

Quaternions and Rigid Movements in 3D

Hamilton found the quaternions $\mathbb{H}$ as a very useful tool for modelling Mechanics in 3D. The quaternions are in fact the vectors in $\mathbb{R}^4$ equipped with a product, and with a complex structure ($\mathbb{C}^2$). More importantly $\mathbb{S}^3$ are the quaternions of norm 1. $\mathbb{S}^3$ with this product is a Lie group, the geometry is the one coming from the Euclidean 4D-space, (à la Riemann).

If we want to do animation we are interested in expressing isometries (movements) of the Euclidean space in 3D. but we know that any isometry in 3D is the product of at most 4 reflections (for exactly 4 reflections the isometry is a screw motion). In fact, I am interested in the isometries of $\mathbb{S}^2$, but I can see them as the isometries in 3D that fix the origin (à la Riemann). If we are interested in orientation preserving isometries that fix the origin, then we have the product of two reflections in intersecting reflection planes: i.e. **a rotation whose axis is the intersection line of the reflecting planes**. Remember that a rotation of $\mathbb{R}^3$ is given by an orthogonal 3×3 -matrix with determinant 1.

I learnt about $\mathbb{S}^2, \mathbb{S}^3$ and the group of rotations of $\mathbb{R}^3$: $SO(3)$ in this building (linear algebra, general topology and diff geometry of surfaces).

The goal of this talk is to convince you that the quaternions of norm 1 (I heard Artal saying the norm of a vector), $\mathbb{S}^3$ and $SO(3)$ are the same, pardon, the second is the quotient of the first by the antipodal map. This gives a fast algorithm to draw rotations in 3D: SLERP (Spherical Linear InterPolation).

The following references are good:

- (1) A. Beardon, *Algebra & Geometry*, Cambridge University Press, 2005
- (2) J.N. Cederberg, *A Course in Modern Geometry*, Springer-Verlag 2001
- (3) J.I. Cogolludo, L. Ugarte, *Apuntes de Geometría Lineal*, Universidad de Zaragoza, 2021-2022.
- (4) D. Marsh, *Applied Geometry for Computer Graphics and CAD*, Springer-Verlag, 2005.
- (5) M. Arribas, A. Elipe, M. Palacios, *Quaternions and the rotation of a rigid body*, Celestial Mech Dyn Astr (2006) **96**, 239-251

1. Quaternions

1.1. Definitions. We begin with Hamilton's definition of quaternions $\mathbb{H} = \{q = (a_0, (a_1, a_2, a_3)); a_i \in \mathbb{R}, 0 \leq i \leq 3\}$, we write $q = (a_0, \mathbf{x})$, con $\mathbf{x} = (a_1, a_2, a_3)$.

a_0 is the **scalar, or real part**, and $\mathbf{x} = (a_1, a_2, a_3)$ is the **vectorial part** we say that a quaternion $q = (a_0, (a_1, a_2, a_3))$ es **pure or purely vectorial** if $a_0 = 0$

Some notation: $0 = (0, (0, 0, 0))$, $1 = (1, (0, 0, 0))$, $i = (0, (1, 0, 0))$, $j = (0, (0, 1, 0))$ y $k = (0, (0, 0, 1))$. Recall that $\{1, i, j, k\}$ form a basis for vector space $\mathbb{R}^4$.

The **norm** of a quaternion $q = (a_0, (a_1, a_2, a_3))$ is $|q| = \sqrt{a_0^2 + a_1^2 + a_2^2 + a_3^2}$. Recall that the only quaternion with 0 norm is 0.

Given a quaternion $q = (a_0, \mathbf{x})$, the **conjugate** of q is $\bar{q} = (a_0, -\mathbf{x})$. We have that $|q| = |\bar{q}|$.

We can see the 3-sphere as $\mathbb{S}^3 = \{q = (a_0, \mathbf{x}) \in \mathbb{H}; |q| = 1\}$, the **unitary** quaternions.

We can write a unitary quaternion $q = (\cos(\theta/2), \sin(\theta/2)(u_1, u_2, u_3))$ with $\mathbf{u} = (u_1, u_2, u_3)$ a unit vector.

Now $a_0^2 + |\mathbf{x}|^2 = 1$, so $a_0 = \cos(\theta/2)$, $|\mathbf{x}| = \sin(\theta/2)$. We have the polar coordinates for unitary quaternions $q = (\cos(\theta/2), \sin(\theta/2)\mathbf{n})$, with $\mathbf{n}$ a unit vector in $\mathbb{R}^3$.

