

**Hand-in Exercises 1, TATA49 Geometry with Applications. Fall 2024**

Maple, Python or Matlab may/should be used in the calculations.

The exercises are customized by your birthdays as follows:  $p_1, p_2$  is the month of your birthday, where January is 01, February 02, ..., and December 12.  $q_1, q_2$  is the date in the month, between 13 and 29, nearest to your birthday.  $r_1, r_2$  are the two last digits in the year you were born. However, if you were born in 1990, then  $r_1, r_2 = 9, 1$ ; if you are born in 200*i*  $r_1 = 5 + i, r_2 = 4 + i$ .

**Exercise 1** Determine the affine transformation that takes the line  $l[1, 0, 1]$  to  $l'[p_1 + 4, 2p_2, 1]$ ; the line  $q[3, 4, 1]$  to  $q'[p_1 + 5, p_2 + q_2, 1]$ , and  $r[1, -1, 1]$  to  $r'[r_2 + p_1 + q_1, r_1 + p_2 + q_2, 1]$ . Decompose it as a product of an isometry, a dilation, a strain and a shear. Is there any fixed point or line by the transformation?

**Exercise 2 a** Determine the image of the equilateral hyperbola with equation  $x_1x_2 = 1$  under the affine transformation with matrix

$$\begin{pmatrix} r_1 + q_1 + q_2 & r_1 + p_1 & q_1 + p_1 \\ r_1 + p_1 & r_1 + p_2 & p_2 + q_2 \\ 0 & 0 & 1 \end{pmatrix}. \text{ Determine the eccentricities.}$$

**b** Show that a similarity transforms an ellipse, hyperbola respectively in another ellipse (hyperbola) of the same eccentricity. Recall that if a hyperbola is defined by a symmetric matrix  $C$  with eigenvalues  $\lambda_1 > 0$  and  $\lambda_2 < 0$ , then the eccentricity  $e$  is defined by  $e = \sqrt{1 + \frac{\lambda_1}{|\lambda_2|}}$ . The eccentricity of an ellipse is given by  $e = \sqrt{1 - \frac{\lambda_1}{|\lambda_2|}}$ , where  $0 < \lambda_1 < \lambda_2$ .

**Exercise 3** Determine the isometry that takes the points  $P(p_1, p_2)$  and  $Q(q_1, q_2)$  to  $P'(r_1, r_2)$  and  $Q'(r_1 + \frac{p_1\sqrt{(p_1-q_1)^2+(p_2-q_2)^2}}{\sqrt{p_1^2+p_2^2}}, r_2 + \frac{p_2\sqrt{(p_1-q_1)^2+(p_2-q_2)^2}}{\sqrt{p_1^2+p_2^2}})$  respectively. Which kind of isometry is it? **The** isometry?

**Exercise 4** Consider the square with sides on the lines  $l[2, -1, 0], m[1, 2, 0], n[2, -1, r_1], r[1, 2, r_1]$

**a)** Verify that the lines above determine a square.

**b)** Determine the sides, vertices and inner angles of the image of the square

in **a)** under the affine transformation with matrix  $\begin{pmatrix} r_1 + q_1 + q_2 & r_1 + p_1 & q_1 + p_1 \\ r_1 + p_1 & r_1 + p_2 & p_2 + q_2 \\ 0 & 0 & 1 \end{pmatrix}$ .

**Exercise 5** Determine the images of the points and lines in Exercise 4 above under the affine transformation given in Exercise 1

**Exercise 6** Show that an isometry  $f$ , not the identity, which satisfies that  $f^2 = 1_d$  is a reflection or a rotation with angle  $\pi$ .

**Exercise 7** Determine all the indirect isometries that fix each of the lines  $l_a[r_1 + p_1, r_2 + q_2, ar_2], 0 \leq a$ . Describe them. Are the lines concurrent?

**Exercise 8** Identify the indirect isometries taking the point  $P(p_1, p_2)$  to the point  $P'(r_1 + q_1, r_2 + q_2)$ . Determine the geometric locus formed by the axis of the isometries.

**Exercise 9** Identify the direct isometries that take the point  $P(p_1, p_2)$  to the point  $P'(r_1 + q_1, r_2 + q_2)$ . Determine the geometric locus formed by the centres of the rotations.

