Solutions to Exercises for TATA55, batch 2, 2018

December 6, 2018

1. (3p) Suppose that G is a group, A, B are subgroups of G, and that $g \in G$. Show that $gA \cap gB$ is a left coset in $A \cap B$.

   Solution: Clearly, 'in' should be 'of'. Apologies!

   Then we claim that $g(A \cap B) = gA \cap gB$. If $h \in A \cap B$, then $gh \in LHS$, but since $h \in A$, we also have $gh \in gA$, and similarly for gB. Conversely, if $w \in gA \cap gB$, then $w = gh$ with $h \in A$ and $h \in B$, hence $h \in A \cap B$.

2. (3p) Determine subgroups K, H in D_4 such that

$$\{1\} \triangleleft K \triangleleft H \triangleleft D_4$$

   with all inclusions proper. Determine D_4/K and $(D_4/K)/(H/K)$.

   Solution: I had in mind that K should be normal in D_4, as well. If not, D_4/K is not a group, but rather the set of left cosets forms a D_4-set, i.e. D_4 acts on it. What meaning, if any, can be ascribed to $(D_4/K)/(H/K)$ in this scenario is not clear. I will choose K normal in D_4 below.

   Recalling that r, r^3 are conjugate in D_4, and that r^2 lies in the center, we choose subgroups which are unions of conjugacy classes, to make them normal in D_4: take $K = \{1, r^2\}$,$H = \{1, r, r^2, r^3\}$. Since these subgroups are all abelian, $K \triangleleft H$. Since $[D_4 : H] = 2$, $D_4/H \simeq C_2$. For the same reason, $H/K \simeq C_2$.

   On the other hand, D_4/K has 4 elements, but it is easy to see that no elemen has order 4 (the image of r has order 2). Thus, it is isomorphic to $C_2 \times C_2$. When modding out by a subgroup of size two, the resulting group has two elements, and is thus isomorphic to C_2. This is confirmed by the third isomorphism theorem, which yields that $D_4/K \simeq D_4/K)/(H/K)$.

3. (5p) Let G be a group.

   (a) Suppose that $S \subseteq G$ is a subset of G such that $gsg^{-1} \in S$ for all $g \in G$ and all $s \in S$. Show that $\langle S \rangle$, the subgroup generated by S, is normal in G.

   (b) Put $K = \langle \{ xyx^{-1}y^{-1} | x, y \in G \} \rangle$. Show that $K \triangleleft G$.

(c) Show that G/K is abelian.

(d) If $N \triangleleft G$ and G/N is abelian, show that $K \subseteq N$.

(e) If $K \subseteq H \leq G$, show that $H \triangleleft G$.

Solution:

Appearantly, some of this is covered in Svensson, so those of you that read the textbook industriously were rewarded for your ardour.

(a) Note that there is a smallest superset $\tilde{S} \supseteq S$ such that $\tilde{S}^{-1} \subseteq \tilde{S}$; this is obtained by simply adding all inverses of elements in S. Note furthermore that $\langle \tilde{S} \rangle = \langle S \rangle$.

Thus, we can without loss of generality assume that $S = S^{-1}$.

Then $gs^{-1}g^{-1} = (g^{-1}sg)^{-1} \in S^{-1} = S$. Let $hs_1^{\epsilon_1}s_2^{\epsilon_2}\cdots s_k^{\epsilon_k} \in \langle S \rangle$, where $\epsilon_j \in \{-1, 1\}$. Then $ghg^{-1} = gs_1^{\epsilon_1}g^{-1}gs^{\epsilon_2}s^{-1}\cdots gs^{\epsilon_k}s^{-1} \in \langle S \rangle$.

(b) The set K is closed under taking inverses, so the above result applies.

(c) Take $x, y \in G$. Then $xy = yx$ iff $xyx^{-1}y^{-1} = 1$. Modulo K, the latter identity always hold, so the image of x and y commute in G/K.

