Exercises for TATA55, batch 1, 2019

September 19, 2019

Solutions to the exercises below should be handed in no later than October XXX, 2019.

1. (3p) Assuming Bezout (the gcd is a integral linear combination of its arguments) show that a prime dividing a product divides one of the factors.

2. (3p) Find all solutions to
$$3x + 5y = 999, \qquad x, y \in \mathbb{Z}$$
with y positive and x even.

3. (3p) Solve (by hand, though you may check your answer using machines)
$$\begin{aligned} x &\equiv 57 \mod 96 \\ x &\equiv 95 \mod 98 \end{aligned}$$

4. (1p+2p) Let X be a finite set, and let $\sim$ be an equivalence relation on X. Let $T = \{x_1, \ldots, x_n\}$ be a transversal, i.e., a choice of exactly one element from each equivalence class.
   (a) Define a map $N : X \to X$ such that
      i. $N \circ N = N$, and
      ii. $x \sim y$ iff $N(x) = N(y)$, and
      iii. $N(X) = T$.
   (b) If $N : X \to X$ satisfies the first two of the above conditions, need $N(X)$ be a transversal?

5. (1p+3p) Let $X = \{a, b\}$, and let X^* denote the monoid of all "words" in the letters in X, including the empty word; the operation is concatenation.

   Suppose that u, v are non-empty words in X^*.

   Show that
$$uv = vu$$
   if and only if u, v are both powers of some common word, i.e. if there exists a non-empty word z, and positive integers k, ℓ, such that
$$u = z^k, \qquad v = z^\ell$$
   As an example, $u = abaaba$ and $v = abaabaaba$ commute.

6. (3p) Let M be a monoid, and let $x \in M$. Suppose that there exists positive integers $0 < n < m$ such that $x^n = x^m$. Show that there are positive integers N, s such that, for all non-negative integers a, b, it holds that
$$x^{N+a} = x^{N+b} \quad \iff \quad a \equiv b \mod s$$
   ((2p) If you can't solve this one, give an example of a monoid M and an element x such that $x^7 = x^{11}$ is the earliest coincidence, and show that for non-negative a, b, $x^{7+a} = x^{7+b}$ if and only if $a \equiv b$ mod 4.)