Hand-in Exercises 2, TATA49 Geometry with Applications. Fall 2024

Maple or Matlab may be used in the calculations.

The exercises are customized by your birthdays as in the previous sheet of exercises.

Exercise 1 *Determine the images under the stereographic projection* $\varphi : \widehat{\mathbb{C}} \to \mathbb{S}^2$ *of all the lines in* $\mathbb{C}$ *with coordinates* $l_a[p_1+p_2, r_1-q_1, a]$*, with* a *a parameter.*

Exercise 2
1. *Determine the planes to which the circles in Exercise 1 belong.*
2. *Using spherical coordinates* $(\cos\lambda\cos\theta, \cos\lambda\sin\theta, \sin\lambda)$ *give equations of the circles with variable the latitude* λ.

Exercise 3 *Determine the images* $\mathcal{C}_t$ *under the stereographic projection* $\varphi : \widehat{\mathbb{C}} \to \mathbb{S}^2$ *of all the circles in* $\mathbb{C}$ *with equations* $x^2+y^2-2(x+y)t+t^2=0$*, with* $t \neq 0$ *a parameter.*

Exercise 4 *Determine the planes to which the image-circles* $\mathcal{C}_t$ *in* $\mathbb{S}^2$ *in Exercise 3 belong, and the distances from the circles in Exercise 3 to the North Pole* $N(0,0,1)$

Exercise 5
1. *Calculate the angles in the spherical triangle with vertices* A*(longitude* $(r_1-2)p_2$ *deg, latitude 0 deg),* B*(longitude 0 deg, latitude* q_2q_1+30 *deg) and* C*(longitude* $(r_1-2)p_2$ *deg, latitude 60 deg). Observation* $(r_1-2)p_2$ *is a number with two digits:* p_2 *in the units and* r_1-2 *in the tens.*
2. *Calculate the area of the triangle if we consider that the triangle is on Earth (Radius of Earth is 6371 km).*

Exercise 6
1. *Determine the great circles that support the sides of the spherical triangle in exercise 5 and their images under the inverse of the inverse of the stereographic projection.*
2. *Determine the sizes of the edges of the spherical triangle (on Earth) in exercise 4 (Radius of Earth is 6371 km).*

Exercise 7 *Consider* $\mathbb{R}^4$ *with coordinates* $\mathbf{v}=(v_1,v_2,v_3,v_0)$ *(Observe that the real part of the quaternion is the fourth coordinate). Given the quaternion* $q_0 = a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}+a_0 = ((a_1,a_2,a_3),a_0)$*, show that the multiplication with* q_0 *to the left is a linear transformation in* $\mathbb{R}^4$ *with matrix* $\begin{pmatrix} a_0 & -a_3 & a_2 & a_1 \\ a_3 & a_0 & -a_1 & a_2 \\ -a_2 & a_1 & a_0 & a_3 \\ -a_1 & -a_2 & -a_3 & a_0 \end{pmatrix}$.

Determine the matrix for the multiplication with q_0 *on the right.*

Exercise 8 1. *Use Exercise 7 to determine the matrix of the rotation in* $\mathbb{E}^3$ *with quaternion* $q_0 = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} + a_0$, $|q_0| = 1$.

2. *Use Exercise 7 to determine the matrix of the reflection in* $\mathbb{E}^3$ *on the plane with equation* $\frac{x_1}{3} + \frac{2x_2}{3} - \frac{2x_3}{3} + 0(1) = 0$. *Observe that the last row and column of the matrix equal* $(0, 0, 0, 1)$.

Exercise 9 *Consider an animation of an object with start orientation (time* $t = 0$*) given by the quaternion* $q_s = (q_1 + 1, p_2 + 2, r_1 - 3, r_2)/l$ *where* $l = \sqrt{(q_1 + 1)^2 + (p_2 + 2)^2 + (r_1 - 3)^2 + (r_2)^2}$ *and final orientation (time* $t = 1$*) given by the quaternion* $q_f = (1/2, (1/2, 1/2, 1/2))$. *Give the orientations at times* $t = j/10$, $j = 0, 1, 2, \ldots, 10$

Exercise 10 *Identify and describe completely the isometry of* $\mathbb{E}^3$ *with matrix:* $\begin{pmatrix} 7/9 & 4/9 & -4/9 & -1 \\ 4/9 & 1/9 & 8/9 & 1 \\ 4/9 & -8/9 & -1/9 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. *(Hint: Study first the matrix and possible fixed elements). Decompose it as a product of reflections.*

Exercise 11 *Consider the rotation in* $\mathbb{S}^2$ *given by* $f_\alpha : \mathbb{S}^2 \to \mathbb{S}^2$, $f_\alpha(x, y, z) = (x\cos\alpha - z\sin\alpha, y, x\sin\alpha + z\cos\alpha)$, $x^2 + y^2 + z^2 = 1$.

Show that the function $f* = \varphi^{-1} f \varphi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ *(*φ *is the stereographic projection from the point* $G(0, 1, 0)$ *on the* x_1x_3*-plane) induces an isometry* T *of* $\mathbb{E}^2$. *Determine and describe this isometry.*

Exercise 12 *Let* q_1 *and* q_2 *be unit quaternions. Show that the functions* $R_1(x) = q_1 x q_1^{-1}$, $R_2(x) = q_2 x q_2^{-1}$, $x = (0, \mathbf{x})$ *are the same rotation if and only if* $q_1 = \lambda q_2$ *for some real number* $\lambda \neq 0$.

Exercise 13 *Consider the cuboid (prism)* $\mathcal{C}$ *given by* $\{(x, y, z) \in \mathbb{R}^3 \,|\, |x| \leq a, |y| \leq b, |z| \leq c\}$, *where* a, b *and* c *are distinct. Let* G *be the group of rotations of the cuboid. Show that* $\mathcal{C}$ *consists of 4 elements. Determine the matrices of these rotations as isometries of* $\mathbb{E}^3$.

Exercise 14 *Consider a cube with vertices* $O(0, 0, 0), A(1, 0, 0), B(1, 1, 0), C(0, 1, 0), O'(0, 0, 1), A'(1, 0, 1), B'(1, 1, 1)$ *and* $C'(0, 1, 1)$. *Show that the tetrahedron with vertices* O, B, C' *and* A' *is a regular tetrahedron. Determine the symmetries of the cube that fixes the tetrahedron.*