(d) The above reasoning shows that x, y commute in G/N iff N contains $xyx^{-1}y^{-1}$. Thus G/N is abelian iff $N \supseteq K$.

(e) Since G/K is abelian, and $H/K \leq G/K$, we have that $H/K \triangleleft G/K$. We can thus form $(G/K)/(H/K)$. Consider the surjective group homomorphism

$$\phi : G \to \frac{G/K}{H/K}$$

$$\phi(g) = g\frac{H}{K}$$

Since $\ker \phi = H$, we get that H is normal in G.

4. (3p) Let $G \subseteq S_{\mathbb{R}}$ be given by all affine maps $\phi_{a,b}$, $a, b \in \mathbb{R}$, $a \neq 0$, $\phi_{a,b}(x) = ax + b$.

(a) Show that G is a subgroup. Is it normal?

(b) Let $N = \{\, \phi_{1,b} \,|\, b \in \mathbb{R} \,\}$. Show that $N \triangleleft G$.

(c) Determine G/N.

Solution:

(a) We calculate

$$\phi_{a,b}(\phi_{c,d}(x)) = \phi_{a,b}(cx+d) = a(cx+d)+b = acx+ad+b = \phi_{ac,ad+b}(x) \quad (1)$$

so the set of affine maps are closed under composition. Furthermore, we see that $\phi_{1,0}$ is the identity, and that $\phi_{a,b}^{-1} = \phi_{1/a,-b/a}$. Hence the affine maps form group.

On the other hand, if f is a general bijection from $\mathbb{R}$ to $\mathbb{R}$, then

$$f(\phi_{a,b}(f^{-1}))(x) = f(af^{-1}(x) + b),$$

which is not in general an affine map; take for instance $f(x) = x^3$ with inverse $f^{-1}(x) = \operatorname{sgn}(x)|x|^{1/3}$.

(b) From (1) we get that $a = c = 1$ gives

$$\phi_{1,b} \circ \phi_{1,d} = \phi_{1,b+d},$$

and that $\phi_{1,b}^{-1} = \phi_{1,-b}$. Thus, N is a subgroup. We calculate

$$\begin{aligned}\phi_{a,b}(\phi_{1,d}(\phi_{1/a,-b/a}(x))) = \phi_{a,b}(\phi_{1,d}(x/a - b/a)) &= \phi_{a,b}(x/a - b/a + d) \\ &= a(x/a - b/a + d) = x - b + ad = \phi_{1,ad-b}(x),\end{aligned}$$

so N is normal in G.

(c) Since N denotes the translations, let us look at $K = \{\phi_{a,0} | a \neq 0\}$, the set of pure scalings. Then (1) shows that K is a subgroup. The map

$$G \ni \phi_{a,b} \mapsto \phi_{a,0} \in K$$

maps $\phi_{c,d}$ to $\phi_{c,0}$ and $\phi_{a,b} \circ \phi_{c,d} = \phi_{ac,ad+b}$ to $\phi_{ac,0} = \phi_{a,0} \circ \phi_{c,0}$, so it is a surjective group homomorphism onto K. The kernel is obviously N, and thus $G/N \simeq K$.

5. (4p) Let $[5] = \{1, 2, 3, 4, 5\}$, and let $X = \binom{[5]}{3}$, the set of unordered triplets of [5].

(a) S_5 acts naturally on [5]. Show that the induced action $\phi.\{a, b, c\} = \{\phi(a), \phi(b), \phi(c)\}$ indeed determines an action of S_5 on X.

(b) Determine the number of orbits of this action.

(c) Let $H = \langle(1, 2, 3, 4, 5)\rangle$ act on X as above. Determine the number of orbits.

(d) Same question for $K = \langle(1, 2)\rangle$.

(e) Partial credits if you solve the above questions for S_4 acting on $\binom{[4]}{2}$ instead.

