 - p_2P_3),\\
g_2 &= \frac{1}{2}(-p_2P_0 - p_3P_1 + p_0P_2 + p_1P_3),\\
g_3 &= \frac{1}{2}(-p_3P_0 + p_2P_1 - p_1P_2 + p_0P_3).
\end{aligned}$$

On the other hand, one easily checks that

$$p_0^2 + p_1^2 + p_2^2 + p_3^2 = 1 \tag{16}$$

and also that

$$p_0P_0 + p_1P_1 + p_2P_2 + p_3P_3 = 0.$$

In other words, Euler parameters are on the $\mathcal{S}^3$ sphere, and their conjugate moments belong to the tangent plane to the sphere.

The above relations suggest to use the algebra of quaternions. Thus, we can define the unit quaternion $p = (p_0, \boldsymbol{q}) = (p_0, p_1, p_2, p_3)$, and $P = (P_0, \boldsymbol{P}) = (P_0, P_1, P_2, P_3)$. By means of these quaternions, there results that the product $\tilde{p}P$ is a pure vector, which is precisely twice the angular moment vector expressed in the body frame, that is,

$$\boldsymbol{g} = \frac{1}{2}\tilde{p}P. \tag{17}$$

Incidentally, let us point out that the angular moment vector $\boldsymbol{G}$ in the space frame is

$$\boldsymbol{G} = p\boldsymbol{g}\tilde{p} = \frac{1}{2}P\tilde{p} = \boldsymbol{s}_1 G_1 + \boldsymbol{s}_2 G_2 + \boldsymbol{s}_3 G_3,$$

hence,

$$\begin{aligned}
G_1 &= \frac{1}{2}(-p_1 P_0 + p_0 P_1 - p_3 P_2 + p_2 P_3),\\
G_2 &= \frac{1}{2}(-p_2 P_0 + p_3 P_1 + p_0 P_2 - p_1 P_3),\\
G_3 &= \frac{1}{2}(-p_3 P_0 - p_2 P_1 + p_1 P_2 + p_0 P_3).
\end{aligned}$$

Taking into account expression (17) of the angular momentum vector in the body frame, the kinetic energy (9) is

$$T = \frac{1}{8}(\mathbf{I}^{-1} \otimes \tilde{p}P) \cdot (\tilde{p}P) \qquad \text{or} \qquad T = \frac{1}{8}(\mathbf{I}^{-1} \otimes \tilde{P}p) \cdot (\tilde{P}p)$$

expressions that coincide with the ones in (6).

So far, we obtained two sets of canonical variables for the rigid body motion based on quaternions. The first one (q, Q) derived directly from the Lagrangian in Sect. 3 and the second one (p, P) built by means of a *weakly* canonical transformation in Sect. 5. We just proved that both set of quaternions are the same.

6 Applications to gravity gradients in a Newtonian force field

As applications of the above-exposed theory, we will proceed to numerically integrate Hamilton's equations for some examples, namely, the rigid body in torque free motion, the heavy top and the attitude of a rigid body in a Keplerian orbit.

6.1 Heavy rigid body

Let us consider the motion of a heavy rigid body about a fixed point O. We will assume that the fixed point O is at the origin, and that position $\boldsymbol{x}^c$ of the center of mass of the body frame is

$$\boldsymbol{x}^c = x_1^c \boldsymbol{b}_1 + x_2^c \boldsymbol{b}_2 + x_3^c \boldsymbol{b}_3.$$

Now, let the direction of the acceleration of gravity along the $\boldsymbol{s}_3$-axis, then the force $\boldsymbol{F}$ is

$$\boldsymbol{F}\Big|_{\mathcal{S}} = -Mg\boldsymbol{s}_3,$$

where M is the total mass of the body and g is the gravity acceleration. Hence, the force in the body frame is

$$\begin{aligned}
\boldsymbol{F} = \boldsymbol{F}\Big|_{\mathcal{B}} &= \tilde{q}\,\boldsymbol{F}\Big|_{\mathcal{S}}\,q\\
&= -Mg(0,\, 2(q_1 q_3 - q_0 q_2),\, 2(q_0 q_1 + q_2 q_3),\, q_0^2 - q_1^2 - q_2^2 + q_3^2)
\end{aligned}$$

and the potential is

$$U = \boldsymbol{x}^c \cdot \boldsymbol{F}, \tag{18}$$

which have to be added to the unperturbed Hamiltonian (6), obtaining in this way, the total Hamiltonian in terms of quaternions and the equations of motion (7).

