Exercises for TATA55, batch 1, 2021

September 10, 2021

Solutions to the exercises below should be handed in no later than October XXX, 2021.

1. (3p) Find all solutions to
$$13x + 17y = 999, \qquad x, y \in \mathbb{Z}$$
Which solutions have both x and y positive?

2. (3p) Solve (by hand, though you may check your answer using machines)
$$\begin{aligned} x &\equiv 112 \mod 95 \\ x &\equiv 115 \mod 97 \end{aligned}$$

3. (3p) For any positive integer n, solve the linear Diophantine equation
$$x_1 + x_2 + \cdots + x_n = 1$$

4. (3p) Let m, n be positive integers. Find a square matrix A such that $A^m = A^{m+kn}$ for all $k \in \mathbb{N}$, and such that furthermore $A^0, \ldots, A^{m-1}$ are all distinct.

5. (3p) For $2 \le n \le 20$, check whether $U_n = Z_n^\times$ is cyclic. Formulate a bold hypothesis.

6. (6p) Let $\mathcal{T} = \{ z \in \mathbb{C} \mid |z| = 1 \}$ be the group of complex numbers of unit modulus, under multiplication.
   (a) For any positive integer n, show that there is a unique subgroup of $\mathcal{T}$ of cardinality n, and that this subgroup is cyclic.
   (b) Show that there are infinitely many different subgroups of $\mathcal{T}$ that are isomorphic to the infinite cyclic group C_∞.
   (c) Show that, for each positive integer r, there are infinitely many different subgroups of $\mathcal{T}$ that are isomorphic to the direkt product C_∞^r.
   (d) Show that, for each positive integers n, r, there are infinitely many different subgroups of $\mathcal{T}$ that are isomorphic to the direkt product $C_n \times C_\infty^r$.
   (e) Show that any “interval” in $\mathcal{T}$ containing 1 generates $\mathcal{T}$.
   (f) Must any generating set of $\mathcal{T}$ contain such an interval?

7. (4p) Let G be the group generated by the permutations $(1, 2)$, $(2, 3)$, and $(4, 5)$.
   (a) List all elements in G
   (b) Is G cyclic?
   (c) Is G abelian?
   (d) Which well-known group is G isomorphic to?