The sum of quaternions is the sum as vectors:

$$q_1 = (a_0, \mathbf{x}), q_2 = (b_0, \mathbf{y}), q_1 + q_2 = (a_0 + b_0, \mathbf{x} + \mathbf{y}).$$

The external action of $\mathbb{R}$ is the same ($\mathbb{H}$ is the vector space $\mathbb{R}^4$).

But now we have a product that satisfies that any quaternion with non-zero norm has multiplicative inverse:

$$q_1 q_2 = (a_0 b_0 - \mathbf{x} \bullet \mathbf{y}, a_0 \mathbf{y} + b_0 \mathbf{x} + \mathbf{x} \times \mathbf{y})$$

1.2. Properties and Examples.

- (1) $q\bar{q} = |q|^2$
- (2) $q^{-1} = \frac{\bar{q}}{|q|^2}$
- (3) If $|q| = 1$, then $q^{-1} = \bar{q}$
- (4) The **unitary** quaternions: $\mathbb{S}^3$, the quaternions with norm 1, form a group with this product; moreover the product, and the inverse, are differentiable maps for $\mathbb{S}^3$.
- (5) A quaternion $q = (a_0, \mathbf{x})$ is a pure quaternion iff $q^2 = -a^2, a \in \mathbb{R}$.
- (6) If $q_1 = (0, \mathbf{x}), q_2 = (0, \mathbf{y})$ are pure quaternions, then $q_1 q_2 = (-\mathbf{x} \bullet \mathbf{y}, \mathbf{x} \times \mathbf{y})$, the energy and the moment in mechanics!!!

Notice that $q_2 q_1 = (-\mathbf{x} \bullet \mathbf{y}, -\mathbf{x} \times \mathbf{y})$

- (7) $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \quad \mathbf{ij} = \mathbf{k}, \mathbf{jk} = \mathbf{i}, \mathbf{ki} = \mathbf{j}, \quad \mathbf{ji} = -\mathbf{k}, \mathbf{kj} = -\mathbf{i}, \mathbf{ik} = -\mathbf{j}$
- (8) Given the quaternion $q = (a_0, (a_1, a_2, a_3))$; the multiplication with q to the left is a linear transformation, L_q , in $\mathbb{R}^4$ with matrix

$$\begin{pmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & -a_3 & a_2 \\ a_2 & a_3 & a_0 & -a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{pmatrix}.$$

- (9) In the same way: the multiplication with q to the right is a linear transformation, D_q , in $\mathbb{R}^4$

$$\text{with matrix } \begin{pmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & a_3 & -a_2 \\ a_2 & -a_3 & a_0 & a_1 \\ a_3 & a_2 & -a_1 & a_0 \end{pmatrix}.$$

- (10) Finally the linear map $\hat{R}_q : \mathbb{H} \rightarrow \mathbb{H}$ given by $R_q(x) = qxq^{-1}$ has matrix

$$R_q = \begin{pmatrix} 1 & 0 \\ 0 & \hat{A} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{a_0^2 + a_1^2 - a_2^2 - a_3^2}{|q|^2} & \frac{2a_1 a_2 - 2a_0 a_3}{|q|^2} & \frac{2a_1 a_3 + 2a_0 a_2}{|q|^2} \\ 0 & \frac{2a_1 a_2 + 2a_0 a_3}{|q|^2} & \frac{a_0^2 - a_1^2 + a_2^2 - a_3^2}{|q|^2} & \frac{2a_2 a_3 - 2a_0 a_1}{|q|^2} \\ 0 & \frac{2a_1 a_3 - 2a_0 a_2}{|q|^2} & \frac{2a_0 a_1 + 2a_2 a_3}{|q|^2} & \frac{a_0^2 - a_1^2 - a_2^2 + a_3^2}{|q|^2} \end{pmatrix}$$

Observations:

- (a) Observe first that the “fourth dimension ”(the scalar part) is fixed by R_q , as R_q is invertible it means that R_q preserves the vectorial part (pure quaternions are taken to pure quaternions), i.e. $R_q | : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is an invertible linear map (an affinity fixing the origin)
- (b) Notice that if $q = \lambda \hat{q}$, with λ real, ($\neq 0$,) then $R_q = R_{\hat{q}}$. Have you seen this kind of quotient in general topology or linear geometry?
- (c) The property above tells us that we need to work with $\mathbb{S}^3$, the unitary quaternions.
- (d) The matrix $\hat{A}$ given above is orthogonal, i.e. $R_q |$ is an isometry of $\mathbb{R}^3$ with the Euclidean metric. Moreover $\det(\hat{A}) = 1$, so we are talking of rotations with center the origin, i.e. rotations of $\mathbb{S}^2$.