In the case here considered, we took as dimensionless principal moments of inertia $(I_1, I_2, I_3) = (1.25, 1.0, 0.75)$, and initial conditions

$$q(t_0) = (0.5, -1/\sqrt{2}, 0, 0.5), \qquad Q(t_0) = (0.3, -0.848528, 0.141421, -1.5)$$

or equivalently in Euler variables

$$(\phi, \vartheta, \psi)(t_0) = (\pi/4, 3\pi/2, \pi/4), \qquad (\Phi, \Theta, \Psi)(t_0) = (0.5, 0.6, 0.4).$$

Note that when $\boldsymbol{x}^c = 0$ we have the rigid body in torque free rotation. This problem is integrated numerically with the above initial conditions, and the evolution of the two quaternions, coordinates q and moments Q appears in Fig. 1.

For the numerical integration of the equations of motion of the heavy top, we choose

$$\boldsymbol{x}^c = (1, 0, 0), \quad Mg = 0.5.$$

In Fig. 2 we present the evolution of q and Q.

6.2 Rigid body in a Keplerian orbit

Let us now consider the attitude of a rigid body moving on an orbit around a central spherical body (the Earth). We also make the assumption that both mass and dimensions of the satellite are much smaller that the position vector and the central

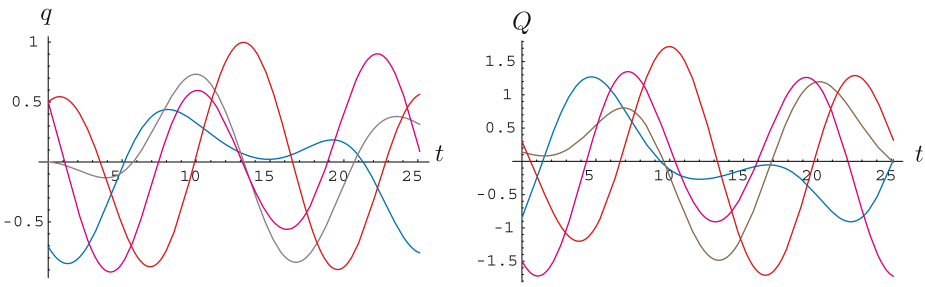

Fig. 1 Torque free rotation: time evolution of the components of quaternions q (*left*) and Q (*right*)

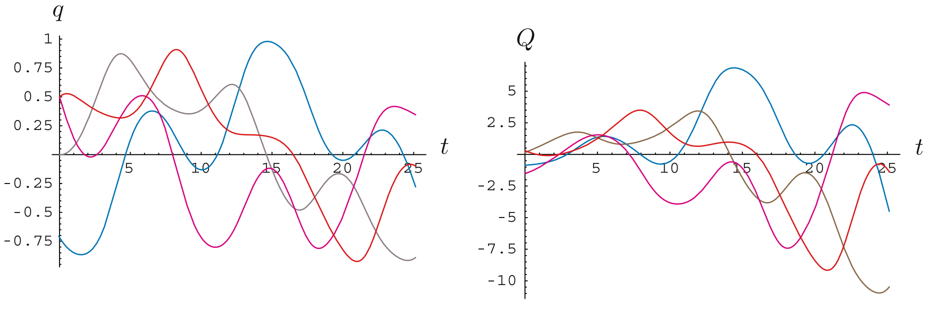

Fig. 2 Heavy rigid body: time evolution of the components of quaternions q (*left*) and Q (*right*)

body mass; consequently, we are assuming that satellite's rotation have no influence on the orbit, but not the converse.

The potential acting on the satellite, attracted by the Earth, may be expressed by the MacCullagh formula (Danby 1992)

$$U = -GM\left[\frac{m}{r} + \frac{1}{2r^3}(I_1 + I_2 + I_3 - 3J)\right], \tag{19}$$

where $r = \|\boldsymbol{x}\|$ is the radial distance, M the central body mass, m the satellite mass, and J is the moment of inertia of the satellite about the line joining both centres of masses.