Solution:

(a) The identity acts trivially, and

$$\xi.\phi.\{a, b, c\} = \xi.\{\phi(a), \phi(b), \phi(c)\} = \{\xi(\phi(a)), \xi(\phi(b)), \xi(\phi(c))\} = \xi\circ\phi.\{a, b, c\}.$$

(b) Given $A = \{a, b, c\}$ and $B = \{u, v, w\}$ there are precisely two bijections ϕ with $\phi(a) = u$, $\phi(b) = v$, $\phi(c) = w$. Thus all triplets live in one big happy orbit.

(c) In this case, all group elements except the identity have empty fixedpoint, so Burnside's lemma gives that the number of orbits is $\binom{5}{3}/5 = 10/5 = 2$. To identify these two orbits, we look at subsets as vectors $(a_1, a_2, a_3, a_4, a_5)$, where $a_i = 1$ if $i \in A$, and $a_i = 0$ otherwise. Then the action permutes this vector cyclically, and the vectors of weight 3 are divided into two orbits: one orbit, consisting of

$$(0,0,1,1,1),(1,0,0,1,1),(1,1,0,0,1),(1,1,1,0,0),(0,1,1,1,0)$$

where the zeroes are "cyclically adjacent", and another orbit, consisting of

$$(0,1,0,1,1),(1,0,1,0,1),(1,1,0,1,0),(0,1,1,0,1),(1,0,1,1,0)$$

where they are not.

(d) In this case, the generator $g = (12)$ has fixed points consisting of all triplets A such that either $1,2 \in A$ or $\{1,2\} \cap A = \emptyset$. There are 3 triplets of the first type, and one of the second type, so the fixed point of g has size $3+1 = 4$. Since the identity element fixes all 10 elements in X, Burnside's lemma tells us that the number of orbits is $\frac{1}{2}(10+4) = 7$.

6. (5p) Show that the number of conjugacy classes in a finite group G is given by

$$\frac{1}{|G|}\sum_{g\in G} |C_G(g)|, \qquad C_G(g) = \{h \in G | gh = hg\}.$$

Determine the number of conjugacy classes in D_8 and D_9.

Solution: The stabilizer of g, when G atcs on itself via conjugation, is precisely $C(g)$, so the statement is precisely Burnside's lemma.

In D_9, we have the relations $r^9 = 1, s^2 = 1$, $sr = r^8 s = r^{-1}s$. The identity obviously commutes with everything, and a rotation r^k certainly commutes with any other rotation r^ℓ. Furthermore, it does not commute with any reflection, since

$$r^k r^\ell s = r^m s, \qquad m \equiv k + \ell \mod 9$$

but

$$r^\ell s r^k = r^\ell r^{-1} s r^{k-1} = \cdots = r^{\ell-k} s$$

and this is equal to the previous expression iff

$$\ell + k \equiv \ell - k \mod 9,$$

hence, if $2k \equiv 0 \mod 9$, hence, if $r^k = 1$.

We conclude that $|C(r^k)| = 9$.

The above calculation shows that a reflection $h = r^\ell s$ does not commute with any rotation. It certainly commutes with itself. If $C(h)$ would contain any other reflection, then (since it is a subgroup) it would contain their product, which is a rotation. But $C(h)$ contains no rotations! Thus $|C(h)| = 2$.

Burnside's lemma now gives that the number of conjugacy classes in D_9 is

$$\frac{1}{18}(18 + 8 * 9 + 9 * 2) = 6.$$

For D_8, the rotation r^4 commutes with everything, whereas the other rotations commute with other rotations and the identity.

A reflection $h = r^\ell s$ now commutes with itself and r^4, thus $C(h) = \{1, h, r^4, r^4 h\}$, and Burnside's lemma gives the number of orbits as

$$\frac{1}{16}(16 + 16 + 6 * 8 + 8 * 4) = 7.$$

7. (1p+2p) Let $\mathbf{u} = (u_1, u_2)^t$ and $\mathbf{v} = (v_1, v_2)^t$ be two linearly independent vectors in $\mathbb{R}^2$, and let $B = \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix}$. Put $L = \{\, a\mathbf{u} + b\mathbf{v} | \, a, b \in \mathbb{Z}\}$. This is called the lattice spanned by $\mathbf{u}$ and $\mathbf{v}$.

