Hand-in Exercises 2, TATA49 Geometry with Applications. Fall 2025

Maple or Matlab may be used in the calculations.

The exercises are customized by your birthdays as in the previous sheet of exercises.

Exercise 1 *Determine the images* $\mathcal{C}_t$ *under the stereographic projection* $\varphi : \widehat{\mathbb{C}} \to \mathbb{S}^2$ *of all the circles in* $\mathbb{C}$ *with equations* $x^2 + y^2 - (x + \sqrt{3}y)t = 0$, *with* $t > 0$ *a parameter. Do the centres of the circles on* $\mathbb{S}^2$ *belong to a great circle? Which one if so is the case?*

Exercise 2

1. *Determine the planes to which the circles on the sphere in Exercise 1 belong.*
2. *Using spherical coordinates* $(\cos\lambda\cos\theta, \cos\lambda\sin\theta, \sin\lambda)$ *give equations of the circles with variable the latitude* λ.

Exercise 3 *Determine the images under the stereographic projection* $\varphi : \widehat{\mathbb{C}} \to \mathbb{S}^2$ *of all the circles in* $\mathbb{C}$ *with equations* $x^2 + y^2 + 4\tan(t)x - 2\sqrt{3}y = 1$, *with* $t \in (-\pi/2, \pi/2)$ *a parameter.*

Exercise 4 *Determine the planes to which the image-circles* $\mathcal{C}_t$ *in* $\mathbb{S}^2$ *in Exercise 3 belong, and the distances from these* $\mathcal{C}_t$ *circles in Exercise 3 to the North Pole* $N(0, 0, 1)$

Exercise 5

1. *Calculate the angles in the spherical triangle with vertices* A*(longitude* $(r_1 - 2)p_2$ *deg, latitude 0 deg),* B*(longitude 0 deg, latitude* q_2q_1+30 *deg) and* C*(longitude* $(r_1-2)p_2$ *deg, latitude 60 deg). Observation* $(r_1 - 2)p_2$ *is a number with two digits:* p_2 *in the units and* $r_1 - 2$ *in the tens.*
2. *Calculate the area of the triangle if we consider that the triangle is on Earth (Radius of Earth is 6371 km).*

Exercise 6

1. *Determine the great circles that support the sides of the spherical triangle in exercise 5 and their images under the inverse of the inverse of the stereographic projection.*
2. *Determine the sizes of the edges of the spherical triangle (on Earth) in exercise 4 (Radius of Earth is 6371 km).*

Exercise 7 *Consider* $\mathbb{R}^4$ *with coordinates* $\mathbf{v} = (v_1, v_2, v_3, v_0)$ *(Observe that the real part of the quaternion is the fourth coordinate). Given the quaternion* $q_0 = a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}+a_0 = ((a_1, a_2, a_3), a_0)$, *show that the multiplication with* q_0 *to the left is a linear transformation in* $\mathbb{R}^4$ *with matrix* $\begin{pmatrix} a_0 & -a_3 & a_2 & a_1 \\ a_3 & a_0 & -a_1 & a_2 \\ -a_2 & a_1 & a_0 & a_3 \\ -a_1 & -a_2 & -a_3 & a_0 \end{pmatrix}$.

Determine the matrix for the multiplication with q_0 *on the right.*

Exercise 8 1. *Use Exercise 7 to determine the matrix of the rotation in* $\mathbb{E}^3$ *with quaternion* $q_0 = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} + a_0$, $|q_0| = 1$.

2. *Use Exercise 7 to determine the matrix of the reflection in* $\mathbb{E}^3$ *on the plane with equation* $\frac{x_1}{3} + \frac{2x_2}{3} - \frac{2x_3}{3} + 0(1) = 0$. *Observe that the last row and column of the matrix equal* $(0, 0, 0, 1)$.

