Hand-in Exercises 1, TATA49 Geometry with Applications. Fall 2025

Maple, Python or Matlab may/should be used in the calculations.

The exercises are customized by your birthdays as follows: p_1, p_2 is the month of your birthday, where January is 01, February 02,..., and December 12. q_1, q_2 is the date in the month, between 13 and 29, nearest to your birthday. r_1, r_2 are the two last digits in the year you were born. However, if you were born in 1990, then $r_1, r_2 = 9, 1$; if you are born in 200i $r_1 = 5 + i, r_2 = 4 + i$.

Exercise 1 1. *Show that the lines* $l[1, 1, 1]$, $m[r_1, r_2, p_1 + q_1]$, $n[q_2 + 4r_1, q_2 + 4r_2, q_2 + 4p_1 + 4q_1]$ *are concurrent and determine the intersection point* Int.

2. *Show that the points* $P(-1/2, -1/2, 1)$, $Q(-5, 10, 1)$, $R(-2, 3, 1)$ *are collinear and give the equation of the line* $t[t_1, t_2, t_3]$ *they belong to.*

3. *Determine the angles at* Int *formed by* l, m *and* n *(of course pairwise).*

4. *Show that there is a triangle whose sides lie on the lines* l, m, t.

Exercise 2 *Determine the affine transformation that takes the lines* l, m, t *in Exercise 1 to the lines* $l'[0, 1, 1]$, $m'[2, 0, 4]$, *respectively* $t'[-3, 3, 3]$. *Decompose it as a product of an isometry, a dilation, a strain and a shear. Is there any fixed point or line by this transformation?*

Exercise 3 a *Determine the image of the circle with equation* $(r_1^2 + r_2^2)x_1^2 + (r_1^2 + r_2^2)x_2^2 - 4x_1 - 6x_2 = 9$ *under the affine transformation given in Exercise 2. Determine the eccentricities.*

b *Show that a similarity transforms an ellipse, hyperbola respectively in another ellipse (hyperbola) of the same eccentricity. Recall that if a hyperbola is defined by a symmetric matrix* C *with eigenvalues* $\lambda_1 > 0$ *and* $\lambda_2 < 0$, *then the eccentricity* e *is defined by* $e = \sqrt{1 + \frac{\lambda_1}{|\lambda_2|}}$. *The eccentricity of an ellipse is given by* $e = \sqrt{1 - \frac{\lambda_1}{|\lambda_2|}}$, *where* $0 < \lambda_1 < \lambda_2$.

Exercise 4 *Determine the isometry that takes the points* $P(p_1, p_2)$ *and* $Q(q_1, q_2)$ *to* $P'(r_1, r_2)$ *and* $Q'(r_1 + \frac{p_1\sqrt{(p_1-q_1)^2+(p_2-q_2)^2}}{\sqrt{p_1^2+p_2^2}}, r_2 + \frac{p_2\sqrt{(p_1-q_1)^2+(p_2-q_2)^2}}{\sqrt{p_1^2+p_2^2}})$ *respectively. Which kind of isometry is it?* ***The*** *isometry?*

Exercise 5 *Consider the positiver real numbers* $\mathbb{R}^+$. *We can equipe* $\mathbb{R}^+$ *with a structure of affine space using the following direction map:* $\phi : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ *given by* $\phi(a, b) = \ln(b) - \ln(a)$ *(we denote* $\mathbf{v}_{ab} \ln(b) - \ln(a)$*). Observe that* $\mathbb{R}$ *is the 1-dim vector space over the reals*

a) *Show that we can identify* $\mathbb{R}^+$ *with the affine line* $\mathcal{R} = \{(x, 1)\}; x \in \mathbb{R}$ *using the logarithm function on the first coordinate. Verify that* $\mathbb{R}^+$ *is in fact an affine space by controlling:*

i) Fixing any $a \in \mathbb{R}^+$ *one has that for every* $b \in \mathbb{R}^+$ ***there is a unique*** $\mathbf{v}_{ab} \in \mathbb{R}$.