Let us denote by $\hat{\boldsymbol{x}} = \boldsymbol{x}/r$ the unit vector in the radial direction expressed in the body frame $\mathcal{B}$. Then, $J = \hat{\boldsymbol{x}} \cdot \mathbf{I} \otimes \hat{\boldsymbol{x}}$.

By using the orbital elements: node Ω, inclination i, and $\omega + f$, we define the quaternion p representing the rotation from the space frame $\mathcal{S}$ to the orbital frame as

$$p_0 = \cos\frac{i}{2}\cos\frac{\Omega + (\omega + f)}{2}, \qquad p_1 = \sin\frac{i}{2}\cos\frac{\Omega - (\omega + f)}{2},$$
$$p_2 = \sin\frac{i}{2}\sin\frac{\Omega - (\omega + f)}{2}, \qquad p_3 = \cos\frac{i}{2}\sin\frac{\Omega + (\omega + f)}{2}.$$

Then, in the space frame the vector $\hat{\boldsymbol{x}}$ is

$$\hat{\boldsymbol{x}}\Big|_{\mathcal{S}} = p\,\boldsymbol{s}_1\,\tilde{p}$$

and thus,

$$\hat{\boldsymbol{x}}\Big|_{\mathcal{B}} = \tilde{q}\,\hat{\boldsymbol{x}}\Big|_{\mathcal{S}}\,q = \tilde{q}\,(p\,\boldsymbol{s}_1\,\tilde{p})\,q,$$

expression that has to be replaced in

$$J = \hat{\boldsymbol{x}} \cdot \mathbf{I} \otimes \hat{\boldsymbol{x}}.$$

Since the quaternion p and r are known functions of t, and in the Hamilton equations we only need the partial derivatives with respect to the quaternion q, we can drop from the potential (19) those additive terms independent of q, resulting into

$$U = \frac{3GM}{2r^3}\Big(\tilde{q}\,(p\,\boldsymbol{s}_1\,\tilde{p})\,q\Big) \cdot \mathbf{I} \otimes \Big(\tilde{q}\,(p\,\boldsymbol{s}_1\,\tilde{p})\,q\Big). \tag{20}$$

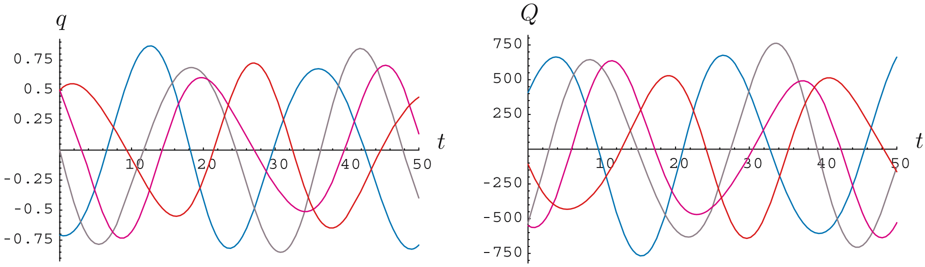

Fig. 3 Rigid body in Keplerian orbit: time evolution of the components of quaternions q (*left*) and Q (*right*)

For the Keplerian orbit, we take an elliptic orbit with initial conditions

$$a = 7,000\,\text{km}, \qquad e = 0.2, \qquad i = \pi/6, \qquad \Omega = \pi/4, \qquad \omega = \pi/12, \qquad f(t_0) = 0$$

and for the moments of inertia of the satellite we take

$$I_1 = 400\,\text{kg/km}^2, \qquad I_2 = 500\,\text{kg/km}^2, \qquad I_3 = 600\,\text{kg/km}^2.$$

The initial conditions for the rotation quaternions are the same as in the above example. The results of the integration appear in Fig. 3.

7 Conclusions

We derived the Hamiltonian function of the attitude dynamics of a rigid body under external torques in terms of quaternions by two different ways. The corresponding Hamilton equations are formally simple, they are singularity-free and fast to manipulate, since addition, and multiplication are the only arithmetic operations involved.

Acknowledgments We are deeply indebted to Professor André Deprit, whose seminars at the University of Zaragoza motivated our interest in quaternions. Supported by the Spanish Ministry of Science and Technology (Projects nos ESP2005-07107 and BFM2003-02137).

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