Exercise 9 *Consider an animation of an object with start orientation (time* $t = 0$*) given by the quaternion* $q_s = (q_1 + 2, p_2 + 1, r_1 - 1, r_2 - 1)/l$ *where* $l = \sqrt{(q_1 + 2)^2 + (p_2 +1)^2 + (r_1 - 1)^2 + (r_2 - 1)^2}$ *and final orientation (time* $t = 1$*) given by the quaternion* $q_f = (3/5, (4/5)(1/3, -2/3, 2/3))$. *Give the orientations at times* $t = j/10$, $j = 0, 1, 2, \ldots, 10$

Exercise 10 1. *Let* q_1 *and* q_2 *be unit quaternions. Show that the functions* $R_1(x) = q_1 x q_1^{-1}$, $R_2(x) = q_2 x q_2^{-1}$, $x = (0, \mathbf{x})$ *are the same rotation if and only if* $q_1 = \lambda q_2$ *for some real number* $\lambda \neq 0$.

2. *Consider a quaternion* $q = (a\cos(\theta), a\sin(\theta)(a_2, a_3, a_4))$ *with* $a_2^2 + a_3^2 + a_4^2 = 1$. *Show that n a positive integer we have* $q = (a^n \cos(n\theta), a^n \sin(n\theta)(a_2, a_3, a_4))$

Exercise 11 *Identify and describe completely the isometry of* $\mathbb{E}^3$ *with matrix:* $\begin{pmatrix} 1/2 & -3/4 & \sqrt{3}/4 & 1/2 \\ 3/4 & 5/8 & \sqrt{3}/8 & -3/4 \\ -\sqrt{3}/4 & \sqrt{3}/8 & 7/8 & \sqrt{3}/4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. *(Hint: Study first the matrix and possible fixed elements). Decompose it as a product of reflections.*

Exercise 12 *Consider the rotation in* $\mathbb{S}^2$ *given by* $f_\alpha : \mathbb{S}^2 \to \mathbb{S}^2$, $f_\alpha(x, y, z) = (x\cos(\alpha) - \frac{y\sin(\alpha)\sqrt{3}}{2} + \frac{z\sin(\alpha)}{2}, \frac{x\sin(\alpha)\sqrt{3}}{2} + \frac{y(1+3\cos(\alpha))}{4} + \frac{\sqrt{3}z(1-\cos(\alpha))}{4}, \frac{-x\sin(\alpha)}{2} + \frac{\sqrt{3}y(1-\cos(\alpha))}{4} + \frac{z(3+1\cos(\alpha))}{4})$, $x^2 + y^2 + z^2 = 1$ *Show that the function* $f* = \varphi^{-1} f \varphi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ *(*φ *is the stereographic projection from the point* $P(0, 1/2, \sqrt{3}/2)$ *on the plane* $x_2/2 + \sqrt{3}x_3/2 = 0$*) induces an isometry* T *of* $\mathbb{E}^2$. *Determine and describe this isometry. (Hint: Compare with Exercises 3, 4 and 11)*

Exercise 13 a) *Consider the model of* $\mathbb{R}$ *given by the vertical line with equation* $x = 0$. *Show that the circumference* $\mathbb{S}^1$ ***is*** *the space* $\mathbb{R}P^1$ *of straight lines* $l_t = \{y = \tan(t)x, x \in \mathbb{R}\}$ *together with the line* $l_\infty = \{x = 0\}$. *Observe that the points in* $\mathbb{R}P^1$ *are lines on the Euclidean plane.*

b) *Express Part a) as a stereographic projection.*

Exercise 14 *Determine the latitude* λ, $0 < \lambda < \pi/2$, *such that the points* $A_1(\cos\lambda, 0, \sin\lambda)$, $A_2(0, \cos\lambda, \sin\lambda)$, $A_3(-\cos\lambda, 0, \sin\lambda)$, $A_4(0, -\cos\lambda, \sin\lambda)$, $A_5(\cos\lambda, 0, -\sin\lambda)$, $A_6(0, \cos\lambda, -\sin\lambda)$, $A_7(-\cos\lambda, 0, -\sin\lambda)$, $A_8(0, -\cos\lambda, -\sin\lambda)$ *are the vertices of a cube.*