ii) $\forall a, b, c \in \mathbb{R}^+$ *the equality* $\phi(a,b) + \phi(b,c) = \phi(a,c)$ *holds.*

b) *Show now that the map* $f_t : \mathbb{R}^+ \to \mathbb{R}^+$, $t \in \mathbb{R}^+$ *given by* $f_t(a) = ta$ *is an affine transformation of* $\mathbb{R}^+$ *by proving that* $\ln f_t \ln^{-1} : \mathcal{R} \to \mathcal{R}$ *is an affine transformation of* $\mathcal{R}$. *Give the matrix of the affine transformation.*

Exercise 6 *Show that a direct isometry* f, *not the identity, which satisfies that* $f^2 = 1_d$ *is a rotation with angle* π.

Exercise 7 *Determine all the direct isometries that fix each of the lines* $l_a[r_1 + p_1, r_2 + q_2, ar_2]$, $0 \leq a$. *Describe these isometries if existing. Are the lines concurrent?*

Exercise 8 *Identify the indirect isometries taking the point* $P(p_1, p_2)$ *to the point* $P'(r_1 + q_1, r_2 + q_2)$. *Determine the geometric locus (the set of points) formed by the axis of the isometries.*

Exercise 9 *Identify the direct isometries that take the point* $P(p_1, p_2)$ *to the point* $P'(r_1 + q_1, r_2 + q_2)$. *Determine the geometric locus formed by the centres of the rotations.*
Compare geometrically the geometric loci given in the two previous Exercises.

Exercise 10 **a)** *Show that any direct isometry* T *of the plane can be expressed in terms of complex numbers as* $T(z) = az + v$, *where* $a = (\cos(\theta), \sin(\theta))$, $v = v_1 + iv_2$, $z = x + iy$.

b) *Give the matrix of the isometry above.*

c) *Let* T' *be another direct isometry given by* $T'(z) = az + w$, *where* $a = (\cos(\phi), \sin(\phi))$, $w = w_1 + iw_2$. *show that* $TT'T^{-1}T'^{-1}$ *is a translation. Which is the translation vector?*

Exercise 11 *Determine the similarity that takes the points* $P(r_2 + p_1, r_1 + p_2), Q(r_2 + q_2, r_1 + q_2)$ *to the points* $P'(p_1 + q_2 + r_1, p_2 + q_1 + r_2)$ *and* $Q'(r_1 + p_1, r_2 + q_1)$ *respectively. What is the ratio of the similarity?*

Exercise 12 *Consider two glide-reflections* G_m, G_n *with axes the lines* m *and* n *respectively. Show that* $T = G_n G_m$ *is a translation. Determine the translation vector of* T *in terms of the of the axes* $m[m_1, m_2, m_3]$ *and* $n[n_1, n_2, n_3]$ *and translation vectors* $\mathbf{v}_m = (\lambda_1 m_2, -\lambda_1 m_1)$, $\mathbf{v}_n = (\lambda_2 n_2, -\lambda_2 n_1)$, $\lambda_1, \lambda_2 \neq 0$ *of* G_m *and* G_n.

Exercise 13 *Consider the affine transformation* T *with matrix*

$$\begin{pmatrix} a & b & 1-a \\ b & -a & -2b \\ 0 & 0 & 1 \end{pmatrix},$$

where a *and* b *are positive real numbers such that* $a^2 + b^2 = 1$. *Show that the affine transformation* T *is a reflection and determine its axis* m, *i.e. the angle that the axis forms with the* x_1-*axis and the intersection point of* m *with the* x_1-*axis.*

Exercise 14 *1. The arm of a robot consists of two equally long links with length 30 cm. with junctions J_1 and J_2. The hand is rigidly linked to the end of the second link (the junction J_3 is rigid with angle 0 rad). The arm can move in a plane. The junctions are the origins of the corresponding coordinate systems. Determine the position and orientation of the hand of the robot, i.e. give the position of a point on the plane in terms of the angles at the junctions.*

2. *Considering the robot arm above: What are the angles at the junctions to reach a point with coordinates $(2, 2)$ in the coordinate system associated to the hand and $(-1, -1 + 30\sqrt{3} + \sqrt{3})$ in the controller coordinate system?.*