2. The rotations of $\mathbb{S}^2$

2.1. Isometries of the Euclidean space in 3D. The rotations of $\mathbb{S}^2 = \{\mathbf{x} \in \mathbb{R}^3; |\mathbf{x}|^2 = 1\}$ are in fact rotations of the **vector** space $\mathbb{R}^3$ (fixing the origin). Why is it so interesting to see rotations of $\mathbb{R}^3$ fixing the origin as rotations of $\mathbb{S}^2$? Because of the stereographic projection(s) we recover **all** rotations in 2D (with any centre!!)

But if we are interested in applications (as we are in this lecture), to see the rotations of $\mathbb{S}^2$ is to see angles needed in navigation.

You have learnt, or will learn, in Linear Geometry two fundamental facts on isometries in 3D:

- a) First, the translations $T_{\mathbf{v}} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_{\mathbf{v}}(\mathbf{x}) = (\mathbf{x}) + (\mathbf{v})$, with vector $\mathbf{v} = (v_1, v_2, v_3)$ and matrix:
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ v_1 & 1 & 0 & 0 \\ v_2 & 0 & 1 & 0 \\ v_3 & 0 & 0 & 1 \end{pmatrix}$$
- b) Any isometry T is given by $T(\mathbf{x}) = \hat{T}(\mathbf{x}) + \mathbf{v}$, with $\hat{T}$ an orthogonal map of $\mathbb{R}^3$. In the way we write the matrix: $\begin{pmatrix} 1 & 0 \\ \mathbf{v} & \hat{A} \end{pmatrix}$.
- c) Any isometry is the product of at most four reflections.
- d) A reflection in a plane with equation $\mathbf{x} \bullet \mathbf{n} = d$, with $\mathbf{n}$ a unit normal vector to the plane (a point in $\mathbb{S}^2$, or a pure, unitary quaternion $n = (0, \mathbf{n})$) is the transformation $S_n(\mathbf{x}) = \mathbf{x} + 2(d - \mathbf{x} \bullet \mathbf{n})\mathbf{n}$. The matrix

$$\begin{pmatrix} 1 & 0 \\ 2d\mathbf{n} & \hat{A} \end{pmatrix}, \quad \det(\hat{A}) = -1, \quad \hat{A}\hat{A}^t = I_3, \quad \hat{A} = \hat{A}^t.$$

We can calculate the matrix $\hat{A}$ of the reflection with unitary pure quaternions:

- e) **Exercise:** Given a pure unitary quaternion $n = (0, \mathbf{n})$, the map $S_n : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $S_n(x) = -nx\bar{n} = n\mathbf{x}n$, $x = (0, \mathbf{x})$ is the reflection in the plane with equation $\mathbf{x} \bullet \mathbf{n} = 0$.

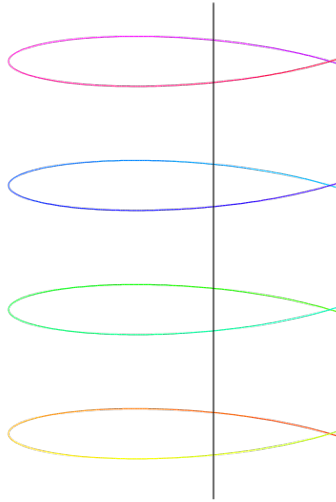
$$S_i(j) = iji = ki = j, \quad S_i(k) = iki = -ji = k, \quad S_i(i) = iiii = -i.$$

The example is the reflection in the yz -plane expressed in quaternions.

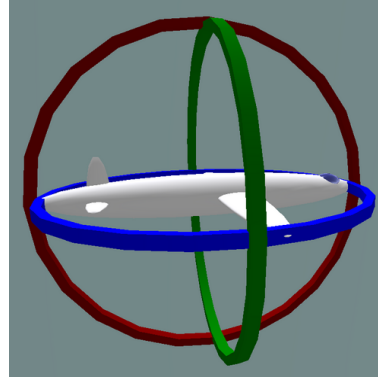
- f) Product of two reflections fixing the origin $R_{n_2 n_1}(x) = (-1)^2 n_2 n_1 x (n_2 n_1)^{-1}$, with $q = n_2 n_1$ a unitary quaternion.
- g) Considering orientation preserving isometries: they are the product of two or four reflections. The product of two reflections are **translations** (reflections in parallel planes) or **rotations** (reflections in intersecting planes, the axis is the intersecting line). The product of four reflections is a **screw motion**: $T(\mathbf{x}) = R(\mathbf{x}) + \mathbf{v}$, with R a rotation and $\mathbf{v}$ a directing vector of the axis of R .
- h) Of course we can express any rotation of $\mathbb{S}^2$ as a product of the so called **Euler angles** (Tait-Bryan in navigation), rotations around the axis i, j, k :

$$\text{yaw} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix}, \text{pitch} \cdot \begin{pmatrix} \cos(\varphi) & 0 & \sin(\varphi) \\ 0 & 1 & 0 \\ -\sin(\varphi) & 0 & \cos(\varphi) \end{pmatrix}, \text{roll} \begin{pmatrix} \cos(\psi) & -\sin(\psi) & 0 \\ \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

A first issue is to choose the order of multiplication. But a major problem is that we have singularities when any of the angles is $\pi/2, -\pi/2$, then we loose degree of freedom. This singularity is called the **gimbal lock**. By the way the problem of Apollo 11.