   (a) Show that $L \leq \mathbb{R}^2$, and that $\mathbb{R}^2/L \simeq (\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z})$.

   (b) If $\mathbf{f}, \mathbf{g}$ are two other linearly independent vectors in $\mathbb{R}^2$, with associated lattice M and matrix C, show that $L = M$ if and only if $B = CU$ for some two-by-two matrix U with integral entries, and determinant ± 1.

Solution: Clearly

$$\begin{aligned} (a\mathbf{u} + b\mathbf{v}) + (c\mathbf{u} + d\mathbf{v}) &= (a + c)\mathbf{u} + (b + d)\mathbf{v} \\ -(a\mathbf{u} + b\mathbf{v}) &= (-a)\mathbf{u} + (-b)(\mathbf{v}), \end{aligned}$$

so L is a subgroup.

Since $\mathbf{u}, \mathbf{v}$ are linearly independent, they form an $\mathbb{R}$-basis for $\mathbb{R}^2$. Given a vector $\mathbf{w} \in \mathbb{R}^2$, we can uniquely write $\mathbf{w} = BY$, with $Y = (y_1, y_2)^t \in \mathbb{R}^2$. The map

$$\begin{aligned} F : \mathbb{R}^2 &\to (\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z}) \\ F(\mathbf{w}) &= (y_1 + \mathbb{Z}, y_2 + \mathbb{Z}) \end{aligned}$$

is a surjective group homomorphism with kernel L, so the first isomorphism theorem gives the desired result.

For the second part, we first recall that a two-by-two integer matrix is invertible, with an inverse that is also an integer matrix, if and only if it has determinant ± 1. This follows e.g. from the formula for the inverse of a two-by-two matrix.

Now assume that $B = CU$. (In the first drafts there was a misprint, with $B = UC$, apologies!) Then the columns of B are integer linear combinations of the columns of C, so any integer linear combination of the columns of B are also integer linear combinations of the columns of C. This proves that $L \subseteq M$. Since $C = BU^{-1}$, we get similarly that $M \subseteq L$.

If on the other hand $M = L$, then in particular the columns of B are in L, hence are integer linear combinations of the columns of C, so $B = CU$ for some two-by-two integer matrix U. Similarly, $C = BV$ for some two-by-two integer matrix V. Combining, we have that

$$BI = B = CU = (BV)U = B(VU),$$

so

$$B(I - UV) = 0.$$

Since B's columns are linearly independent, it is invertible (as a matrix over $\mathbb{R}$) so we conclude that

$$UV = I$$

hence that U, V are invertible, are each other's inverses, and since thay have integer entries, have determinant ± 1.

8. (4p) Denote by K the hypercube $K = \{ (x_1, x_2, x_3, x_4) | 0 \leq x_1, x_2, x_3, x_4 \leq 1 \}$, and let $V = \{ (x_1, x_2, x_3, x_4) | x_1, x_2, x_3, x_4 \in \{0, 1\} \}$ be the set of its vertices. Let $\mathbf{e}_1 = (1, 0, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0, 0)$, $\mathbf{e}_3 = (0, 0, 1, 0)$, $\mathbf{e}_4 = (0, 0, 0, 1)$. Let $\Delta = \text{conv}(\mathbf{0}, \mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4)$.

   Let $\sigma \in S_4$ act on K by $\sigma.(x_1, x_2, x_3, x_4) = (x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)})$.

   (a) What are the sizes of the orbits?