Compare geometrically the geometric loci given in Exercises 8 and 9.

**Exercise 10 a)** Show that any direct isometry  $T$  of the plane can be expressed in terms of complex numbers as  $T(z) = az + v$ , where  $a = (\cos(\theta), \sin(\theta))$ ,  $v = v_1 + iv_2$ ,  $z = x + iy$ .

**b)** Give the matrix of the isometry above.

**c)** Let  $T'$  be another direct isometry given by  $T'(z) = az + w$ , where  $a = (\cos(\phi), \sin(\phi))$ ,  $w = w_1 + iw_2$ . show that  $TT'T^{-1}T'^{-1}$  is a translation. Which is the translation vector?

**Exercise 11** Determine the similarity that takes the points  $P(r_2 + p_1, r_1 + p_2)$ ,  $Q(r_2 + q_2, r_1 + q_2)$  to the points  $P'(p_1 + q_2 + r_1, p_2 + q_1 + r_2)$  and  $Q'(r_1 + p_1, r_2 + q_1)$  respectively. What is the ratio of the similarity?

**Exercise 12** Give the fractal formed by the union images of  $\{(0, 0)\}$  under a sequence of compositions of these three similarity contractions:  $f_1$  is a dilation of ratio  $\frac{2}{5}$  followed by a rotation of angle  $\frac{2\pi}{r_2 + q_1}$ , and a translation of vector  $\mathbf{v}_1 = (0, 0)$ ;  $f_2$  is a dilation of ratio  $\frac{3}{5}$  followed by a rotation of angle  $\frac{2\pi}{r_2 + 2}$ , and a translation of vector  $\mathbf{v}_1 = (1, 0)$ , and  $f_3$  is a dilation of ratio  $\frac{1}{2}$  followed by a rotation of angle  $\frac{2\pi}{r_1 + p_1}$ , and a translation of vector  $\mathbf{v}_1 = (0, 1)$ . The probabilities for the appearances of  $f_1, f_2, f_3$  are 0.4, 0.4 and 0.2 respectively. You may use the maple commands given in the examples of fractals.

**Exercise 13** Consider the affine transformation  $T$  with matrix

$$\begin{pmatrix} \frac{q_1}{\sqrt{q_1^2 + q_2^2}} & \frac{q_2}{\sqrt{q_1^2 + q_2^2}} & \frac{r_1(\sqrt{q_1^2 + q_2^2} - q_1) - r_2 q_2}{\sqrt{q_1^2 + q_2^2}} \\ \frac{q_2}{\sqrt{q_1^2 + q_2^2}} & \frac{-q_1}{\sqrt{q_1^2 + q_2^2}} & \frac{r_2(\sqrt{q_1^2 + q_2^2} + q_1) - r_1 q_2}{\sqrt{q_1^2 + q_2^2}} \\ 0 & 0 & 1 \end{pmatrix}. \text{ Describe it geometrically and identify it. Is there an axis for the transformation? If existing, identify the axis}$$

**Exercise 14** 1. I Figure A below you see the arm of a planar robot. Give the function  $f$  that determines the configuration of the hand depending on the parameters of the arm.

2. With the robot arm as in the figure below, determine, if possible, the parameters for the angles at the junctions (links) the controller should get if the coordinates of the point are  $P_{(H)}(1/4, 1/4)$ , relative to the local coordinates  $(x_H, y_H)$  at the hand, and  $P_0((1 + \sqrt{2} - \sqrt{3})/4\sqrt{2}, (2 + \sqrt{2})/4\sqrt{2})$ , relative to the coordinates  $(x_0, y_0)$  for the controller, and total angle for the hand (the part of the arm with the hand) is  $7\pi/12$  rad. Observe that it could be no solution.

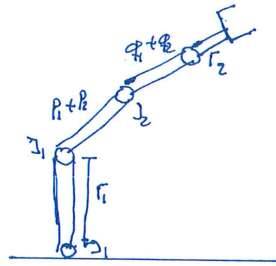


Figure A

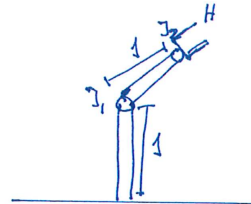


Figure B