The trace of a screw motion (a helix)



The three Euler angles

2.2. Rotations, Quaternions and SLERP. Now, we look at Properties e) and f) above with detail, to obtain the beautiful result:

THEOREM 2.1. *Let $q = (\cos(\theta/2), \sin(\theta/2)\mathbf{u})$ with $\mathbf{u} = (u_1, u_2, u_3)$ a unit vector. The map $R_q : \mathbb{R}^3 \rightarrow \mathbb{R}^3, R_q(x) = qxq^{-1}$ is a clockwise rotation of $\mathbb{R}^3$ around the axis directed by $\mathbf{u}$ and by angle θ .*

To prove it we have to look again at Properties **e)** and **f)**: Take two pure quaternions n_1, n_2 in $\mathbb{S}^2$ (two unit vectors in $\mathbb{R}^3$) such that $\mathbf{n}_2 \bullet \mathbf{n}_1 = \cos(\theta/2)$ and $\mathbf{n}_2 \times \mathbf{n}_1 = \sin(\theta/2)\mathbf{u}$. We can calculate $n_2 n_1 = (0 - \cos(\theta/2), \sin(\theta/2)\mathbf{u}) = -q$. Now $S_{n_2} S_{n_1}(\mathbf{x}) = (-1)^2 (n_2 n_1) x (n_2 n_1)^{-1} = -q x (-q)^{-1} = q x \bar{q} = R_q(\mathbf{x})$.

This allows us to see rotations of $\mathbb{S}^2$ as points in $\mathbb{S}^3$, in fact classes, of points $\{q, -q\}$, and it gives us a fast procedure to render animation (or to control with a computer the orientations of a rigid body for navigation). The procedure is called **SLERP, Spherical Linear Interpolation**.

We have seen these animations for instance when in television it is shown the most probable trajectory of a throw in basket.

If we see an animation of a rotation of a rigid body as the fastest way of going in $\mathbb{S}^3$ from the start orientation (a rotation from the standard reference system in 3D) given by a (unitary) quaternion q_s to the final orientation of the rigid body, given by a (unitary) quaternion q_f . We can think that we have an arc of a great circle in $\mathbb{S}^3$ joining q_s and q_f . An intermediate orientation/quaternion

$$q_t = \alpha(t)q_s + \mu(t)q_f, \quad t \in [0, 1].$$

In $\mathbb{S}^3$ as subspace of $\mathbb{R}^4$ we have $q_s \bullet q_f = \cos(\theta)$. If $q_s \bullet q_t = \cos(t\theta)$, $q_f \bullet q_t = \cos((1-t)\theta)$, $t \in [0, 1]$, we have:

$$\cos(t\theta) = \alpha(t) + \mu(t) \cos(\theta), \quad \cos((1-t)\theta) = \alpha(t) \cos(\theta) + \mu(t)$$

Solving the system of LINEAR equations we obtain

$$\alpha = \frac{\sin((1-t)\theta)}{\sin(\theta)}, \quad \mu = \frac{\sin(t\theta)}{\sin(\theta)}$$

Then

$$q_t = \frac{\sin((1-t)\theta)}{\sin(\theta)} q_s + \frac{\sin(t\theta)}{\sin(\theta)} q_f$$

Observation/Exercise: In fact $q_t \in \mathbb{S}^3$ for $t \in [0, 1]$ (Hint: We must remember some trigonometric equalities).

Notice that from the point of view of the calculations, the animation consists in multiplying a number of 3D-points/3-vectors by a “double” family of unitary quaternions, each of them a linear combination (with easy functional coefficients) of two given quaternions and their conjugates.

Disclaimer: The functional coefficients are trigonometric functions. Any numerical calculation will begin with a truncation, so if one does not use high resolution, the animation will be deficient.

Finally, I used the 4x4-matrices because in that way I can write at once rotations and translations in 3D. But to do that I must look at the affine space in 3D as the affine part of the (real) projective 3-space. But the Theorem tells us that

$$SO(3) \cong \mathbb{R}P^3 = \mathbb{S}^3 / \langle \text{antipodal} \rangle$$