   (b) Put $\Delta_\sigma = \{ \sigma.(x_1, x_2, x_3, x_4) | (x_1, x_2, x_3, x_4) \in \Delta \}$. Determine the volume of this simplex, and show that

   $$K = \cup_{\sigma \in S_4} \Delta_\sigma,$$

   with $\Delta_\sigma \cap \Delta_\tau$ a simplex of dimension < 4, hence of volume zero, for $\sigma \neq \tau$.

   (c) Partial credit if you solve the corresponding questions for $n = 3$, even more partial if you look at $n = 2$.

Solution:

The simplex Δ is given by the inequalities

$$0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq 1$$

For any partition S of $[4] = \{1, 2, 3, 4\}$ into parts which are intervals, we put

$$A_S = \{ (x_1, x_2, x_3, x_4) \in \Delta | x_i = x_j \text{ iff } \{i, j\} \subset P \in S \}.$$

So, we have for instance that

$$\begin{aligned} (1/3,1/3,1/3,1/3) &\in S_{\{[4]\}} = \{ (x_1,x_2,x_3,x_4) \in \Delta | x_1 = x_2 = x_3 = x_4 \} \\ (1/3,1/3,2/3,2/3) &\in S_{\{\{1,2\},\{3,4\}\}} = \{ (x_1,x_2,x_3,x_4) \in \Delta | x_1 = x_2 < x_3 = x_4 \} \end{aligned}$$

We do not have any A_S which demands that $x_1 = x_3$ and $x_2 = x_4$, since the corresponding intervals overlap.

We then have that A_S partition Δ. For a set partition S, there is a corresponding numerical partition λ which records the size of the parts. We let B_λ be the union of the corresponding A_S, so that

$$\begin{aligned} B_4 &= A_{\{[4]\}} \\ B_{3+1} &= A_{\{\{1,2,3\},\{4\}\}} \cup A_{\{\{\{1\},\{2,3,4\}\}\}} \\ B_{2+2} &= A_{\{\{1,2\},\{3,4\}\}} \\ B_{2+1+1} &= A_{\{\{1,2\},\{3\},\{4\}\}} \cup A_{\{\{1\},\{2,3\},\{4\}\}} \cup A_{\{\{1\},\{2\},\{3,4\}\}} \\ B_{1+1+1+1} &= A_{\{\{1\},\{2\},\{3\},\{4\}\}} \end{aligned}$$

Then Δ is also partitioned into the B_λ's, and any orbit of an element in B_λ is contained in B_λ.

- For $(x,x,x,x) \in B_4$, the orbit have size 1.
- For $(x,x,x,y) \in B_{3+1}$, the orbit have size 4.
- For $(x,x,y,y) \in B_{2+2}$, the orbit have size 6.
- For $(x,x,y,z) \in B_{2+1+1}$, the orbit have size 12.
- For $(x,y,z,u) \in B_{1+1+1+1}$, the orbit have size 24.

Since the symmetric group S_4 acts as a group of isometries, the induced maps are volume-preserving, so $\sigma.\Delta$ have the same volume as Δ, which is

$$\frac{1}{4!} \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix} = \frac{1}{24}.$$

We have that

$$\Delta_\sigma = \left\{ (x_1,x_2,x_3,x_4) \middle| 0 \le x_{\sigma^{-1}(1)} \le x_{\sigma^{-1}(2)} \le x_{\sigma^{-1}(3)} \le x_{\sigma^{-1}(4)} \le 1 \right\}$$

It is evident that the Δ_σ's cover the unit cube.

If (i,j) is an inversion of σ, i.e., $i < j$ but $\sigma(i) > \sigma(j)$, then

$$(x_1,x_2,x_3,x_4) \in \Delta \cap \Delta_\sigma \implies x_i = x_j$$

Since all A_S except $A_{\{[4]\}}$ have dimension < 4, and hence 4-dimensional volume zero, we get that $\Delta \cap \Delta_\sigma$ has volume zero. Similarly, $\Delta_\sigma \cap \Delta_\tau$ have volume zero if $\sigma \neq \tau$.